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On Some Fixed Point Theorems through Weak** Commutative in 2-Metric Space

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Abstract

In this present research article, we prove the existence of a common fixed point for three self-mappings defined on a complete 2- metric space through weak **commutativity of maps.

Keywords: Fixed point , 2- metric space , weak** commuting mapping .

AMS Subject Classification: 47H10, 54H25

INTRODUCTION

The notion of 2-metric space was introduced by *Gahler* [1] in 1963 as a generalization of area function for Euclidean triangles. Many fixed point theorems were established by various authors like *Brouwer*, *Banach*, *Schauder* etc. A point $x \in X$ is said to be a *fixed point* of a self-map $f : X \to X$ if $f(x) = x$, where X is a non- empty set. Theorems concerning fixed points of self-maps are known as fixed point theorems. Most of the fixed point theorems were proved for contraction mappings. It is well known that every contraction on a metric space is continuous. The converse is not necessarily true. The identity mapping on [0,1] simply serves the counter example.

In this present work we consider Weak ** Commuting maps on a 2-metric space. Let T_1 and T_2 be two mappings from a metric space (X, d) into itself. T_1 and T_2 are said to commute if $T_1T_2x = T_2T_1x$, for all x in X.

Sessa [5] introduced the concept of weak commutativity in metric spaces. In subsequent years the condition of weak commutativity was again made weaker. Weak* commutativity was introduced in metric space In recent years weak** commutativity has been introduced and some theorems have been established. The existence of fixed point for weak**commutative self-maps in 2-metric space are studied.

In this research article we present the concepts of weak** commutativity in 2-metric space

PRELIMINARIES

In this section we define weak**commutativity, weak* commutativity and weak commutativity. We also present an example to establish the fact that weak** commutativity does not imply commutativity.

1.1 Definition: Two self-maps A and S of a 2-metric space (X, d) are called *weak** commutative*

$(I)\, A(X) \subset S(X)$ and

$$(II)\, d\left(A^2S^2x,\ S^2A^2x,\ a\right) \le d\left(A^2Sx,\ S\,A^2x,\ a\right) \le d\left(AS^2x,\ S^2Ax,\ a\right) \le d\left(ASx,\ SAx,\ a\right) \le d\left(S^2x,\ A^2x,\ a\right)$$ for all x, a in X.

1.2 Definition : Two self-maps A and S define on a 2-metric space (X, d) are said to be *weak* commutative* if

$(I)\ A(X) \subset S(X)$

$(II)\ d\left(A^2S^2x,\ S^2A^2x,\ a\right) \le d\left(S^2x,\ A^2x, a\right)$

for all x, a in X.

1.3 Definition: Two self-maps A and S define on a 2-metric space (X, d) are said to be *weak commutative* if

$(I)\ A(X) \subset S(X)$

$(II)\ d\ (ASx\ , SAx\ , a) \le d(Ax,\ Sx,\ a)$ for all x, a in X.

1.4 Example: let $X = [0,1]$ with 2-metric d-defined as

$$d(x,\ y,\ z) = min\ \{|x - y|,\ |y - z|,\ |z - x|\}$$

Let A and S be defined as

$Ax = \dfrac{x}{x+4}$ and $Sx = \dfrac{x}{2}$ for all x in X

Then A and S are weak** commutative but not weak commutative.

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COMMON FIXED POINT THEOREMS FOR A WEEK ** COMMUTING PAIR OF MAPPINGS

In this section, we have some results on common fixed points for Three self-maps of a 2- complete metric space using the concept of week **commuting maps

Theorem 2.1: let (X,d) be a complete 2-metric space and A, S, T be three mappings of X into X. let $S^2(x)\cup T^2x\subseteq A^2x$ and the pairs of mappings A,S and A,T be

Weak** commutative and

1. $$d\left(S^2x,T^2y,a\right)\le\frac{\alpha\left[d\left(A^2x,S^2x,a\right)\right]^{r+w}\left[d\left(A^2y,T^2y,a\right)\right]^{1-r}}{\left[d\left(A^2x,A^2y,a\right)\right]^{w}}+\beta\left[d\left(A^2x,A^2y,a\right)\right]^{1-r-w}\left[d\left(S^2x,T^2y,a\right)\right]^{r+w}$$

For all x, y, a in X with $AX\neq AY$ and for some $\alpha,\beta,\gamma\in[0,1]$ with $0<(\alpha+\beta)<1$ and $2r+w=1$ when $w\neq 0$. Then A,S,T have a unique common fixed point if either of A,S,T, is continuous.

Proof: For any $x_0\in X$, we have $x_1\in X$, such that $S^2x_0=A^2x_1$ for $S^2(x)\subseteq A^2(x)$

Similarly for this x_1 we get $x_2\in X$ such that $T^2x_1=A^2x_2$ for $T^2(x)\subseteq A^2(x)$ and so on.

Inductively we have a common sequence $\{y_n\}$ defined as $y_{2n+1}=A^2x_{2n+1}=S^2x_{2n}$

$y_{2n+2}=A^2x_{2n+2}=S^2x_{2n+1}$ For $n=0,1,2,3........$

$$d\left(y_{2n},y_{2n+1},a\right)=d\left(A^2x_{2n},A^2x_{2n+1},a\right)=d\left(T^2x_{2n-1},S^2x_{2n},a\right)$$

$$=d\left(S^2x_{2n},Tx_{2n-1},a\right)\le\frac{\alpha\left[d\left(A^2x_{2n},S^2x_{2n},a\right)\right]^{r+w}\left[d\left(A^2x_{2n-1},T^2x_{2n-1},a\right)\right]^{1-r}}{\left[d\left(A^2x_{2n},A^2x_{2n-1},a\right)\right]^{w}}$$

$$+\beta\left[d\left(A^2x_{2n},A^2x_{2n-1},a\right)\right]^{1-r-w}\left[d\left(S^2x_{2n},T^2x_{2n-1},a\right)\right]^{r+w}$$

$$d\left(y_{2n},y_{2n+1},a\right)\le(\alpha+\beta)\left[d\left(y_{2n},y_{2n+1},a\right)\right]^{r+w}\left[d\left(y_{2n-1},y_{2n},a\right)\right]^{1-r-w}$$

$$\left[d\left(y_{2n},y_{2n+1},a\right)\right]^{1-r-w}\le(\alpha+\beta)^{\frac{1}{1-r-w}}d\left(y_{2n-1},y_{2n},a\right)$$

$$d_{2n}\le(\alpha+\beta)^{\frac{1}{1-r-w}}d_{2n-1}$$

Where $d_{2n}=d\left(y_n,y_{2n+1},a\right)$ and $d_{2n-1}=d\left(y_n,y_{2n-1},a\right)$

Again

$$d_{2n+1}=d\left(y_{2n+1},y_{2n+2},a\right)=d\left(A^2x_{2n+1},A^2x_{2n+2},a\right)$$

$$=d\left(S^2x_{2n},Tx_{2n+1},a\right)\le\frac{\alpha\left[d\left(A^2x_{2n},S^2x_{2n},a\right)\right]^{r+w}\left[d\left(A^2x_{2n+1},T^2x_{2n+1},a\right)\right]^{1-r}}{\left[d\left(A^2x_{2n},A^2x_{2n+1},a\right)\right]^{w}}$$

$$+\beta\left[d\left(A^2x_{2n},A^2x_{2n+1},a\right)\right]^{1-r-w}\left[d\left(S^2x_{2n},T^2x_{2n+1},a\right)\right]^{r+w}$$

$$d_{2n+1}=\alpha\left[d_{2n}\right]^{r}\left[d_{2n+1}\right]^{1-r}+\beta\left[d_{2n}\right]^{1-r-w}\left[d_{2n+1}\right]^{r+w}$$

$case(I)$ When $w \neq 0$ then we obtain

$$d_{2n} = (\alpha+\beta)^{\frac{1}{r}} d_{2n-1}$$

And $d_{2n+1} = (\alpha+\beta)[d_{2n}]^r [d_{2n+1}]^{1-r}$

Or $[d_{2n+1}] = (\alpha+\beta)^{\frac{1}{r}} [d_{2n}]$

$case(II)$ When $w = 0$, we get

$$d_{2n} \le (\alpha+\beta)^{\frac{1}{1-r}} d_{2n-1}$$

And $d_{2n+1} = \alpha[d_{2n}]^r [d_{2n+1}]^{1-r} + \beta[d_{2n}]^{1-r} [d_{2n+1}]^r$

We claim that $d_{2n+1} \le d_{2n}$

If it is not so suppose that $d_{2n+1} \ge d_{2n}$ then we have

$$d_{2n+1} \le \alpha[d_{2n}]^r [d_{2n+1}]^{1-r} + \beta[d_{2n}]^{1-r} [d_{2n+1}]^r$$

$= (\alpha+\beta) d_{2n+1}$ This is a contradiction

Since $(\alpha+\beta) \le 1$

Therefore
$d_{2n+1} \le d_{2n} \le q d_{2n-1} \le q^2 d_{2n-2} \le q^3 d_{2n-3} \le .... \le q^n d_0 \to 0$
as $n \to \infty$.

Since $q = (\alpha+\beta)^{\frac{1}{1-r}} < 1$.

Thus in both cases the sequence $\{y_n\}$ is a Cauchy sequence.

Therefore $\{A^2 x_{2n}\}$ is a Cauchy sequence and so converges to a point u in X.

Also as $n \to \infty$ $\{A^2 x_{2n}\} = \{S^2 x_{2n-1}\}$ and $\{A^2 x_{2n+1}\} = \{T^2 x_{2n}\}$ converges to u

Because they are subsequences of the sequence $\{A^2 x_{2n}\}$

First let A is continuous then the sequence $\{A^4 x_{2n}\}$ and $\{A^2 S^2 x_{2n}\}$ converges to a point $A^2 u$

Since A is weak** commute with S we have

$$d(S^2A^2x_{2n}, A^2u, a) \le d(S^2A^2x_{2n}, A^2u, A^2S^2x_{2n}) + d(S^2A^2x_{2n}, A^2S^2x_{2n}, a) + d(A^2S^2x_{2n}, A^2u, a)$$

Which imply on letting $n \to \infty$ that $\{S^2A^2x_{2n}\}$ also converges to A^2u. i.e $S^2u = A^2u$

Now we claim that $T^2u = A^2u$. Suppose not then we have

$$d(S^2A^2x_{2n}, T^2u, a) \le \alpha \frac{\left[d(A^4x_{2n}, S^2A^2x_{2n}, a)^{r+w} d(A^2u, T^2u, a)^{1-r} \right]}{d(A^4x_{2n}, A^2u, a)^w} + \beta\left[d(A^4x_{2n}, A^2u, a)^{1-r-w} d(S^2u, T^2u, a)^{r+w} \right]$$

When $n \to \infty$, we deduce that $d(A^2u, T^2u, a) < 0$, a contradiction. So $T^2u = A^2u$.

Now suppose that $S^2u \ne T^2u$, then

$$d(S^2u, T^2u, a) \le \alpha \frac{\left[d(A^2u, S^2u, a)^{r+w} d(A^2u, T^2u, a)^{1-r} \right]}{d(A^4x_{2n}, A^2u, a)^w}$$

$$+\beta\left[d(A^2u, A^2u, a)^{1-r-w} d(S^2u, T^2u, a)^{r+w} \right]$$

$= 0$, a contradiction. So $S^2u = T^2u$

Thus we have $S^2u = T^2u = A^2u$.

A similar conclusion we get if we assume that A is weak ** commute with S, we deduce that

$$d(A^2Sx_{2n}, Su, a) \le d(A^2Sx_{2n}, Su, S^2Ax_{2n}) + d(A^2Sx_{2n}, S^2Ax_{2n}, a) + d(S^2Ax_{2n}, Su, a)$$

Which implies as $n \to \infty$ that $\{A^2Sx_{2n}\}$ converges to Su

Now using this we have

$$d(S^2Ax_{2n}, T^2x_{2n+1}, a) \le \alpha \frac{\left[d(A^3x_{2n}, S^2Ax_{2n}, a) \right]^{r+w} \left[d(A^2x_{2n+1}, T^2x_{2n+1}, a) \right]^{1-r}}{\left[d(A^3x_{2n}, A^2x_{2n+1}, a) \right]^w}$$

$$+\beta\left[d(A^3x_{2n}, A^2x_{2n+1}, a) \right]^{1-r-w} \left[d(S^2Ax_{2n}, T^2x_{2n+1}, a) \right]^{r-w}$$

Letting $n \to \infty$

$d(Au, u, a) \le \beta d(Au, u, a)$, a contradiction

Thus $Au = u$ so $A^2u = u$

Similarly as done above $Su = Tu = u$ Therefore u is the common fixed point of A, S and T

Now suppose that S is continuous, then the sequence $\left\{A^2Sx_{2n}\right\}$ converges to Su.

A being weak** commute with S we have

$$d\left(A^2Sx_{2n}, Su, a\right) \le d\left(A^2Sx_{2n}, Su, S^2Ax_{2n}\right) + d\left(A^2Sx_{2n}, S^2Ax_{2n}, a\right) + d\left(S^2Ax_{2n}, Su, a\right)$$

Letting $n \to \infty$ we observe that $\left\{A^2Sx_{2n}\right\}$ converges to Su

Now

$$d\left(S^3x_{2n}, T^2x_{2n+1}, a\right) \le \alpha\frac{\left[d\left(A^2Sx_{2n}, S^3x_{2n}, a\right)\right]^{r+w}\left[d\left(A^2x_{2n+1}, T^2x_{2n+1}, a\right)\right]^{1-r}}{\left[d\left(A^2Sx_{2n}, A^2x_{2n+1}, a\right)\right]^{w}}$$
$$+\beta\left[d\left(A^2Sx_{2n}, A^2x_{2n+1}, a\right)\right]^{1-r-w}\left[d\left(S^3x_{2n}, T^2x_{2n+1}, a\right)\right]^{r+w}$$

Letting $n \to \infty$, $d\left(su, u, a\right) \le \beta d\left(su, u, a\right)$, acontrodiction.

Thus $Su = u$ and so $S^2u = u$.

As above we can show that $T^2u = A^2u = S^2u$.

Thus again we have $u = Tu = Su = Au$

i.e. u is the common fixed point of A, S, T.

If the mapping T is continuous instead of S analogously we can show that u is the common fixed point of A, S, T.

Now we claim that u is the unique common fixed point of A, S and T.

For this let $v \ne u$ be another common fixed point, then

$$d\left(u, v, a\right) = d\left(S^2u, T^2v, a\right) \le \alpha\frac{\left[d\left(A^2u, S^2u, a\right)\right]^{r+w}\left[d\left(A^2v, T^2v, a\right)\right]^{1-r}}{\left[d\left(A^2u, A^2v, a\right)\right]^{w}}$$
$$+\beta\left[d\left(A^2u, A^2v, a\right)\right]^{1-r-w}\left[d\left(S^2u, T^2v, a\right)\right]^{r+w}$$
$$= \beta\left[d\left(u, v, a\right)\right]^{1-r-w}\left[d\left(u, v, a\right)\right]^{r+w}$$

$d\left(u, v, a\right) \le \beta d\left(u, v, a\right)$, which is a contradiction.

Thus u is the unique common fixed point of A, S and T.

Theorem 2.2: Let A, S and T be three self mappings of a complete 2- metric space $\left(X, d\right)$. Let $S^2\left(x\right) \cup T^2x \subseteq A^2x$ if A, S and A, T be Weak** commutative pairs and the following condition holds.

1. $$d\left(S^2x, T^2y, a\right) \le \alpha\left[d\left(A^2x, S^2x, a\right)\right]^{1-\beta-w}\left[d\left(A^2y, T^2y, a\right)\right]^{\beta}\left[d\left(A^2x, A^2y, a\right)\right]^{\gamma}$$ for all x, y, a in X, $0 \le \alpha < 1$, $0 \le \beta < 1$, $0 \le \gamma$ with $0 < \beta + \gamma \le 1$. If either of A, S and T is continuous Then A, S and T have a unique common fixed point.

Proof: For any arbitrary point $x_0 \in X$, construct the sequence $\left\{y_{2n}\right\}$ as in theorem (2.1). Using (1) it is easy to show that $\left\{y_n\right\}$ is Cauchy sequence. Since X is complete, So there exist a point u in X.

Now, Let A is continuous, then the sequence $\left\{A^4x_{2n}\right\}$ and $\left\{A^2S^2x_{2n}\right\}$ converge to a point A^2u.

A is weak** commutative with S, we have

$$d\left(S^2A^2x_{2n}, A^2u, a\right) \le d\left(S^2A^2x_{2n}, A^2u, A^2S^2x_{2n}\right) + d\left(S^2A^2x_{2n}, A^2S^2x_{2n}, a\right) + d\left(A^2S^2x_{2n}, A^2u, a\right)$$

Which imply on letting $n \to \infty$ that $\left\{S^2A^2x_{2n}\right\}$ also converges to A^2u. i.e $S^2u = A^2u$

Now we claim that $T^2u = A^2u$. Suppose not then we have

$$d\left(S^2A^2x_{2n}, T^2u, a\right) \le \alpha\left[d\left(A^4x_{2n}, S^2A^2x_{2n}, a\right)\right]^{1-\beta-\gamma}\left[d\left(A^2u, T^2u, a\right)\right]^{\beta}\left[d\left(A^4x_{2n}, A^2u, a\right)\right]^{\gamma}$$

When $n \to \infty$, we deduce that $d\left(A^2u, T^2u, a\right) \le 0$, a contradiction, So $T^2u = A^2u$.

Now suppose that $S^2u \ne T^2u$, then

$$d\left(S^2u, T^2u, a\right) \le \alpha\left[d\left(A^2u, S^2u, a\right)\right]^{1-\beta-\gamma}\left[d\left(A^2u, T^2u, a\right)\right]^{\beta}\left[d\left(A^2u, A^2u, a\right)\right]^{\gamma}$$

$= 0$, a contradiction.

So $S^2u = T^2u$

Thus we have $S^2u = T^2u = A^2u$.

A similar conclusion we get if we assume that A is weak ** commute with T.

A being weak ** commute with S, we deduce that

$$d\left(A^2Sx_{2n},Su,a\right)\le d\left(A^2Sx_{2n},Su,S^2Ax_{2n}\right)+d\left(A^2Sx_{2n},S^2Ax_{2n},a\right)+d\left(S^2Ax_{2n},Su,a\right)$$ Which

implies as $n\to\infty$ that $\left\{A^2Sx_{2n}\right\}$ converges to Su

Now using this we have

$$d\left(S^2Ax_{2n},T^2x_{2n+1},a\right)\le\alpha\left[d\left(A^3x_{2n},S^2Ax_{2n},a\right)\right]^{1-\beta-\gamma}\left[d\left(A^2x_{2n},T^2x_{2n+1},a\right)\right]^{\beta}\left[d\left(A^3x_{2n},A^2x_{2n+1},a\right)\right]^{\gamma}$$

Letting $n\to\infty$

$d\left(Au,u,a\right)\le\beta d\left(Au,u,a\right)$, a contradiction

Thus $Au=u$ so $A^2u=u$

Similarly as done above $Su=Tu=u$

Therefore u is the common fixed point of A,S and T

Now suppose that S is continuous, then the sequence $\left\{A^2Sx_{2n}\right\}$ converges to Su.

A being weak** commute with S we have

$$d\left(A^2Sx_{2n},Su,a\right)\le d\left(A^2Sx_{2n},Su,S^2Ax_{2n}\right)+d\left(A^2Sx_{2n},S^2Ax_{2n},a\right)+d\left(S^2Ax_{2n},Su,a\right)$$

Letting $n\to\infty$ we observe that $\left\{A^2Sx_{2n}\right\}$ converges to Su

Now

$$d\left(S^3x_{2n},T^2x_{2n+1},a\right)\le\ \alpha\left[d\left(A^2Sx_{2n},S^3x_{2n},a\right)\right]^{1-\beta-\gamma}\left[d\left(A^2x_{2n+1},T^2x_{2n+1},a\right)\right]^{\beta}\ \left[d\left(A^2Sx_{2n},A^2x_{2n+1},a\right)\right]^{\gamma}$$

Letting $n\to\infty$, $d\left(su,u,a\right)\le\beta d\left(su,u,a\right)$, acontrodiction.

Thus $Su=u$ and so $S^2u=u$.

Suppose $S^2u\ne T^2u$ then

$$d\left(S^2u,T^2u,a\right)\le\alpha\left[d\left(A^2u,S^2u,a\right)\right]^{1-\beta-\gamma}\left[d\left(A^2u,T^2u,a\right)\right]^{\beta}\left[d\left(A^2u,A^2u,a\right)\right]^{\gamma}$$

$=0$, a contradiction. So $S^2u=T^2u$

Since A is weak** commutative with S, we have $S^2Au=A^3u=Au$

Now

$$d\left(Au,u,a\right)=d\left(A^3u,T^2u,a\right)\le\alpha\left[d\left(A^3u,S^2Au,a\right)\right]^{1-\beta-\gamma}\left[d\left(A^2u,T^2u,a\right)\right]^{\beta}\left[d\left(A^3u,A^2u,a\right)\right]^{\gamma}$$

$=0$, which implies that $Au=u$.

Therefore $Au=Su=u$.

Since A is weak** commutative with T, we have

$A^2Tu=TA^2u$ and $T^2Au=T^3u$

Now $d\left(Tu,u,a\right)=d\left(T^3u,S^2u,a\right)=d\left(S^2u,T^2u,a\right)$

$$\le\alpha\left[d\left(A^2u,S^2u,a\right)\right]^{1-\beta-\gamma}\left[d\left(A^2Tu,T^3u,a\right)\right]^{\beta}\left[d\left(A^2u,A^2Tu,a\right)\right]^{\gamma}$$

$=0$, which implies that $Tu=u$.

Therefore $Au=Su=Tu=u$

i.e. u is the common fixed point of A,S and T.

If the mapping T is continuous instead of S analogously we can show that u is the common fixed point of A,S and T

Now we claim that u is the unique common fixed point of A,S and T.

For this let $v\ne u$ be another common fixed point, then

$$d\left(u,v,a\right)=d\left(S^2u,T^2v,a\right)\le\alpha\left[d\left(A^2u,S^2u,a\right)\right]^{1-\beta-\gamma}\left[d\left(A^2v,T^2v,a\right)\right]^{\beta}\left[d\left(A^2u,A^2v,a\right)\right]^{\gamma}$$

i.e. $d\left(u,v,a\right)\le 0$, which implies that $u=v$

Thus u is the unique common fixed point of A,S and T.

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