2 – Equitable Domination in Fuzzy Graphs

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Abstract
In this paper, 2 – equitable dominating set and 2 – equitable domination number of a fuzzy graph is introduced. Some results on 2 – equitable dominating sets are proved upper and lower bounds of 2 – equitable domination number are obtained. The new parameters connected 2 – equitable dominating set, connected 2 – equitable domination number and exact 2 – equitable dominating set, exact 2 – equitable domination number of a fuzzy graph are discussed.

Keywords : strong neighbours, 2- dominating set, 2- domination number, 2- equitable dominating set, 2- equitable domination number.

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INTRODUCTION
Zadeh[8] introduced the concept of fuzzy sets in the year 1965. In 1975, fuzzy graph was introduced by Rosenfeld[4]. The notion of domination in fuzzy graphs was developed by A. Somasundaram and S. Somasundaram[6]. Nagroorgani and Chandrasekaran[3] discussed about domination in a fuzzy graph using strong arcs. The concept of degree equitable domination in graphs was introduced by Venkatasubramanian Swaminathan and Kuppusamy Markandan Dharmalingam[7]. Sivakumar.S., Soner.N.D., and Anwar Alwardi[5] introduced the concept 2-equitable domination in graphs. The concept of equitable domination in fuzzy graphs was introduced by Dharmalingam and Rani[2]. In this paper, the 2-equitable domination sets and its numbers are defined and discussed.

PRELIMINARIES

Definition 2.1
Let \( G^* = (V,E) \) be a graph with vertex \( V \) and edge set \( E \subseteq V \times V \). Let \( \sigma \) and \( \mu \) be a fuzzy set of \( V \) and \( E \) respectively. Then \( G = (\sigma, \mu) \) be a fuzzy graph if \( \mu(u,v) \leq \sigma(u) \wedge \sigma(v) \) for all \((u,v) \in E \) and is denoted by \( G = (\sigma, \mu) \).

Definition 2.2
Let \( G = (\sigma, \mu) \) be a fuzzy graph then the order and size are defined as \( p = \sum_{u \in V} \sigma(u) \) and \( q = \sum_{(u,v) \in E} \mu(u,v) \).

Definition 2.3
The neighbourhood degree of a vertex \( u \) is defined to be the sum of the weights of the vertices adjacent to \( u \) and is denoted by \( d_N(u) \). The minimum neighbourhood degree of \( G \) is \( \delta_N(G) = \min\{d_N(u) : u \in V \} \) and the maximum neighbourhood degree of \( G \) is \( \Delta_N(G) = \max\{d_N(u) : u \in V \} \).

Definition 2.4
An arc \((u,v)\) in a fuzzy graph \( G = (\sigma,\mu) \) is said to be strong if \( \mu^o(u,v) = \mu(u,v) \) then \( u, v \) are called strong neighbours.

Definition 2.5
The strong neighbourhood of the vertex \( u \) is defined as \( N^s(u) = \{v \in V | (u,v) \text{ is a strong arc} \} \).

Definition 2.6
A vertex \( u \in V \) dominates \( v \in V \) if \( (u,v) \) is a strong arc. A subset \( D \) of \( V \) is called a dominating set of a fuzzy graph \( G \) if for every \( v \in V - D \) there exists \( u \in D \) such that \( u \) dominates \( v \). The minimum scalar cardinality taken over all dominating set is called domination number and it is denoted by \( \gamma \) of a fuzzy graph \( G \).

Definition 2.7
Let \( u \) and \( v \) be two vertices in a fuzzy graph \( G \). A subset \( D \) of \( V \) is called an equitable dominating set if for every \( v \in V - D \) there exist a vertex \( u \in D \) such that \( uv \in E(G) \) and \( |d(u) - d(v)| \leq 1 \) and \( \mu(u,v) \leq \sigma(u) \wedge \sigma(v) \). The minimum scalar cardinality of an equitable dominating set in a fuzzy graph is called equitable domination number and is denoted by \( \gamma_e(G) \).

Definition 2.8
A subset \( D \) of \( V \) is called a 2 – dominating set of \( G \) if for every vertex \( v \in V - D \) there exist atleast two strong neighbours in \( D \). The 2 – domination number of a fuzzy graph \( G \) is the minimum cardinality of a set of all 2 – dominating set of \( G \) and is denoted by \( \gamma_2 \).
2 – EQUIVALENT DOMINATION IN FUZZY GRAPHS

In this section, 2-equitable dominating set and 2-equitable domination numbers of a fuzzy graph are defined. The relation among domination number, equitable domination number and 2-equitable domination number are also obtained.

Definition 3.1

An equitable dominating set \( D \subseteq V \) of a fuzzy graph \( G = (\sigma, \mu) \) is called 2 – equitable dominating set if for every vertex \( v \in V - D \) there exist \( v \in D \) or \( v \) is equitable dominated by atleast two vertices in \( D \).

The minimum scalar cardinality of an 2 – equitable dominating set of \( G \) is called the 2 – equitable dominating number of \( G \) and is denoted by \( \gamma_{2e}(G) \).

Example 3.2

[Diagram showing vertices a, b, c, d, e with edges and values 0.9, 0.7, 0.8, 0.9, 0.8]

Equitable Dominating set \( D \) of a fuzzy graph are \( \{b, e\}, \{a, d\}, \{e, c\} \Rightarrow \gamma_e = 1.8 \)

2 – equitable dominating set = \( \{e, d, b\}, \gamma_{2e} = 2.7 \)

Proposition 3.3

i) \( \gamma(G) \leq \gamma_e(G) \leq \gamma_{2e}(G) \)

ii) \( \gamma_{2}(G) \leq \gamma_{2e}(G) \)

Proof

From the definition, 2-equitable dominating set of a fuzzy graph \( G \), it is clearly that for any fuzzy graph \( G \) any 2 – equitable dominating set \( D \) is also an equitable dominating set and is also dominating set. Hence \( \gamma(G) \leq \gamma_e(G) \leq \gamma_{2e}(G) \)

Similarly, since every 2 – equitable dominating set is 2 – dominating set for any fuzzy graph \( G \). Hence \( \gamma_{2}(G) \leq \gamma_{2e}(G) \).

Definition 3.4

2 – equitable dominating set \( D \) is said to be minimal if no proper subset of \( D \) is 2 – equitable dominating set.

Proposition 3.5

For any fuzzy graph \( G \) with order \( p \), then \( \sum_{v_i \neq v_j} \min(\sigma(v_i), \sigma(v_j)) \leq \gamma_{2e}(G) \leq p \)

Proof

Let \( D \) be a dominating set of a fuzzy graph \( G \) having atleast two vertices has minimum of \( V \) which is a sum of minimum value of a vertices\( v_i, v_j \in D, \gamma_{2e}(G) \leq p \) it is obviously true.

Theorem 3.6

Let \( G \) be a fuzzy graph, \( \gamma_{2e}(G) = p \) iff the fuzzy graph \( G \) has adjacent to less than two vertices.

Proof

Let \( G \) be a fuzzy graph then \( \gamma_{2e}(G) = p \) then definition of fuzzy graph has all vertices in dominating set \( D \). which shows that every vertex in \( G \) has adjacent to less than two vertices. Conversely, \( G \) be a fuzzy graph has adjacent to less than two vertices then every vertex in are in dominating set. Which is \( \gamma_{2e}(G) = p \)

Theorem 3.7

Let \( D \) is a minimal 2 – equitable dominating set then \( V - S \) contains minimal 2 – equitable dominating set if every vertex of \( V \) in a fuzzy graph \( G \) adjacent to more than two vertices in \( V \)

Proof

Let \( D \) be a minimal 2 – equitable set of \( G \) suppose that \( V - D \) is not an equitable dominating set, then there exist atleast one vertex \( v \in D \) which is not equitable adjacent to any vertex in \( V - D \). Therefore \( V - D \) is equitable adjacent to atleast two vertices in \( D \) then \( D - \{v\} \) is an 2 – equitable dominating set which is a contradiction. Hence every vertex in \( D \) must be equitable adjacent to atleast one vertex in \( V - D \). Hence \( V - D \) is equitable dominating set which contains minimal equitable dominating set.

Corollary 3.8

Every connected fuzzy graph has minimum 2 – equitable dominating set \( D \) then \( V - D \) need not be 2 – equitable dominating set of \( G \).

Proof

Let \( D \) be a 2 – equitable dominating set of \( G \) satisfies the condition also \( |d(u) - d(\nu)| \leq 1 \), suppose \( v \in V \), then \( v \) be in every 2 – equitable dominating set of a fuzzy graph \( G \), since it has only one neighbor vertex. This one neighbor also strong neighbor of \( v \). Which shows that every vertex in \( V - D \) does not has two strong neighbors for \( v \). This implies that \( V - D \) is not a 2 – equitable dominating set of \( G \).
Theorem 3.9
Let $G$ be a connected fuzzy graph has no non–equitable edge and $H$ is spanning subgraph of $G$ then $\gamma_{2e}(G) \leq \gamma_{2e}(H)$

Proof
Let $G$ be a connected fuzzy graph and $H$ is the spanning subgraph of $H$. Consider $D$ is minimum $2$–equitable dominating set of $G$, $D$ also an $2$–equitable dominate all the vertices in $V(H) - D$ that is $D$ is an $2$–equitable dominating set in $H$. Hence $\gamma_{2e}(G) \leq \gamma_{2e}(H)$.

Theorem 3.10
For cyclic fuzzy graph $G$ with odd cycle then,
$$\gamma_{2e} = \begin{cases} \gamma_e + \min \sigma(v_i), & v_i \notin D \text{ if odd cycle} \\ \gamma_e, & \text{if even cycle} \end{cases}$$

Theorem 3.11
For any fuzzy graph $G$, $\gamma_e + \min \sigma(v_i) \leq \gamma_{2e}(G)$, for $v_i \notin D$.

Proof
Let $D$ be $2$–equitable dominating set with minimum cardinality $\gamma_{2e}$. For any vertex $v_i \in D, D - \{v_i\}$ is equitable dominating set. Hence $\gamma_e + \min \sigma(v_i) \leq \gamma_{2e}(G)$.

Theorem 3.12
For any fuzzy graph $G$ of order $p$, size $q$ and maximum neighbourhood degree $\Delta_N$ of fuzzy graph $G$. Which shows that $\Delta_N + \Delta_N(\Delta_N - 1) = \Delta_N^2$. The $2$–equitable dominating set has another vertex with maximum degree of $G$, i.e. $2\Delta_N^2 + 1$. In the same way, an edge $v_1, v_2$ of $G$ enables to dominate atmost itself $2(\Delta_N - 1) + 2 = 2\Delta_N$ and $2(\Delta_N - 1) + 2(\Delta_N - 1)^2 = 2\Delta_N^2 - 2\Delta_N$, which has maximum number of vertices adjacent to it. i.e $2\Delta_N^2 + 1$. The result is $\gamma_{2e}(G) \geq \frac{p + q}{2\Delta_N^2 + 1}$ is obvious.

CONNECTED 2–EQUITABLE DOMINATION IN FUZZY GRAPHS

Definition 4.1
Let $G = (\sigma, \mu)$ be a fuzzy graph. An $2$–equitable dominating set $D \subseteq V(G^*)$ if the subgraph of $G$ induced by $D$ is connected. The connected $2$–equitable dominating of $G$ with minimum cardinality is called connected $2$–equitable dominating number of a fuzzy graph $G$ and it is denoted by $\gamma_{2ce}$.

Observation 4.2
For any fuzzy tree $T$, with size $p$ then $\gamma_{2ce} = p$

Proposition 4.3
For any fuzzy graph $G, \gamma(G) \leq \gamma_e(G) \leq \gamma_{2e}(G) \leq \gamma_{2ce}(G)$

Theorem 4.4
Let $G$ be a fuzzy graph without any equitable isolated vertices, then $\gamma_{2ce}(G) \leq \Delta_N + 1$

Proof
Let $v$ be any vertex with $d(v) = \Delta_N(G)$ then obviously $N(v)$ is connected equitable dominating set and hence $\gamma_{2ce}(G) \leq \Delta_N + 1$.

Example 4.5

![Diagram](image)

Connected $2$–equitable dominating set = {a, b, c, e}, $\gamma_{2ce} = 2.4$
Theorem 4.6
For any fuzzy graph, \( \gamma_{2ce}(G) \geq \frac{2p}{\Delta N + 2} \)

Proof
Let \( D \) be a minimum connected 2 – equitable dominating set and let \( k \) number of strong edges between dominating set \( D \) and \( V - D \). Since the degree of each vertex in \( D \) is atmost \( \Delta N \), which shows that \( \sum \mu(u_i, u_j) \leq \Delta N \). But each vertex in \( V - D \) is adjacent to at least 2 vertices in \( D \). \( \sum \mu(u_i, u_j) \geq 2(p - \gamma_{2ce}) \) combining these two inequalities produce \( \gamma_{2ce}(G) \geq \frac{2p}{\Delta N + 2} \).

Example 5.2

Equitable dominating set of a fuzzy graph \( G = \{v_1, v_4, v_6, v_{10}\} \), \( \gamma_e = 1.9 \)

Exact 2 – equitable dominating set = \( \{v_2, v_4, v_6, v_8, v_{10}\} \),
\( \gamma_{2ee} = 2.7 \)

Theorem 5.3
If \( G \) be a fuzzy graph has an exact 2 – equitable dominating set then all such sets have the same size.

Proof
Let \( D_1, D_2 \) be the two exact 2 – equitable dominating sets of \( G \). Let us write \( C = D_1 \cap D_2 \) and let \( X_0 \) and \( X_1 \) be the subsets of \( D_1 - I \) such that every vertex of \( X_0 \) has no neighbours in \( I \) and every vertex of \( X_1 \) has exactly one strong neighbor in \( I \). \( \gamma_{2ee} \) is an exact 2 – equitable dominating set \( Z \) in which every vertex of \( X_0 \) has exactly one neighbor in \( Y_0 \) and every vertex of \( X_1 \) has exactly two neighbours in \( Y_0 \). This implies \( |X_0| = |Y_0| \) and thus \( |D_1| = |D_2| \). Hence \( \gamma_{2ee} \) is an exact 2 – equitable dominating set, every vertex of \( X_1 \) has exactly one neighbor in \( Y_0 \) and \( Y_1 \), and every vertex of \( X_0 \) has exactly two neighbours in \( Y_0 \) and \( Y_1 \).

Proof
Let \( D \) be an exact 2 – equitable dominating set of a fuzzy graph \( G \), then \( \gamma_{2ee} \leq \frac{2p}{\delta N + 1} \)

Equitable domination number of \( G \) = \( 2(p - \gamma) \) since \( D \) is an exact 2 – equitable dominating set. \( D \) induces the matching of \( G \) if it has two strong neighbours. Then which implies that \( \gamma_{2ee} \leq \frac{2p}{\delta N + 1} \).

Proposition 5.5
Every exact 2 – equitable dominating is 2 – equitable dominating set, but converse need not be true.
REFERENCES


