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A Study on Root Properties of Super Hyperbolic GKM algebra $SHGGH_{71}^{(3)}$

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Abstract:

In this paper, the Super hyperbolic generalized Kac-Moody algebras of indefinite type is defined and the family $SHGGH_{71}^{(3)}$ is also delimited. Different cases under this family is studied with one imaginary simple root. The symmetrizable GGCM of $SHGGH_{71}^{(3)}$ is given by $\begin{pmatrix} -k & -a & -b & -c \\ -d & 2 & -2 & -2 \\ -e & -2 & 2 & -2 \\ -f & -2 & -2 & 2 \end{pmatrix}$ with the conditions $ae = bd \text{ and } cd = af$ where $k,a,b,c,d,e,f \in \Re_+ \cup \{-2\} \forall i$. For the family $SHGGH_{71}^{(3)}$, the properties of roots is computed for different cases. There are 438 non-isomorphic connected Dynkin diagram associated with the Super-hyperbolic generalized Kac-Moody algebra $SHGGH_{71}^{(3)}$. The existence and non existence of purely imaginary roots, strictly imaginary roots and special imaginary roots for this family $SHGGH_{71}^{(3)}$ is elucidated with examples.

INTRODUCTION

The Generalized Kac-Mody algebra (abbreviated as GKM algebra) was introduced by Borcherds in [2]. Bennett and Casperson delimited the special and strictly imaginary roots of KM algebras in [1] and [3]. In KM algebras all simple roots are real and in GKM algebras, imaginary simple roots exists. In GKM algebras, the Generalized Generalized Cartan Matrices (GGCM) are extensions of finite, affine and indefinite types of KM algebras. Stanumoorthy. The purely imaginary, strictly imaginary and special imaginary roots for finite, affine and hyperbolic types for specific families in [5]-[9]. For the familyEB_2, Xinfang Song computed the root multiplicities and also determined the root structure in [10] and [11].

Keywords: Dynkin diagram, real, imaginary, Roots, Super hyperbolic.

AMS MSC 2010 Code: 17B67

PRELIMINARIES

The notations and definitions used in this paper are as in [2].

Definition 2.1 [2]: In GKM algebras the Dynkin diagrams is defined as follows: To every GGCM A is associated a Dynkin diagram S(A) defined as follows : S(A) has n vertices and vertices i and j are connected by max $\{|a_{ij}|,|a_{ji}|\}$ number of lines if $a_{ij}.a_{ji} \leq 4$ and there is an arrow pointing towards i if $|a_{ij}|>1$. If $a_{ij}.a_{ji} > 4$, i and j are connected by a bold faced edge, equipped with the ordered pair $(|a_{ij}|,|a_{ji}|)$ of integers. If $a_{ii} = 2$, i^{th} vertex will be denoted by a white circle and if $a_{ii} = 0$, i^{th} vertex will be denoted by a crossed circle. If $a_{ii} = -k$, $k > 0$, i^{th} vertex will be denoted by a white circle with $-k$ written above the circle within the parenthesis.

Definition 2.2 [4]: We say a GGCM $A = (a_{ij})_{i,j=1}^{n}$ is of Super Hyperbolic type if A is of indefinite type and the Dynkin diagram associated with A has a connected, proper sub diagram of hyperbolic type , whose GGCM is of order n-1. We then say the associated Dynkin diagram and the corresponding GKM algebra to be of Super hyperbolic type (abbreviated as SH type).

Definition 2.3 [1]: Let α be an imaginary root of g(A) which is symmetrizable GKM algebra. We call α a special imaginary root, if α satisfies the following conditions:

i) $(\alpha, \alpha) \neq 0$; ii)$r_\alpha(\Delta) = \Delta$, $r_\alpha(\Delta^{re}) = \Delta^{re}$, $r_\alpha(\Delta^{im}) = \Delta^{im}$; r_α preserves root multiplicities.

Definition 2.4 [3]: A root $\gamma \in \Delta^{im}$ is said to be strictly imaginary if for every $\alpha \in \Delta^{re}$, $\alpha + \gamma \text{ or } \alpha - \gamma$ is a root. The set of all strictly imaginary roots is denoted by Δ^{sim}.

Definition 2.5 [7]: A root $\alpha \in \Delta_+^{im}$ is called purely imaginary if for any $\beta \in \Delta_+^{im}$, $\alpha + \beta \in \Delta_+^{im}$ The KM algebra is said to have the purely imaginary property if every imaginary root is purely imaginary.

Note: In this paper, the GGCM consists of one imaginary simple root.

CLASSIFICATION OF DYNKIN DIAGRAMS OF GKM ALGEBRAS $SHGGH_{71}^{(3)}$:

In this section, the GGCM $\begin{pmatrix} -k & -a & -b & -c \\ -d & 2 & -2 & -2 \\ -e & -2 & 2 & -2 \\ -f & -2 & -2 & 2 \end{pmatrix}$ is considered, where $k, p_i, q_i \in \Re_+ \cup \{-2\} \forall i$

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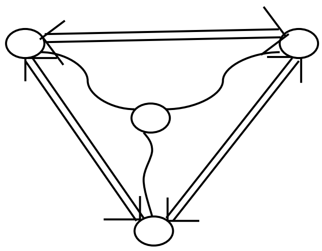
Proposition 3.1 : There are 438 non-isomorphic connected Dynkin diagrams associated with the GGCM of indefinite Super hyperbolic type $SHGGH_{71}^{(3)}$.

Proof : The associated Dynkin diagram with the hyperbolic family $H_{71}^{(3)}$ is



We extend the 4[th] vertex with $H_{71}^{(3)}$ and all possible combinations of connected non-isomorphic Dynkin diagrams are determined for the associated GGCM of Super hyperbolic type $SHGGH_{71}^{(3)}$. Here  can be represented by one of the possible 9 edges:

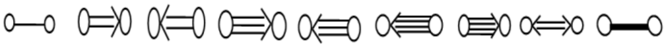
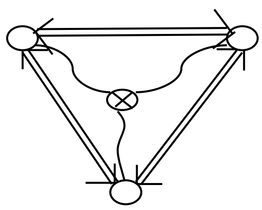
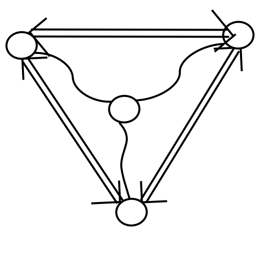
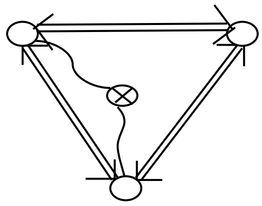
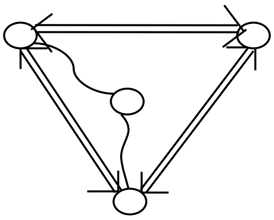
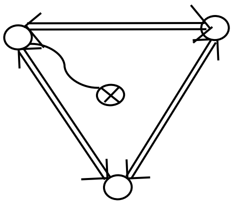
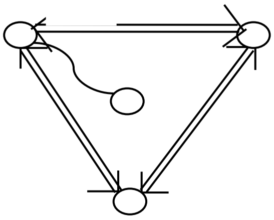


Table 1

Extended Dynkin diagram of Super hyperbolic type $SHGGH_{18}^{(3)}$	**Corresponding GGCM**	**Number of possible Dynkin diagrams**
When k=0, 	$\begin{pmatrix} 0 & -a & -b & -c \\ -d & 2 & -2 & -2 \\ -e & -2 & 2 & -2 \\ -f & -2 & -2 & 2 \end{pmatrix}$	By connecting the fourth vertex to all the other three vertices, there exists 9^3 connected Dynkin diagrams in which 564 are isomorphic Dynkin diagrams. Excluding these, we get 165 connected non-isomorphic Dynkin diagrams, for both the case, when k = 0 and k > 0 (i.e), In total we get 230 Dynkin diagrams for both the cases.
When k>0, 	$\begin{pmatrix} -k & -a & -b & -c \\ -d & 2 & -2 & -2 \\ -e & -2 & 2 & -2 \\ -f & -2 & -2 & 2 \end{pmatrix}$	
When k=0, 	$\begin{pmatrix} 0 & -a & 0 & -c \\ -d & 2 & -2 & -2 \\ 0 & -2 & 2 & -2 \\ -f & -2 & -2 & 2 \end{pmatrix}$	Among the 3 vertices, we connect the fourth vertex with any of the two vertices with different combinations by the 9 possible edges. Therefore, in this case, the associated connected Dynkin diagrams are 2x 9^2 = 243 and excluding the 198 isomorphic Dynkin diagrams, we get 45 connected non-isomorphic Dynkin diagrams, for both

Extended Dynkin diagram of Super hyperbolic type $SHGGH_{18}^{(3)}$	**Corresponding GGCM**	**Number of possible Dynkin diagrams**
When k>0, 	$\begin{pmatrix} -k & -a & 0 & -c \\ -d & 2 & -2 & -2 \\ 0 & -2 & 2 & -2 \\ -f & -2 & -2 & 2 \end{pmatrix}$	the case, when k = 0 and k > 0.
When k=0, 	$\begin{pmatrix} 0 & -a & 0 & 0 \\ -d & 2 & -2 & -2 \\ 0 & -2 & 2 & -2 \\ 0 & -2 & -2 & 2 \end{pmatrix}$	In this case, we connect the fourth vertex independently to the other three vertices by the 9 possible edges. Thus, the possible number of connected Dynkin diagrams associated with $SHGGH_{71}^{(3)}$ is 3 x 9 = 27. But by joining the vertices 2 and 4, we get 18 isomorphic Dynkin diagrams. Thus, by deleting these isomorphic diagrams we get, 9 diagrams when k = 0 and 9 diagrams when k > 0.
When k>0, 	$\begin{pmatrix} -k & -a & 0 & 0 \\ -d & 2 & -2 & -2 \\ 0 & -2 & 2 & -2 \\ 0 & -2 & -2 & 2 \end{pmatrix}$	

Thus there exists 438 types of connected, non isomorphic Dynkin diagrams associated with the GGCM of $SHGGH_{71}^{(3)}$.

PROPERTIES OF ROOTS

Consider the symmetrizable GGCM of $SHGGH_{71}^{(3)} = \begin{pmatrix} -k & -a & -b & -c \\ -d & 2 & -2 & -2 \\ -e & -2 & 2 & -2 \\ -f & -2 & -2 & 2 \end{pmatrix}$ with the conditions $ae = bd$ and $cd = af$ where $k,a,b,c,d,e,f \in \Re_+ \cup \{-2\} \forall i$.

We have $\coprod = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$, $\prod^{re} = \{\alpha_2, \alpha_3, \alpha_4\}$ and $\prod^{im} = \{\alpha_1\}$.

The non-degenerate symmetric bilinear form are given as,

$$(\alpha_1,\alpha_1) = -kd, (\alpha_1,\alpha_2) = -ad, (\alpha_1,\alpha_3) = -ae, (\alpha_1,\alpha_4) = -af, (\alpha_2,\alpha_2) = 2a,$$

$$(\alpha_2,\alpha_3) = -2a, (\alpha_2,\alpha_4) = -2a, (\alpha_3,\alpha_3) = 2a, (\alpha_3,\alpha_4) = -2a, (\alpha_4,\alpha_4) = 2a$$

The fundamental reflections are

$$r_2(\alpha_1) = \alpha_1 + ad\alpha_2, r_2(\alpha_2) = -\alpha_2, r_2(\alpha_3) = \alpha_3 + \alpha_2, r_2(\alpha_4) = \alpha_4 + 2\alpha_2,$$

$$r_3(\alpha_1) = \alpha_1 + ae\alpha_3, r_3(\alpha_2) = \alpha_2 + 2\alpha_3, r_3(\alpha_3) = -\alpha_3, r_3(\alpha_4) = \alpha_4 + 2\alpha_3,$$

$$r_4(\alpha_1) = \alpha_1 + af\alpha_4, r_4(\alpha_2) = \alpha_2 + 2\alpha_4, r_4(\alpha_3) = \alpha_3 + 2\alpha_4, r_4(\alpha_4) = -\alpha_4$$

Here, $\Delta_+^{im} = \bigcup_{w \in W} w(K)$ where K is given by

$$\begin{aligned}
&K = \{k_1\alpha_1 + k_2\alpha_2 + k_3\alpha_3 + k_4\alpha_4 / k_1 \in \mathrm{N}, k_2, k_3, k_4 \in \mathrm{Z}_+,\\
&2k_2 \le dk_1 + 2k_3 + 2k_{4,} 2k_3 \le ek_1 + 2k_2 + 2k_4 \ \ and \ \ 2k_4 \le fk_1 + 2k_2 + 2k_3 \ \ with\\
&k_2 = 0 \Rightarrow 2k_3 \le ek_1 + 2k_4 \ \ and \ \ 2k_4 \le fk_1 + 2k_3, k_3 = 0 \Rightarrow 2k_2 \le dk_1 + 2k_4 \ \ and \ \ 2k_4 \le fk_1 + 2k_2\\
&k_4 = 0 \Rightarrow 2k_2 \le dk_1 + 2k_3 \ \ and \ \ 2k_3 \le ek_1 + k_2, k_2 = k_3 = 0 \Rightarrow 2k_4 \le fk_1; \ \ k_2 = k_4 = 0 \Rightarrow 2k_3 \le ek_1\\
&k_3 = k_4 = 0 \Rightarrow 2k_2 \le dk_1; \ \ k_2 = k_3 = k_4 = 0 \Rightarrow k_1 = 1\}
\end{aligned}$$

Note that, the Weyl group is infinite.

Root System of $SHGGH_{71}^{(3)}$ **:** In this section, we discuss the root properties for the Super Hyperbolic family $SHGGH_{71}^{(3)}$.

Real Roots: All simple real roots have same length 2a.

Roots of Height 2:

1) $(\alpha_1+\alpha_2, \alpha_1+\alpha_2) = -kd+2a-2ad$

Case (i): When $k \neq 0$

$$(\alpha_1 + \alpha_2, \alpha_1 + \alpha_2) = \begin{cases} -kd + 2a - 2ad & is \ imaginary \ if \ a < d \\ 0 & is \ isotropic \ if \ d = 0 \end{cases}$$

Case (ii): When k=0 ;

$(\alpha_1+\alpha_2, \alpha_1+\alpha_2) = 2a-2ad$

$$(\alpha_1 + \alpha_2, \alpha_1 + \alpha_2) = \begin{cases} 2a - 2ad & is \ imaginary \ if \ a < d \\ 0 & is \ isotropic \ if \ a = 0 \end{cases}$$

2) $(\alpha_1+\alpha_4, \alpha_1+\alpha_4) = -kd+2a-2cd$

Case (i): When $k \neq 0$

$$(\alpha_1 + \alpha_4, \alpha_1 + \alpha_4) = \begin{cases} -kd + 2a - 2cd & is \ real \ if \ k = 1, a > c \ \& \ a > d \\ -kd + 2a - 2cd & is \ real \ if \ k = 1, d = c \ \& \ a > c \\ -kd + 2a - 2cd & is \ imaginary \ if \ a < d \\ -kd + 2a - 2cd & is \ imaginary \ if \ k > 1, d = c \ \& \ a > c \\ 0 & is \ isotropic \ if \ a = c = 0 \end{cases}$$

Case (ii): When k=0 ;

$(\alpha_1+\alpha_4, \alpha_1+\alpha_4) = 2a-2cd$

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$$(\alpha_1+\alpha_4,\alpha_1+\alpha_4)=\begin{cases}2a-2cd & \text{is imaginary if } a>c \;\&\; d>a\\ 0 & \text{is isotropic if } a \,\&\, d=0 \text{ or } a=d=1\\ 2a & \text{is real if } a\neq 0, cd<a \,\&\, a>c\end{cases}$$

3) $(\alpha_2+\alpha_3, \alpha_2+\alpha_3) = 2a+2a-4a = 0$. Therefore, $\alpha_2+\alpha_3$ is an isotropic root.

4) $(\alpha_2+\alpha_4, \alpha_2+\alpha_4) = 0$ so that $\alpha_2+\alpha_4$ is isotropic.

Similarly, the other cases of height 2 roots $\alpha_1+\alpha_4$, $\alpha_3+\alpha_4$ can be discussed.

Roots of Height 3:

1) $(2\alpha_1+\alpha_2, 2\alpha_1+\alpha_2) = -kd+2a-4ad$

Case (i): When $k \neq 0$

$$(2\alpha_1+\alpha_2,2\alpha_1+\alpha_2)=\begin{cases}-kd+2a-4ad & \text{is imaginary if } a<d\\ 0 & \text{is isotropic if } d=0\end{cases}$$

Case (ii): When k=0 ;

$(2\alpha_1+\alpha_2, 2\alpha_1+\alpha_2) = 2a- 4ad$

$$(2\alpha_1+\alpha_2,2\alpha_1+\alpha_2)=\begin{cases}2a-4ad & \text{is imaginary if } a\geq d\\ 0 & \text{is isotropic if } a=0\end{cases}$$

2) $(\alpha_2+\alpha_3+\alpha_4, \alpha_2+\alpha_3+\alpha_4) = -6a$

$$(\alpha_2+\alpha_3+\alpha_4,\alpha_2+\alpha_3+\alpha_4)=\begin{cases}-6a & \text{is imaginary if } a>0\\ 0 & \text{is isotropic if } a=0\end{cases}$$

Similarly, roots of height 3 of all other cases can be delimited.

Roots of Height 4:

1) $(2\alpha_1+2\alpha_2, 2\alpha_1+2\alpha_2) = -4kd+8a-8ad$

Case (i): When $k \neq 0$

$$(2\alpha_1+\alpha_2,2\alpha_1+\alpha_2)=\begin{cases}-kd+2a-4ad & \text{is imaginary if } a<d\\ 0 & \text{is isotropic if } d=0\end{cases}$$

Case (ii): When k=0 ;

$(2\alpha_1+2\alpha_2, 2\alpha_1+2\alpha_2) = 8a- 8ad$

$$(2\alpha_1+2\alpha_2,2\alpha_1+2\alpha_2)=\begin{cases}8a-8ad & \text{is imaginary if } ad>1 \text{ or } a<d\\ 0 & \text{is isotropic if } a=d=1 \text{ or } a>d \text{ and } d=1\end{cases}$$

2) $(3\alpha_2+2\alpha_3, 3\alpha_2+\alpha_3) = 18a+2a-12a = 8a$, which is a real root.

Proposition 4.1: Let A= $\begin{pmatrix} -k & -a & -b & -c \\ -d & 2 & -2 & -2 \\ -e & -2 & 2 & -2 \\ -f & -2 & -2 & 2 \end{pmatrix}$ be the symmetrizable GGCM of $SHGGH_{71}^{(3)}$, where $k, p_i, q_i \in \Re_+ \cup \{-2\} \forall i$.

There exists no special imaginary root of g(A).

Proof: Suppose $\alpha = \sum_{i-1}^{n+1} k_i\alpha_i \in K \subseteq \Delta_+^{im}, k_i \in Z_+ \forall i$ be a special imaginary root of g(A). We have

$$(\alpha,\alpha) = 2ak_2^2 + 2ak_3^2 + 2ak_4^2 - ekk_1^2 - 2aek_1k_2 - 2bek_1k_3 - 2cek_1k_4 - 4ak_2k_3 \\ - 4ak_2k_4 - 4ak_3k_4 \quad < 0$$

Let *(α,α)=A*. By the reflection of imaginary root definition, we have

$$\left.\begin{aligned} r_\alpha(\alpha_1) = \alpha_1 + \frac{2a}{A}(kk_1 + ak_2 + bk_3 + ck_4)\alpha, r_\alpha(\alpha_2) = \alpha_2 - \frac{2a}{A}(2k_2 - dk_1 - 2k_3 - 2k_4)\alpha \\ r_\alpha(\alpha_3) = \alpha_3 - \frac{2a}{A}(2k_3 - ek_1 - 2k_2 - 2k_4)\alpha, r_\alpha(\alpha_4) = \alpha_4 - \frac{2a}{A}(2k_4 - fk_1 - 2k_2 - 2k_3)\alpha \end{aligned}\right\} \ldots(1)$$

Then for a special imaginary root α, we have $r_\alpha(\alpha_2) = \alpha_2; r_\alpha(\alpha_3) = \alpha_3; r_\alpha(\alpha_4) = \alpha_4$ … (2)

From the above equations (1) and (2), we get

$$(2k_2 - dk_1 - 2k_3 - 2k_4)\alpha = (2k_3 - ek_1 - 2k_2 - 2k_4)\alpha = (2k_4 - fk_1 - 2k_2 - 2k_3)\alpha = 0$$

Then, $3k_4 = k_1(2d - 2e + 5f)$ is absurd. Therefore, no special imaginary root exists for g(A), where A is a symmetrizable $SHGGH_{71}^{(3)}$.

Proposition 4.2: The Super hyperbolic GKM algebra $SHGGH_{71}^{(3)}$ satisfies the purely imaginary property.

Proof: There are some generalized Kac-Moody algebras, in which the purely imaginary roots exists and does not exists. Here the, SHGKM algebra $SHGGH_{71}^{(3)}$ proves the purely imaginary property.

Here, α_1 is an imaginary simple root and if any root added with α_1, the resultant is also an imaginary root and also support of α is connected. Therefore, all imaginary roots are purely imaginary and hence, for any $\alpha \in \Delta_+^{im}$ and for any $\beta \in \Delta_+^{im}$ we get $\alpha + \beta \in \Delta_+^{im}$ is a root.

Example : $2\alpha_1+\alpha_2$ satisfies the purely imaginary property.

(i.e). $\alpha = \alpha_1$, $\beta = \alpha_1+\alpha_2$; we get $\alpha+\beta =$ 2a-4ad (where a ≥ d), which satisfies the purely imaginary property.

Proposition 4.3: The Super hyperbolic GKM algebra $SHGGH_{71}^{(3)}$ satisfies the strictly imaginary property.

Proof: Since support of α is connected, the addition or subtraction of any combination of α_i (i=1,2,3&4) is a root. Hence, for any $\alpha \in \Delta^{re}$ and for any $\gamma \in \Delta_+^{im}$ we get $\alpha + \gamma$ is a root. Therefore, the SHGKM algebra $SHGGH_{71}^{(3)}$ satisfies the strictly imaginary property.

Example : $\alpha_2+\alpha_3+\alpha_4$ satisfies the strictly imaginary property.

(i.e). $\alpha = \alpha_2$, $\gamma = \alpha_3+\alpha_4$; we get $\alpha+\gamma =$ -6a is a root, which proves the strictly imaginary property.

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