On Some Fixed Point Theorems through Weak** Commutative in 2-Metric Space

Sujatha Kurakula¹*, Dr. V. Srinivaskumar ²

¹Research Scholar in JNTUH, Hyderabad-500070, Telangana State, India. (*corresponding Author)
²Assistant Professor, Department of Mathematics, JNTU College of Engineering, JNTU Hyderabad, Kukatpally -500044, Telangana State, India.

Abstract
In this present research article, we prove the existence of a common fixed point for three self-mappings defined on a complete 2-metric space through weak** commutativity of maps.

Keywords: Fixed point , 2-metric space , weak** commuting mapping .

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INTRODUCTION
The notion of 2-metric space was introduced by Gahler [1] in 1963 as a generalization of area function for Euclidean triangles. Many fixed point theorems were established by various authors like Brouwer, Banach, Schauder etc. A point \( x \in X \) is said to be a fixed point of a self-map \( f : X \to X \) if \( f(x) = x \), where \( X \) is a non-empty set. Theorems concerning fixed points of self-maps are known as fixed point theorems. Most of the fixed point theorems were proved for contraction mappings. It is well known that every contraction on a metric space is continuous. The converse is not necessarily true. The identity mapping on \([0,1]\) simply serves the counter example.

In this present work we consider Weak ** Commuting maps on a 2-metric space. Let \( T_1 \) and \( T_2 \) be two mappings from a metric space \((X,d)\) into itself. \( T_1 \) and \( T_2 \) are said to commute if \( T_1 T_2 x = T_2 T_1 x \), for all \( x \) in \( X \).

Sessa [5] introduced the concept of weak commutativity in metric spaces. In subsequent years the condition of weak commutativity was again made weaker. Weak* commutativity was introduced in metric space. In recent years weak** commutativity has been introduced and some theorems have been established. The existence of fixed point for weak**commutative self-maps in 2-metric space are studied.

In this research article we present the concepts of weak** commutativity in 2-metric space.
COMMON FIXED POINT THEOREMS FOR A WEEK **COMMUTING PAIR OF MAPPINGS

In this section, we have some results on common fixed points for Three self-maps of a 2-complete metric space using the concept of week **commuting maps

**Theorem 2.1:** let \((X, d)\) be a complete 2-metric space and \(A, S, T\) be three mappings of \(X\) into \(X\). Let \(S^2(x) \cup T^2(x) \subseteq A^2x\) and the pairs of mappings \(A, S\) and \(A, T\) be Weak** commutative and

1. \[d(S^2x, T^2y, a) \leq \alpha \left[ d(A^2x, S^2x, a)^w \right] + \beta \left[ d(A^2x, A^2y, a)^r \right] \]

For all \(x, y, a\) in \(X\) with \(AX \neq AY\) and for some \(\alpha, \beta, \gamma \in [0, 1]\) with \(0 < (\alpha + \beta) < 1\) and \(2r + w = 1\) when \(w \neq 0\). Then \(A, S, T\) have a unique common fixed point if either of \(A, S, T\), is continuous.

**Proof:** For any \(x_0 \in X\), we have \(x_1 \in X\), such that \(S^2x_0 = A^2x_1\) for \(S^2(x) \subseteq A^2(x)\) and so on.

Inductively we have a common sequence \(\{y_n\}\) defined as \(y_{2n+1} = A^2x_{2n+1} = S^2x_{2n}\) for \(n = 0, 1, 2, 3, \ldots\)

\[d(y_{2n+2}, y_{2n+1}, a) = d(A^2x_{2n+2}, A^2x_{2n+1}, a) = d(T^2x_{2n-1}, S^2x_{2n}, a)\]

\[= d(S^2x_{2n}, T^2x_{2n}, a) \leq \alpha \left[ d(A^2x_{2n}, S^2x_{2n}, a)^w \right] \]

\[+ \beta \left[ d(A^2x_{2n}, A^2x_{2n-1}, a)^r \right] \]

\[d(y_{2n+1}, y_{2n}, a) \leq (\alpha + \beta) \left[ d(y_{2n}, y_{2n+1}, a)^w \right] + \left[ d(y_{2n-1}, y_{2n}, a)^r \right] \]

\[d_{2n} \leq (\alpha + \beta)^{1-r-w} d_{2n-1}\]

Where \(d_{2n} = d(y_n, y_{2n+1}, a)\) and \(d_{2n-1} = d(y_n, y_{2n-1}, a)\)

Again

\[d_{2n+1} = d(y_{2n+1}, y_{2n+2}, a) = d(A^2x_{2n+1}, A^2x_{2n+2}, a)\]

\[= d(S^2x_{2n}, T^2x_{2n+1}, a) \leq \alpha \left[ d(A^2x_{2n}, S^2x_{2n}, a)^w \right] \]

\[+ \beta \left[ d(A^2x_{2n}, A^2x_{2n+1}, a)^r \right] \]

\[d_{2n+1} = \alpha [d_{2n}]^{1+w} + \beta [d_{2n}]^{1-r-w} [d_{2n+1}]^{1+w}\]
case (I) When \( w \neq 0 \) then we obtain

\[
d_{2n} = (\alpha + \beta) \frac{1}{r} d_{2n-1}
\]

And

\[
d_{2n+1} = (\alpha + \beta) \frac{1}{r} \left[ d_{2n} \right]^{1-r} \left[ d_{2n+1} \right]^{r}
\]

Or

\[
\left[ d_{2n+1} \right] = (\alpha + \beta) \frac{1}{r} \left[ d_{2n} \right]
\]

case (II) When \( w = 0 \), we get

\[
d_{2n} \leq (\alpha + \beta) \frac{1}{r} d_{2n-1}
\]

And

\[
d_{2n+1} = \alpha \left[ d_{2n} \right]^{1-r} + \beta \left[ d_{2n} \right]^{r} \left[ d_{2n+1} \right]^{r}
\]

We claim that

\[
d_{2n+1} \leq d_{2n}
\]

If it is not so suppose that \( d_{2n+1} \geq d_{2n} \) then we have

\[
d_{2n+1} \leq \alpha \left[ d_{2n} \right]^{1-r} + \beta \left[ d_{2n} \right]^{r} \left[ d_{2n+1} \right]^{r}
\]

\[
= (\alpha + \beta) d_{2n+1} \quad \text{This is a contradiction}
\]

Since \((\alpha + \beta) \leq 1\)

Therefore

\[
d_{2n+1} \leq d_{2n} \leq q d_{2n-1} \leq q^2 d_{2n-2} \leq \ldots \ldots \leq q^a d_0 \to 0
\]

as \( n \to \infty \).

Since \( q = (\alpha + \beta) \frac{1}{r} < 1 \).

Thus in both cases the sequence \( \{y_n\} \) is a Cauchy sequence.

Therefore \( \{A^2 x_{2n}\} \) is a Cauchy sequence and so converges to a point \( u \) in \( X \).

Also as \( n \to \infty \) \( \{A^2 x_{2n}\} = \{S^2 x_{2n-1}\} \) and \( \{A^2 x_{2n+1}\} = \{T^2 x_{2n}\} \) converges to \( u \).

Because they are subsequences of the sequence \( \{A^2 x_{2n}\} \)

First let A is continuous then the sequence \( \{A^2 x_{2n}\} \) and \( \{A^2 S^2 x_{2n}\} \) converges to a point \( A^2 u \)

Since A is weak** commute with S we have

\[
d \left( S^2 A^2 x_{2n}, A^2 u, a \right) \leq d \left( S^2 A^2 x_{2n}, A^2 u, A^2 S^2 x_{2n} \right) + d \left( S^2 A^2 x_{2n}, A^2 S^2 x_{2n}, a \right) + d \left( A^2 S^2 x_{2n}, A^2 u, a \right)
\]

Which imply on letting \( n \to \infty \) that \( \{S^2 A^2 x_{2n}\} \) also converges to \( A^2 u \). i.e \( S^2 u = A^2 u \)

Now we claim that \( T^2 u = A^2 u \). Suppose not then we have

\[
d \left( S^2 A^2 x_{2n}, T^2 u, a \right) \leq \alpha \left[ d \left( A^4 x_{2n}, S^2 A^2 x_{2n}, a \right) \right]^{r-w} + \beta \left[ d \left( A^4 x_{2n}, A^2 u, a \right) \right]^{r-w} + d \left( A^4 x_{2n}, A^2 u, a \right) \]

When \( n \to \infty \), we deduce that \( d \left( A^2 u, T^2 u, a \right) < 0 \), a contradiction. So \( T^2 u = A^2 u \).

Now suppose that \( S^2 u \neq T^2 u \), then

\[
d \left( S^2 u, T^2 u, a \right) \leq \alpha \left[ d \left( A^2 u, S^2 u, a \right) \right]^{r-w} + d \left( A^2 u, T^2 u, a \right) \]

Thus we have \( S^2 u = T^2 u = A^2 u \).

A similar conclusion we get if we assume that A is weak ** commute with S, we deduce that

\[
d \left( A^2 S^2 x_{2n}, Su, a \right) \leq d \left( A^2 S^2 x_{2n}, S^2 A^2 x_{2n} \right) + d \left( A^2 S^2 x_{2n}, Su, a \right)
\]

Which implies as \( n \to \infty \) that \( \{A^2 S^2 x_{2n}\} \) converges to \( Su \)

Now using this we have

\[
d \left( S^2 A^2 x_{2n}, T^2 x_{2n+1}, a \right) \leq \alpha \left[ d \left( A^4 x_{2n}, S^2 A^2 x_{2n}, a \right) \right]^{r-w} + d \left( A^4 x_{2n}, T^2 x_{2n+1}, a \right) \]

Letting \( n \to \infty \)

\[
d \left( Au, u, a \right) \leq \beta d \left( Au, u, a \right), \text{ a contradiction}
\]

Thus \( Au = u \) so \( A^2 u = u \)
Similarly as done above \( Su = Tu = u \) Therefore \( u \) is the common fixed point of \( A, S \) and \( T \).

Now suppose that \( S \) is continuous, then the sequence \( \{ A^2 S x_{2n} \} \) converges to \( Su \).

\( A \) being weak** commute with \( S \) we have

\[
\begin{align*}
d \left( A^2 S x_{2n}, Su, a \right) & \leq d \left( A^2 S x_{2n}, Su, S^2 A x_{2n} \right) \\
+ d \left( A^2 S x_{2n}, S^2 A x_{2n}, a \right) + d \left( S^2 A x_{2n}, Su, a \right)
\end{align*}
\]

Letting \( n \to \infty \) we observe that \( \{ A^2 S x_{2n} \} \) converges to \( Su \).

Thus \( Su = u \) and so \( S^2 u = u \).

As above we can show that \( T^2 u = A^2 u = S^2 u \).

Thus again we have \( u = Tu = Su = Au \)

i.e. \( u \) is the common fixed point of \( A, S, T \).

If the mapping \( T \) is continuous instead of \( S \) analogously we can show that \( u \) is the common fixed point of \( A, S, T \).

Now we claim that \( u \) is the unique common fixed point of \( A, S \) and \( T \).

For this let \( v \neq u \) be another common fixed point, then

\[
d \left( u, v, a \right) = d \left( S^2 u, T^2 v, a \right) \leq \\
\alpha \left[ d \left( A^2 u, S^2 u, a \right) \right]^{\gamma} \left[ d \left( A^2 v, T^2 v, a \right) \right]^{1-\gamma} \\
+ \beta \left[ d \left( A^2 u, T^2 u, a \right) \right]^{\gamma-\gamma} \left[ d \left( S^2 u, T^2 v, a \right) \right]^{\beta-\gamma} \\
= \beta \left[ d \left( u, v, a \right) \right]^{\gamma-\gamma} \left[ d \left( u, v, a \right) \right]^{\beta-\gamma} \\
d \left( u, v, a \right) \leq \beta d \left( u, v, a \right), \text{ which is a contradiction.}
\]

Thus \( u \) is the unique common fixed point of \( A, S \) and \( T \).

**Theorem 2.2:** Let \( A, S \) and \( T \) be three self mappings of a complete 2- metric space \( (X, d) \). Let

\( S^2(\chi) \cup T^2(\chi) \subseteq A^2 \chi \) if \( A, S \) and \( T \) be Weak** commutative pairs and the following condition holds.

\[
d \left( S^2(\chi), T^2(\chi), a \right) \leq \alpha \left[ d \left( A^2(\chi), S^2(\chi), a \right) \right]^{\beta-\gamma} \left[ d \left( A^2(\chi), T^2(\chi), a \right) \right]^{\beta-\gamma} \\
X, 0 \leq \alpha < 1, 0 \leq \beta < 1, 0 \leq \gamma \text{ with } 0 < \beta + \gamma \leq 1. \]

If either of \( A, S \) and \( T \) is continuous Then \( A, S \) and \( T \) have a unique common fixed point.

**Proof:** For any arbitrary point \( x_0 \in X \), construct the sequence \( \{ y_{2n} \} \) as in theorem (2.1). Using (1) it is easy to show that \( \{ y_{2n} \} \) is Cauchy sequence. Since \( X \) is complete, So there exist a point \( u \) in \( X \).

Now, Let \( A \) is continuous, then the sequence \( \{ A^2 x_{2n} \} \) and \( \{ A^2 S^2 x_{2n} \} \) converge to a point \( A^2 u \).

\( A \) is weak** commutative with \( S \), we have

\[
d \left( S^2 A^2 x_{2n}, A^2 u, a \right) \leq d \left( S^2 A^2 x_{2n}, S^2 A^2 x_{2n}, a \right) + d \left( A^2 S^2 x_{2n}, A^2 u, a \right) \\
\]

Which imply on letting \( n \to \infty \) that \( \{ S^2 A^2 x_{2n} \} \) also converges to \( A^2 u \) i.e \( S^2 u = A^2 u \).

Now we claim that \( T^2 u = A^2 u \). Suppose not then we have

\[
d \left( S^2 A^2 x_{2n}, T^2 u, a \right) \leq \\
\alpha \left[ d \left( A^2 x_{2n}, S^2 A^2 x_{2n}, a \right) \right]^{\gamma-\gamma} \left[ d \left( A^2 x_{2n}, T^2 u, a \right) \right]^{\beta-\gamma} \\
\]

When \( n \to \infty \), we deduce that \( d \left( A^2 u, T^2 u, a \right) \leq 0 \), a contradiction. So \( T^2 u = A^2 u \).

Now suppose that \( S^2 u = T^2 u \), then

\[
d \left( S^2 u, T^2 u, a \right) \leq \alpha \left[ d \left( A^2 u, S^2 u, a \right) \right]^{\gamma-\gamma} \left[ d \left( A^2 u, T^2 u, a \right) \right]^{\gamma-\gamma} \left[ d \left( A^2 u, A^2 u, a \right) \right]^{\beta-\gamma} = 0, \text{ a contradiction.}
\]

So \( S^2 u = T^2 u \)

Thus we have \( S^2 u = T^2 u = A^2 u \).
A similar conclusion we get if we assume that A is weak ** commute with T. 

A being weak ** commute with S, we deduce that

\[ d\left( A^2 S x_{2n}, S u, a \right) \leq d\left( A^2 S x_{2n}, S u, S^2 A x_{2n} \right) \]

which implies as \( n \to \infty \) that \( \{ A^2 S x_{2n} \} \) converges to \( S u \)

Now using this we have

\[ d\left( S^2 A x_{2n}, T^2 x_{2n+1}, a \right) \leq \alpha \left[ d\left( A^3 x_{2n}, S^2 A x_{2n}, a \right) \right]^{\gamma^{1-\beta}} \left[ d\left( A^3 x_{2n}, A^3 x_{2n+1}, a \right) \right]^{\gamma} \]

Letting \( n \to \infty \)

\[ d\left( A u, u, a \right) \leq \beta d\left( A u, u, a \right), \] a contradiction

Thus \( A u = u \) so \( A^2 u = u \)

Similarly as done above \( S u = T u = u \)

Therefore \( u \) is the common fixed point of \( A, S \) and \( T \)

Now suppose that \( S \) is continuous, then the sequence \( \{ A^2 S x_{2n} \} \) converges to \( S u \).

\( A \) being weak ** commute with \( S \) we have

\[ d\left( A^2 S x_{2n}, S u, a \right) \leq d\left( A^2 S x_{2n}, S u, S^2 A x_{2n} \right) + d\left( A^2 S x_{2n}, S^2 A x_{2n}, a \right) + d\left( S^2 A x_{2n}, S u, a \right) \]

Letting \( n \to \infty \) we observe that \( \{ A^2 S x_{2n} \} \) converges to \( S u \)

Now

\[ d\left( S^3 x_{2n}, T^2 x_{2n+1}, a \right) \leq \alpha \left[ d\left( A^2 S x_{2n}, S^3 x_{2n}, a \right) \right]^{\gamma^{1-\beta}} \left[ d\left( A^2 S x_{2n}, A^2 x_{2n+1}, a \right) \right]^{\gamma} \]

Letting \( n \to \infty \), \( d\left( su, u, a \right) \leq \beta d\left( su, u, a \right) \), a contradiction.

Thus \( S u = u \) and so \( S^2 u = u \).

Suppose \( S^2 u \neq T^2 u \) then

\[ d\left( S^2 u, T^2 u, a \right) \leq \alpha \left[ d\left( A u, S^2 u, a \right) \right]^{\gamma^{1-\beta}} \left[ d\left( A^2 u, T^2 u, a \right) \right] \left[ d\left( A^2 u, A^2 u, a \right) \right] \]

\[ = 0 \] a contradiction. So \( S^2 u = T^2 u \)

Since \( A \) is weak** commutative with \( S \), we have

\[ S^2 Au = A^3 u = Au \]

Now

\[ d\left( A u, u, a \right) = d\left( A^2 u, T^2 u, a \right) \leq \alpha \left[ d\left( A^2 u, S^2 u, a \right) \right]^{\gamma^{1-\beta}} \left[ d\left( A^2 u, T^2 u, a \right) \right] \left[ d\left( A^2 u, A^2 u, a \right) \right] \]

\[ = 0 \] which implies that \( A u = u \).

Therefore \( A u = S u = u \).

Since \( A \) is weak** commutative with \( T \), we have

\[ A^2 Tu = T A^2 u \text{ and } T^2 Au = T^3 u \]

Now

\[ d\left( Tu, u, a \right) = d\left( T^3 u, S^2 u, a \right) = d\left( S^2 u, T^2 u, a \right) \]

\[ \leq \alpha \left[ d\left( A^2 u, S^2 u, a \right) \right]^{\gamma^{1-\beta}} \left[ d\left( A^2 u, T^3 u, a \right) \right] \left[ d\left( A^2 u, A^2 Tu, a \right) \right] \]

\[ = 0 \] which implies that \( Tu = u \).

Therefore \( A u = S u = Tu = u \)

i.e. \( u \) is the common fixed point of \( A, S \) and \( T \).

If the mapping \( T \) is continuous instead of \( S \) analogously we can show that \( u \) is the common fixed point of \( A, S \) and \( T \).

Now we claim that \( u \) is the unique common fixed point of \( A, S \) and \( T \).

For this let \( v \neq u \) be another common fixed point, then

\[ d\left( u, v, a \right) = d\left( S^2 u, T^2 v, a \right) \leq \alpha \left[ d\left( A^2 u, S^2 u, a \right) \right]^{\gamma^{1-\beta}} \left[ d\left( A^2 v, T^2 v, a \right) \right] \left[ d\left( A^2 u, A^2 v, a \right) \right] \]

i.e. \( d\left( u, v, a \right) \leq 0 \), which implies that \( u = v \).

Thus \( u \) is the unique common fixed point of \( A, S \) and \( T \).

REFERENCES


