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An Application of Wilf's Subordinating Factor Sequence on Certain Subclasses of Analytic Functions

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Abstract

In this work, we derive several subordination results for a certain subclasses $M_{\mu}^{a,c}(\alpha;A,B)$ and $N_{\mu}^{a,c}(\alpha;A,B)$ of analytic functions defined on the open unit disc *U.* Making use of the familiar principle of differential subordination, several properties and relationships involving the functions.

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1. INTRODUCTION

Let $\mathcal{A}$ denoted to class of function of the form

$$f(z)=z+\sum_{k=2}^{\infty}a_k z^k \quad (a_k\geq 0), \tag{1.1}$$

which are analytic function in the open disc $U=\{z\in\mathbb{C}:|z|<1\}$, Also let K denote the familiar class of functions $f\in\mathcal{A}$ which are also univalent and convex in U.

For $\mu>0$ and $a,c\in\mathbb{C}$, are such that $\Re\{c-a\}\geq 0$, Raina and Sharma [5] defined the integral operator $J_{\mu}^{a,c}:\mathcal{A}\rightarrow\mathcal{A}$, as following:

(i) for $\Re\{c-a\}>0$ and $\Re\{a\}>-\mu$ by

$$J_{\mu}^{a,c}f(z)=\frac{\Gamma(c+\mu)}{\Gamma(a+\mu)\Gamma(c-a)}\int_0^1(1-t)^{c-a-1}t^{a-1}f(zt^{\mu})dt; \tag{1.2}$$

(ii) for $a=c$ by

$$J_{\mu}^{a,a}f(z)=f(z), \tag{1.3}$$

where Γ stands for Euler's Gamma function (which is valid for all complex numbers except the non-positive integers).

For $f(z)$ defined by (1.1), it is easily from (1.2) and (1.3) that:

$$J_{\mu}^{a,c}f(z)=z+\frac{\Gamma(c+\mu)}{\Gamma(a+\mu)}\sum_{k=2}^{\infty}\frac{\Gamma(a+k\mu)}{\Gamma(c+k\mu)}a_k z^k, \tag{1.4}$$

$$(\mu>0,\Re\{c\}\geq\Re\{a\}>-\mu).$$

Let $M_{\mu}^{a,c}(\alpha;A,B)$ be the subclass of functions $f\in\mathcal{A}$ for which:

$$\frac{z(J_{\mu}^{a,c}f(z))'}{J_{\mu}^{a,c}f(z)}\prec(1-\alpha)\frac{1+Az}{1+Bz}+\alpha, \tag{1.5}$$

$$(-1\leq B<A\leq 1, 0\leq\alpha<1),$$

that is, that

$$M_{\mu}^{a,c}(\alpha;A,B)=\left\{f\in\mathcal{A}:\left|\frac{\frac{z(J_{\mu}^{a,c}f(z))'}{J_{\mu}^{a,c}f(z)}-1}{B\frac{z(J_{\mu}^{a,c}f(z))'}{J_{\mu}^{a,c}f(z)}-[B+(A-B)(1-\alpha)]}\right|<1, z\in U\right\}. \tag{1.6}$$

Also, for $-1\leq B<A\leq 1$ and $0\leq\alpha<1$, let $N_{\mu}^{a,c}(\alpha;A,B)$ be the subclass of functions $f\in\mathcal{A}$ for which:

$$1+\frac{z(J_{\mu}^{a,c}f(z))''}{(J_{\mu}^{a,c}f(z))'}\prec(1-\alpha)\frac{1+Az}{1+Bz}+\alpha,$$

form (1.5) and (1.6), it is clear that

$$f(z)\in N_{\mu}^{a,c}(\alpha;A,B)\Leftrightarrow zf'(z)\in M_{\mu}^{a,c}(\alpha;A,B). \tag{1.7}$$

It is easily to see that:

(i) $M_{\mu}^{a,a}(\alpha;A,B)=S^*(A,B,\alpha)$ and $N_{\mu}^{a,c}(\alpha;A,B)=K(A,B,\alpha)$, see [1, with p=1];

(ii) $M_{\mu}^{a,a}(0;A,B)=S^*(A,B)$ and $N_{\mu}^{a,c}(0;A,B)=K(A,B)$, see [3];

(iii) $M_{\mu}^{a,a}(\alpha;\beta,-\beta)=S^*(\alpha,\beta)$ and $N_{\mu}^{a,a}(\alpha;\beta,-\beta)=C(\alpha,\beta)$ the subclasses of starlike and convex of order $0\leq\alpha<1$ and type $0\leq\alpha<1$, see [2];

(iv) $M_{\mu}^{a,a}(\alpha;1,-1)=S^*(\alpha)$ and $N_{\mu}^{a,a}(\alpha;1,-1)=K(\alpha)$ the subclasses of starlike and convex of order $0\leq\alpha<1$, see [6].

Definition 1 (Hadamard Product or Convolution). Given two functions $f, g \in \mathcal{A}$, where $f(z)$ is given by (1.1) and $g(z)$ is defined by

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k,$$

the Hadamard product (or convolution) $f * g$ is defined (as usual) by

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k = (g * f)(z)$$

Definition 2 (Subordination Principle). For two functions f and g, analytic in U, we say that the function $f(z)$ is subordinate to $g(z)$ in U, and write

$$f \prec g \qquad \text{or} \qquad f(z) \prec g(z) \qquad (z \in U),$$

if there exists a Schwarz function $w(z)$, analytic in U with

$$w(0) = 0 \quad \text{and} \quad |w(z)| < 1 \qquad (z \in U),$$

such that

$$f(z) = g(w(z)) \qquad (z \in U),$$

In particular, if the function g is univalent in U, the above subordination is equivalent to

$$f(0) = g(0) \quad \text{and} \quad f(U) \subset g(U).$$

Definition 3 (Subordinating Factor Sequence). A sequence $\{b_k\}_{k=1}^{\infty}$ of complex numbers is said to be a subordinating factor sequence if, whenever $f(z)$ of the form (1.1) is analytic, univalent and convex in U, we have the subordination given by

$$\sum_{k=1}^{\infty} a_k b_k z^k \prec (f)(z) \qquad (z \in U; a_1 = 1).$$

MAIN RESULT

We will make use of the following lemmas. Unless otherwise mentioned, we assume in the reminder of this paper that, $0 \le \alpha < 1, -1 \le B < A \le 1, \mu > 0,$ $a, c \in \mathrm{R}, c > a > -\mu$ and $z \in U$.

Lemma 1 [4]. *Let the function* $f(z)$ *be given by (1.1). Then* $f \in M_{\mu}^{a,c}(\alpha; A, B)$, *if*

$$\sum_{k=2}^{\infty} \frac{\Gamma(c+\mu)}{\Gamma(a+\mu)} \left[(1-B)(k-1) + (A-B)(1-\alpha)\right] \frac{\Gamma(a+k\mu)}{\Gamma(c+k\mu)} |a_k| \le (A-B)(1-\alpha). \tag{2.1}$$

Lemma 2 [4]. *Let the function* $f(z)$ *be given by (1.1). Then* $f \in N_{\mu}^{a,c}(\alpha; A, B)$, *if*

$$\sum_{k=2}^{\infty} \frac{\Gamma(c+\mu)}{\Gamma(a+\mu)} \left[(1-B)(k-1) + (A-B)(1-\alpha)\right] \frac{\Gamma(a+k\mu)}{\Gamma(c+k\mu)} k\, |a_k| \le (A-B)(1-\alpha). \tag{2.2}$$

Lemma 3 [7]. *The sequence* $\{b_k\}_{k=1}^{\infty}$ *is a subordinating factor sequence if and only if*

$$\Re\left\{1 + 2\sum_{k=1}^{\infty} b_k z^k\right\} > 0, \qquad (z \in U)$$

Theorem 1. *Let the function* $f(z)$ *defined by (1.1) be in the class* $M_{\mu}^{a,c}(\alpha; A, B)$. *Then*

$$\frac{\frac{\Gamma(c+\mu)}{\Gamma(a+\mu)}\left[(1-B) + (A-B)(1-\alpha)\right]\frac{\Gamma(a+2\mu)}{\Gamma(c+2\mu)}}{2\left[\frac{\Gamma(c+\mu)}{\Gamma(a+\mu)}\left[(1-B) + (A-B)(1-\alpha)\right]\frac{\Gamma(a+2\mu)}{\Gamma(c+2\mu)} + (A-B)(1-\alpha)\right]} (f * g)(z) \prec g(z) \tag{2.3}$$

for every function g *in* K, *and*

$$\Re\{f(z)\} > -\frac{\frac{\Gamma(c+\mu)}{\Gamma(a+\mu)}\left[(1-B)+(A-B)(1-\alpha)\right]\frac{\Gamma(a+2\mu)}{\Gamma(c+2\mu)}+(A-B)(1-\alpha)}{\frac{\Gamma(c+\mu)}{\Gamma(a+\mu)}\left[(1-B)+(A-B)(1-\alpha)\right]\frac{\Gamma(a+2\mu)}{\Gamma(c+2\mu)}}, \qquad (z\in U). \tag{2.4}$$

The following constant factor $\frac{\frac{\Gamma(c+\mu)}{\Gamma(a+\mu)}[(1-B)+(A-B)(1-\alpha)]\frac{\Gamma(a+2\mu)}{\Gamma(c+2\mu)}}{2\left[\frac{\Gamma(c+\mu)}{\Gamma(a+\mu)}[(1-B)+(A-B)(1-\alpha)]\frac{\Gamma(a+2\mu)}{\Gamma(c+2\mu)}+(A-B)(1-\alpha)\right]}$ *in the subordination result (2.3) cannot be replaced by a larger one*.

Proof. Let $f\in M_{\mu}^{a,c}(\alpha;A,B)$ and let $g(z)=z+\sum_{k=2}^{\infty}q_k z^k\in K$. Then we have

$$\begin{aligned}&\frac{\frac{\Gamma(c+\mu)}{\Gamma(a+\mu)}[(1-B)+(A-B)(1-\alpha)]\frac{\Gamma(a+2\mu)}{\Gamma(c+2\mu)}}{2\left[\frac{\Gamma(c+\mu)}{\Gamma(a+\mu)}[(1-B)+(A-B)(1-\alpha)]\frac{\Gamma(a+2\mu)}{\Gamma(c+2\mu)}+(A-B)(1-\alpha)\right]}(f*g)(z)\\&=\frac{\frac{\Gamma(c+\mu)}{\Gamma(a+\mu)}[(1-B)+(A-B)(1-\alpha)]\frac{\Gamma(a+2\mu)}{\Gamma(c+2\mu)}}{2\left[\frac{\Gamma(c+\mu)}{\Gamma(a+\mu)}[(1-B)+(A-B)(1-\alpha)]\frac{\Gamma(a+2\mu)}{\Gamma(c+2\mu)}+(A-B)(1-\alpha)\right]}\left(z+\sum_{k=2}^{\infty}a_k q_k z^k\right).\end{aligned}\tag{2.5}$$

Thus, by Definition 3, subordination result (2.3) will hold true if the sequence

$$\left\{\frac{\frac{\Gamma(c+\mu)}{\Gamma(a+\mu)}[(1-B)+(A-B)(1-\alpha)]\frac{\Gamma(a+2\mu)}{\Gamma(c+2\mu)}}{2\left[\frac{\Gamma(c+\mu)}{\Gamma(a+\mu)}[(1-B)+(A-B)(1-\alpha)]\frac{\Gamma(a+2\mu)}{\Gamma(c+2\mu)}+(A-B)(1-\alpha)\right]}a_k\right\}_{k=1}^{\infty}, \tag{2.6}$$

is a subordinating factor sequence (with, of course, $a_1=1$). In view of Lemma 3, this is equivalent to the following inequality

$$\Re\left\{1+\sum_{k=1}^{\infty}\frac{\frac{\Gamma(c+\mu)}{\Gamma(a+\mu)}[(1-B)+(A-B)(1-\alpha)]\frac{\Gamma(a+2\mu)}{\Gamma(c+2\mu)}}{\frac{\Gamma(c+\mu)}{\Gamma(a+\mu)}[(1-B)+(A-B)(1-\alpha)]\frac{\Gamma(a+2\mu)}{\Gamma(c+2\mu)}+(A-B)(1-\alpha)}a_k z^k\right\}>0, \qquad (z\in U). \tag{3.7}$$

Now, since

$$\Phi(k)=\frac{\Gamma(c+\mu)}{\Gamma(a+\mu)}\left[(1-B)(k-1)+(A-B)(1-\alpha)\right]\frac{\Gamma(a+k\mu)}{\Gamma(c+k\mu)},$$

is an increasing function of $k\ (k\geq 2)$, we have

$$\begin{aligned}&\Re\left\{1+\sum_{k=1}^{\infty}\frac{\frac{\Gamma(c+\mu)}{\Gamma(a+\mu)}[(1-B)+(A-B)(1-\alpha)]\frac{\Gamma(a+2\mu)}{\Gamma(c+2\mu)}}{\frac{\Gamma(c+\mu)}{\Gamma(a+\mu)}[(1-B)+(A-B)(1-\alpha)]\frac{\Gamma(a+2\mu)}{\Gamma(c+2\mu)}+(A-B)(1-\alpha)}a_k z^k\right\}\\
&=\Re\left\{1+\frac{\frac{\Gamma(c+\mu)}{\Gamma(a+\mu)}[(1-B)+(A-B)(1-\alpha)]\frac{\Gamma(a+2\mu)}{\Gamma(c+2\mu)}}{\frac{\Gamma(c+\mu)}{\Gamma(a+\mu)}[(1-B)+(A-B)(1-\alpha)]\frac{\Gamma(a+2\mu)}{\Gamma(c+2\mu)}+(A-B)(1-\alpha)}z+\right.\\
&\left.\frac{1}{\frac{\Gamma(c+\mu)}{\Gamma(a+\mu)}[(1-B)+(A-B)(1-\alpha)]\frac{\Gamma(a+2\mu)}{\Gamma(c+2\mu)}+(A-B)(1-\alpha)}\sum_{k=2}^{\infty}\frac{\Gamma(c+\mu)}{\Gamma(a+\mu)}\left[(1-B)+(A-B)(1-\alpha)\right]\frac{\Gamma(a+2\mu)}{\Gamma(c+2\mu)}a_k z^k\right\}\\
&\geq 1-\frac{\frac{\Gamma(c+\mu)}{\Gamma(a+\mu)}[(1-B)+(A-B)(1-\alpha)]\frac{\Gamma(a+2\mu)}{\Gamma(c+2\mu)}}{\frac{\Gamma(c+\mu)}{\Gamma(a+\mu)}[(1-B)+(A-B)(1-\alpha)]\frac{\Gamma(a+2\mu)}{\Gamma(c+2\mu)}+(A-B)(1-\alpha)}r-\\
&\frac{1}{\frac{\Gamma(c+\mu)}{\Gamma(a+\mu)}[(1-B)+(A-B)(1-\alpha)]\frac{\Gamma(a+2\mu)}{\Gamma(c+2\mu)}+(A-B)(1-\alpha)}\sum_{k=2}^{\infty}\frac{\Gamma(c+\mu)}{\Gamma(a+\mu)}\left[(1-B)(k-1)+(A-B)(1-\alpha)\right]\frac{\Gamma(a+k\mu)}{\Gamma(c+k\mu)}|a_k|r^k\\
&>1-\frac{\frac{\Gamma(c+\mu)}{\Gamma(a+\mu)}[(1-B)+(A-B)(1-\alpha)]\frac{\Gamma(a+2\mu)}{\Gamma(c+2\mu)}}{\frac{\Gamma(c+\mu)}{\Gamma(a+\mu)}[(1-B)+(A-B)(1-\alpha)]\frac{\Gamma(a+2\mu)}{\Gamma(c+2\mu)}+(A-B)(1-\alpha)}r-\frac{(A-B)(1-\alpha)}{\frac{\Gamma(c+\mu)}{\Gamma(a+\mu)}[(1-B)+(A-B)(1-\alpha)]\frac{\Gamma(a+2\mu)}{\Gamma(c+2\mu)}+(A-B)(1-\alpha)}r\\
&=1-r>0 \qquad (|z|=r<1),\end{aligned}$$

where we have also made use of assertion (2.1) of Lemma 1. Thus (2.7) holds true in U. This proves the inequality (2.3). The inequality (2.4) follows from (2.3) by taking the convex function $g(z)=\frac{z}{1-z}=z+\sum_{k=2}^{\infty} z^k$. To prove the sharpness of the constant

$$\frac{\frac{\Gamma(c+\mu)}{\Gamma(a+\mu)}[(1-B)+(A-B)(1-\alpha)]\frac{\Gamma(a+2\mu)}{\Gamma(c+2\mu)}}{2\left[\frac{\Gamma(c+\mu)}{\Gamma(a+\mu)}[(1-B)+(A-B)(1-\alpha)]\frac{\Gamma(a+2\mu)}{\Gamma(c+2\mu)}+(A-B)(1-\alpha)\right]},$$

we consider the function $f_0 \in M_{\mu}^{a,c}(\alpha;A,B)$ given by

$$f_0(z)=z+\frac{(A-B)(1-\alpha)}{\frac{\Gamma(c+\mu)}{\Gamma(a+\mu)}\left[(1-B)+(A-B)(1-\alpha)\right]\frac{\Gamma(a+2\mu)}{\Gamma(c+2\mu)}}z^2. \qquad (2.8)$$

Thus from (2.3), we have

$$\frac{\frac{\Gamma(c+\mu)}{\Gamma(a+\mu)}[(1-B)+(A-B)(1-\alpha)]\frac{\Gamma(a+2\mu)}{\Gamma(c+2\mu)}}{2\left[\frac{\Gamma(c+\mu)}{\Gamma(a+\mu)}[(1-B)+(A-B)(1-\alpha)]\frac{\Gamma(a+2\mu)}{\Gamma(c+2\mu)}+(A-B)(1-\alpha)\right]}f_0(z)\prec\frac{z}{1-z}, \qquad (z\in U). \qquad (2.9)$$

Moreover, it can easily be verified for the function $f_0(z)$ given by (2.8) that

$$\min_{|z|\le r}\left\{\Re\left\{\frac{\frac{\Gamma(c+\mu)}{\Gamma(a+\mu)}[(1-B)+(A-B)(1-\alpha)]\frac{\Gamma(a+2\mu)}{\Gamma(c+2\mu)}}{2\left[\frac{\Gamma(c+\mu)}{\Gamma(a+\mu)}[(1-B)+(A-B)(1-\alpha)]\frac{\Gamma(a+2\mu)}{\Gamma(c+2\mu)}+(A-B)(1-\alpha)\right]}f_0(z)\right\}\right\}=-\frac{1}{2}, \qquad (2.10)$$

see Figure 1.

This shows that the constant $\frac{\frac{\Gamma(c+\mu)}{\Gamma(a+\mu)}[(1-B)+(A-B)(1-\alpha)]\frac{\Gamma(a+2\mu)}{\Gamma(c+2\mu)}}{2\left[\frac{\Gamma(c+\mu)}{\Gamma(a+\mu)}[(1-B)+(A-B)(1-\alpha)]\frac{\Gamma(a+2\mu)}{\Gamma(c+2\mu)}+(A-B)(1-\alpha)\right]}$ is the best possible.

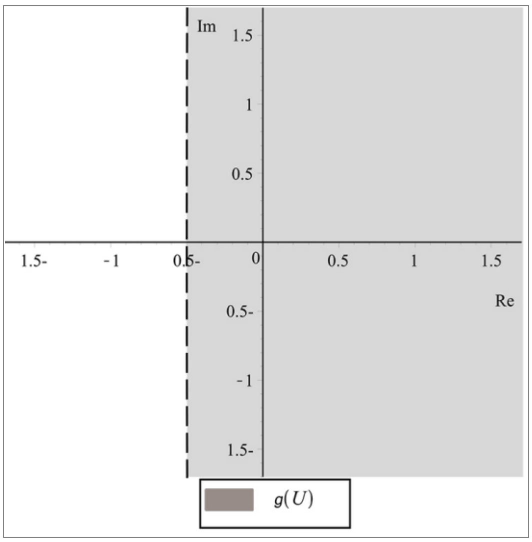

Figure 1

Putting $a=c$ in Theorem 1, we have:

Corollary 1. Let the function $f(z)$ defined by (1.1) be in the subclass $S^*(A,B,\alpha)$ and suppose that $g(z)\in K$. Then

$$\frac{(1-B)+(A-B)(1-\alpha)}{2\left[(1-B)+2(A-B)(1-\alpha)\right]}(f*g)(z)\prec g(z) \qquad (2.11)$$

and

$$\Re\{f(z)\}>-\frac{(1-B)+2(A-B)(1-\alpha)}{(1-B)+(A-B)(1-\alpha)}, \qquad (z\in U). \qquad (2.12)$$

The constant factor $\frac{(1-B)+(A-B)(1-\alpha)}{2[(1-B)+2(A-B)(1-\alpha)]}$ in the subordination result (2.11) cannot be replaced by a larger one.

Putting $a=c$ and $\alpha=0$ in Theorem 1, we have:

Corollary 2. Let the function $f(z)$ defined by (1.1) be in the subclass $S^*(A,B)$ and suppose that $g(z)\in K$. Then

$$\frac{A-2B+1}{2\left[(1-B)+2(A-B)\right]}(f*g)(z)\prec g(z) \qquad (2.13)$$

and

$$\Re\{f(z)\}>-\frac{(1-B)+2(A-B)}{A-2B+1}, \qquad (z\in U). \qquad (2.14)$$

The constant factor $\frac{A-2B+1}{2[(1-B)+2(A-B)]}$ in the subordination result (2.13) cannot be replaced by a larger one.

Putting $a=c$ and $A=\beta, B=-\beta$ in Theorem 1, we have:

Corollary 3. Let the function $f(z)$ defined by (1.1) be in the subclass $S^*(\alpha;\beta)$ and suppose that $g(z)\in K$. Then

$$\frac{1+\beta(3-2\alpha)}{2\left[1+\beta(5-4\alpha)\right]}(f*g)(z)\prec g(z) \qquad (2.15)$$

and

$$\Re\{f(z)\}>-\frac{1+\beta(5-4\alpha)}{1+\beta(3-2\alpha)}, \qquad (z\in U). \qquad (2.16)$$

The constant factor $\frac{1+\beta(3-2\alpha)}{2[1+\beta(5-4\alpha)]}$ in the subordination result (2.15) cannot be replaced by a larger one.

Putting $a=c$ and $A=1, B=-1$ in Theorem 1, we have:

Corollary 4. Let the function $f(z)$ defined by (1.1) be in the subclass $S^*(\alpha)$ and suppose that $g(z)\in K$. Then

$$\frac{2-\alpha}{2(3-2\alpha)}(f*g)(z)\prec g(z) \qquad (2.17)$$

and

$$\Re\{f(z)\}>-\frac{3-2\alpha}{2-\alpha}, \qquad (z\in U). \qquad (2.18)$$

The constant factor $\frac{2-\alpha}{2(3-2\alpha)}$ in the subordination result (2.17) cannot be replaced by a larger one.

Theorem 2. Let the function $f(z)$ defined by (1.1) be in the class $N_{\mu}^{a,c}(\alpha;A,B)$. Then

$$\frac{\frac{\Gamma(c+\mu)}{\Gamma(a+\mu)}\left[(1-B)+(A-B)(1-\alpha)\right]\frac{\Gamma(a+2\mu)}{\Gamma(c+2\mu)}}{\frac{\Gamma(c+\mu)}{\Gamma(a+\mu)}2\left[(1-B)+(A-B)(1-\alpha)\right]\frac{\Gamma(a+2\mu)}{\Gamma(c+2\mu)}+(A-B)(1-\alpha)}(f*g)(z)\prec g(z) \qquad (2.19)$$

for every function g in K, and

$$\Re\{f(z)\}>-\frac{\frac{\Gamma(c+\mu)}{\Gamma(a+\mu)}2\left[(1-B)+(A-B)(1-\alpha)\right]\frac{\Gamma(a+2\mu)}{\Gamma(c+2\mu)}+(A-B)(1-\alpha)}{\frac{\Gamma(c+\mu)}{\Gamma(a+\mu)}2\left[(1-B)+(A-B)(1-\alpha)\right]\frac{\Gamma(a+2\mu)}{\Gamma(c+2\mu)}}, \quad (z\in U). \qquad (2.20)$$

The following constant factor $\frac{\frac{\Gamma(c+\mu)}{\Gamma(a+\mu)}[(1-B)+(A-B)(1-\alpha)]\frac{\Gamma(a+2\mu)}{\Gamma(c+2\mu)}}{\frac{\Gamma(c+\mu)}{\Gamma(a+\mu)}2[(1-B)+(A-B)(1-\alpha)]\frac{\Gamma(a+2\mu)}{\Gamma(c+2\mu)}+(A-B)(1-\alpha)}$ in the subordination result (2.19) cannot be replaced by a larger one.

Putting $a=c$ in Theorem 2, we have:

Corollary 5. Let the function $f(z)$ defined by (1.1) be in the subclass $K(A,B,\alpha)$ and suppose that $g(z)\in K$. Then

$$\frac{(1-B)+(A-B)(1-\alpha)}{2(1-B)+3(A-B)(1-\alpha)}(f*g)(z)\prec g(z) \tag{2.21}$$

and

$$\Re\{f(z)\}>-\frac{2(1-B)+3(A-B)(1-\alpha)}{2\left[(1-B)+(A-B)(1-\alpha)\right]}, \qquad (z\in U). \tag{2.22}$$

The following constant factor $\frac{(1-B)+(A-B)(1-\alpha)}{2(1-B)+3(A-B)(1-\alpha)}$ in the subordination result (2.21) cannot be replaced by a larger one.

Putting $a=c$ and $\alpha=0$ in Theorem 2, we have:

Corollary 6. Let the function $f(z)$ defined by (1.1) be in the subclass $K(A,B)$ and suppose that $g(z)\in K$. Then

$$\frac{A-2B+1}{3A-5B+1}(f*g)(z)\prec g(z) \tag{2.23}$$

and

$$\Re\{f(z)\}>-\frac{3A-5B+1}{2\left[A-2B+1\right]}, \qquad (z\in U). \tag{2.24}$$

The constant factor $\frac{A-2B+1}{3A-5B+1}$ in the subordination result (2.23) cannot be replaced by a larger one.

Putting $a=c$ and $A=\beta, B=-\beta$ in Theorem 2, we have:

Corollary 7. Let the function $f(z)$ defined by (1.1) be in the subclass $C(\alpha;\beta)$ and suppose that $g(z)\in K$. Then

$$\frac{1+\beta(3-2\alpha)}{2\left[1+\beta(4-3\alpha)\right]}(f*g)(z)\prec g(z) \tag{2.25}$$

and

$$\Re\{f(z)\}>-\frac{1+\beta(4-3\alpha)}{1+\beta(3-2\alpha)}, \qquad (z\in U). \tag{2.26}$$

The constant factor $\frac{1+\beta(3-2\alpha)}{2\left[1+\beta(4-3\alpha)\right]}$ in the subordination result (2.25) cannot be replaced by a larger one.

Putting $a=c$ and $A=1, B=-1$ in Theorem 2, we have:

Corollary 8. Let the function $f(z)$ defined by (1.1) be in the subclass $K(\alpha)$ and suppose that $g(z)\in K$. Then

$$\frac{2-\alpha}{5-3\alpha}(f*g)(z)\prec g(z) \tag{2.27}$$

and

$$\Re\{f(z)\}>-\frac{5-3\alpha}{2(2-\alpha)}, \qquad (z\in U). \tag{2.28}$$

The constant factor $\frac{2-\alpha}{5-3\alpha}$ in the subordination result (2.27) cannot be replaced by a larger one.

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