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The Forcing Restrained Monophonic Number of a Graph

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Abstract

For the connected graph $G = (V, E)$ of order at least two, a chord of a path P is an edge joining two non-adjacent vertices of P is called a monophonic path if it is a chordless path. A set M of vertices of a connected graph G is a monophonic set if every vertex of G lies on a $x - y$ monophonic path for some elements x and y in M. The minimum cardinality of a monophonic set of G is the monophonic number of G, denoted by $m(G)$. A set M of vertices of a connected graph G is a restrained monophonic set if either $V = M$ or M is a monophonic set with the subgraph $G[V - M]$ induced by $V - M$ has no isolated vertices. The minimum cardinality of a restrained monophonic set of G is the restrained monophonic number of G, denoted by $m_r(G)$. Let G be a connected graph and M is a minimum restrained monophonic set of G. A subset $T \subseteq M$ is called a forcing subset for M if M is the unique minimum restrained monophonic set containing T. A forcing subset for M of minimum cardinality is a minimum forcing subset of M. The forcing restrained monophonic number of M, denoted by $f_{m_r}(M)$, is the cardinality of a minimum forcing subset of M. The forcing restrained monophonic number of G, denoted by $f_{m_r}(G)$, is $f_{m_r}(G) = min\,\{f_{m_r}(M)\}$, Where the minimum is taken over all minimum restrained monophonic sets M in G. We determine bounds for if and find the forcing restrained monophonic number of certain classes of graphs. For every pair a, b of integers with $0 \leq a \leq b$ and $b \geq 2$, there exists a connected graph G such that $f_{m_r}(G) = f_m(G) = 0, m(G) = a$ and $m_r(G) = b$. For every integers a, b and c with $0 \leq a \leq b \leq c$ and $c > a + b$, then there exists a connected graph G such that $f_m(G) = 0$, $f_{m_r}(G) = a$, $m(G) = b$ and $m_r(G) = c$.

Keywords and Phrases: monophonic set, monophonic number, restrained monophonic set, restrained monophonic number, forcing restrained monophonic set, forcing restrained monophonic number.

AMS Subject Classification: 05C12

INTRODUCTION

By a graph $G = (V, E)$, we mean a finite undirected connected graph without loops or multiple edges. The order and size of G are denoted by p and q, respectively.The neighborhood of a vertex v is the set $N(v)$ consisting of all vertices u which are adjacent with v. The closed neighborhood of a vertex v is the set $N[v] = N(v) \cup \{v\}$. A vertex v is an extreme vertex if the subgraph induced by its neighbors is complete. A vertex v is a semi-extreme vertex of G if the subgraph induced by its neighbors has a full degree vertex in $N(v)$. In particular, every extreme vertex is a semi-extreme vertex and a semi-extreme vertex need not be an extreme vertex. A chord of a path $u_1, u_2, \ldots, u_k$ in G is an edge $u_i u_j$ with $j \geq i + 2$. A $u - v$ path P is called a monophonic path if it is a chordless path. A set M of vertices is a monophonic set if every vertex of G lies on a monophonic path joining some pair of vertices in M, and the minimum cardinality of a monophonic set is the monophonic number $m(G)$ of G. A monophonic set of cardinality $m(G)$ is called a m-set of G. The monophonic domination number of a graph G was studied in [9]. A set M of vertices of a connected graph G is a restrained monophonic set if either $V = M$ or M is a monophonic set with the subgraph $G[V - M]$ induced by $V - M$ has no isolated vertices. The minimum cardinality of a restrained monophonic set of G is the restrained monophonic number of G, and is denoted by $m_r(G)$.

Theorem 1. Each extreme vertex of a connected graph G belongs to every restrained monophonic set of G.

FORCING RESTRAINED MONOPHONIC NUMBER OF A GRAPH

Definition 2. Let G be a connected graph and M is a minimum restrained monophonic set of G. A subset $T \subseteq M$ is called a forcing subset for M if M is the unique minimum restrained monophonic set containing T. A forcing subset for M of minimum cardinality is a minimum forcing subset of M. The forcing restrained monophonic number of M, denoted by $f_{m_r}(M)$, is the cardinality of a minimum forcing subset of M. The forcing restrained monophonic number of G, denoted by $f_{m_r}(G)$, is $f_{m_r}(G) = min\,\{f_{m_r}(M)\}$, Where the minimum is taken over all minimum restrained monophonic sets M in G.

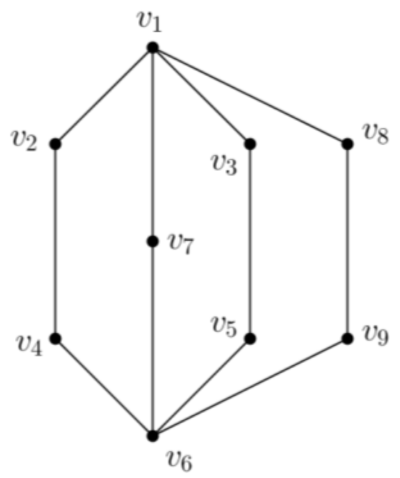

Figure 1: The graph G is Forcing Restrained Monophonic Number of a Graph

Example 3. For the graph G given in Figure 1, $M_1 = \{v_1, v_2, v_4\}$, $M_2 = \{v_1, v_6, v_7\}$, $M_2 = \{v_1, v_3, v_5\}$, $M_4 = \{v_3, v_5, v_6\}$, $M_5 = \{v_2, v_4, v_6\}$, $M_6 = \{v_1, v_8, v_9\}$, and $M_7 = \{v_6, v_8, v_9\}$ are the only seven minimum restrained monophonic sets of G. It is clear that $f_{m_r}(M_1) = 2$, $f_{m_r}(M_2) = 1, f_{m_r}(M_3) = f_{m_r}(M_4) = f_{m_r}(M_5) = f_{m_r}(M_6) = f_{m_r}(M_7) = 2$ so that $f_{m_r}(G) = 1$.

Theorem 4. For any connected graph $G, 0 \leq f_{m_r}(G) \leq m_r(G) \leq p$

Theorem 5. Let G be a connected graph. Then

(a) $f_{m_r}(G) = 0$ if and only if G has a unique minimum restrained monophonic set.

(b) $f_{m_r}(G) = 1$ if and only if G has atleast two minimum restrained monophonic sets, one of which is a unique minimum restrained monophonic set containing one of its elements,

(c) $f_{m_r}(G) = m_r(G)$ if and only if no minimum restrained monophonic set of G is the unique minimum restrained monophonic set of containing any of its proper subsets.

Proof. (a) Let $f_{m_r}(G) = 0$. Then , by definition, $f_{m_r}(M) = 0$ for some minimum restrained monophonic set M of G so that the empty set φ is the minimum forcing subset for M. Since the empty set φ is a subset of every set , it follows that M is the unique minimum restrained monophonic set of G. The converse is clear.

(b) Let $f_{m_r}(G) = 1$. Then by Theorem 5(a), G has atleast two minimum restrained monophonic sets. Also since $f_{m_r}(G) = 1$, there is a singleton subset T of a minimum restrained monophonic set M of G such that T is not a subset of any other minimum restrained monophonic set containing one of its elements. The converse is clear.

(c) $f_{m_r}(G) = m_r(G)$. Then $f_{m_r}(M) = m_r(G)$ for every minimum restrained monophonic set M in G. Also, by Theorem 5, $m_r(G) \geq 2$ and hence $f_{m_r}(G) \geq 2$. Then by theorem 5(a), G has atleast two minimum restrained monophonic sets and so the empty set φ is not a forcing for any minimum restrained monophonic set of G. Since $f_{m_r}(M) = m_r(G)$, no proper subset of M is a forcing subset of M. Thus, no minimum restrained monophonic set of G is the unique minimum restrained monophonic set containing any of its proper subsets. Conversly, G contains more than one minimum restrained monophonic set and no subset of any minimum restrained monophonic set M other than M is a forcing subset for M. Hence it follows that $f_{m_r}(G) = m_r(G)$.

Definition 6. A vertex v of a connected graph G is said to be a restrained monophonic vertex of G if v belongs to every minimum restrained monophonic set of G.

Example 7. For the graph G given in Figure 2 $M_1 = \{v_1, v_2, v_4\}$ and $M_2 = \{v_1, v_2, v_3\}$ are the only two restrained monophonic sets of G. It is clear that v_1 and v_2 are restrained monophonic vertices of G.

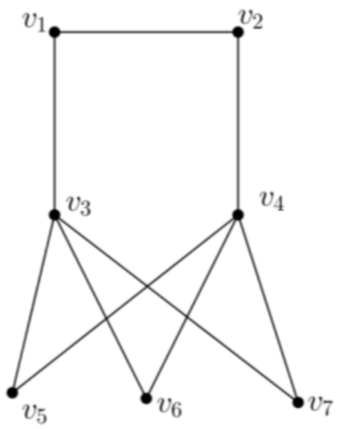

Figure 2: The graph G is Restrained Monophonic Vertices of a Graph

Theorem 8. Let G be a connected graph and let $\Im$ be the set of relative complements of the minimum forcing subsets in their respective minimum restrained monophonic sets in G. Then $\bigcap_{F \in \Im} F$ is the set of restrained monophonic vertices of G.

Proof. Let W be the set of all restrained monophonic vertices of G. We show that $W = \bigcap_{F \in \Im} F$. Let v is a restrained monophonic vertex of G that belongs to every minimum restrained monophonic set M of G. Let $T \subseteq M$ be any minimum forcing subset for any minimum restrained monophonic set M of G. We claim that $v \notin T$. If $v \in T$, then $T' = T - \{v\}$ is a proper subset of T such that M is the unique minimum restrained monophonic set containing T' so that T' is a forcing subset for M with $|T'| < |T|$, which is a contradiction to T is a minimum forcing subset for M. Thus $v \notin T$ and so $v \in F$, where F in the relative complement of T in M. Hence $v \in \bigcap_{F \in \Im} F$ so that $W \subseteq \bigcap_{F \in \Im} F$. Conversly, let v $\in \bigcap_{F \in \Im} F$. Then v belongs to the relative complement of T in M for every T and every M such that $T \subseteq M$, where T is a minimum forcing subset for M. Since F is the relative complement of T in M, we have $F \subseteq M$ for every M, which implies that v is a restrained monophonic vertex of G. Thus $v \in W$ and so $\bigcap_{F\in\Im} F \subseteq W$. Hence $W = \bigcap_{F\in\Im} F$.

Theorem 9. Let G be a connected graph and W be the set of all restrained monophonic vertices of G. Then $f_{m_r}(G) \leq m_r(G) - |W|$.

Proof. Let M be any minimum restrained monophonic set M of G. Then $m_r(G) = |M|$, $W \subseteq M$ and M is the unique minimum restrained monophonic set containing $M - W$. Thus $f_{m_r}(G) \leq |M - W| = |M| - |W| = m_r(G) - |W|$.

Remark 10. The bound in Theorem 9 is sharp. For the graph G in Figure 2 , $M_1 = \{v_1, v_2, v_4\}$ and $M_2 = \{v_1, v_2, v_3\}$ are the only two restrained monophonic sets so that $m_r(G) = 3$ and $f_{m_r}(G) = 1$. Also $W = \{v_1, v_2\}$ is the set of all restrained monophonic vertices of G and so $f_{m_r}(G) = m_r(G) - |W|$. Also the inequality in Theorem 9 can be strict, For the graph G given in Figure 3, $M_1 = \{v_1, v_3, v_9\}$, $M_2 = \{v_1, v_5, v_8\}$, $M_3 = \{v_1, v_6, v_7\}$ are the only three restrained monophonic sets so that $m_r(G) = 3$ and $f_{m_r}(G) = 1$. Now v_1 is the only restrained monophonic vertex of G and so $f_{m_r}(G) = m_r(G) - |W|$.

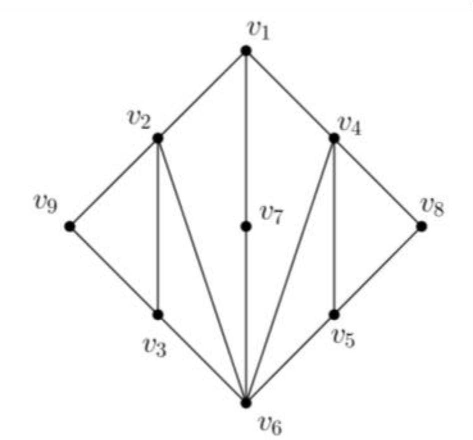

Figure 3: The graph G is Restrained Monophonic Vertex of a Graph

Theorem 11. For a cycle $G = C_p(p \geq 5)$, $f_{m_r}(G) = 1$.

REALIZATION RESULTS

Theorem 12. For every pair a, b of integers with $0 \leq a \leq b$ and $b \geq 2$, there exists a connected graph G such that $f_{m_r}(G) = f_m(G) = 0, m(G) = a$ and $m_r(G) = b$.

Proof. Let P_i: v_i , u_i , be a path of order 2. Let G be the graph obtained from P_i by adding the new verices $x, y, z_1, z_2, \ldots, z_{a-2}$, $l_1, l_2, \ldots, l_{b-a}$ and join each $z_i(1 \leq i \leq a-2)$ to u_a and join each $l_i(1 \leq i \leq b-a)$ to both x and y, and join x, y to v_1 The graph G is given in Figure 4.

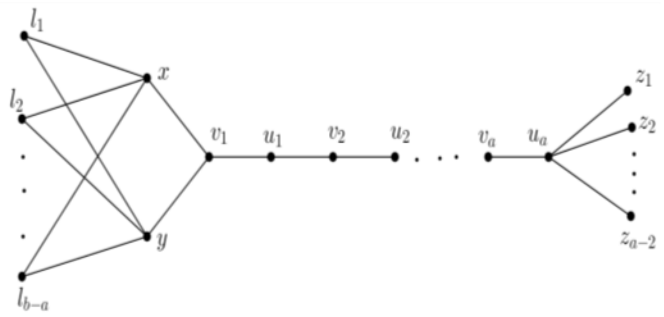

Figure 4: The graph G in Theorem 12

Let $Z = \{z_1, z_2, \ldots, z_{a-2}\}$ be the set of all end vertices of G. By Theorem 1, Z is a subset of every monophonic set of G. It is clear that Z is not a monophonic set of $G. W = Z \cup \{x, y\}$ Then it is clear that Z is the unique monophonic set of G so that $m(G) = a$ and $f_m(G) = 0$ by Theorem 5(a) . By Theorem 1, Z is a subset of every restrained monophonic set of G. We see that W is not a restrained monophonic set of G. Now, it is easily seen that $W_1 = W \cup \{l_1, l_2, \ldots, l_{b-a}\}$ is the unique restrained monophonic set of G so that $m_r(G) = b$ and $f_{m_r}(G) = 0$ by Theorem 5(a).

Theorem 13. For every integers a, b and c with $0 \leq a \leq b \leq c$ and $c > a + b$, then there exists a connected graph G such that $f_m(G) = 0, f_{m_r}(G) = a, m(G) = b$ and $m_r(G) = c$.

Proof. Let C_i: u_i , v_i, w_i , be a cycle of order 3, and let $K_{2,a}$: $x, y, k_1, k_2, \ldots, k_a$ be a copy of $K_{2,a}$ Let G be the graph obtained from adding the new vertices $z_1, z_2, \ldots, z_{b-a-2}$, $l_1, l_2, \ldots, l_{c-b-a}$ and join each $z_i(1 \leq i \leq b-a-2)$ to w_a. and join each $l_i(1 \leq i \leq c-b-a)$ to both v_1 and v_a, and join x, y to u_1. The graph G is given in Figure 5.

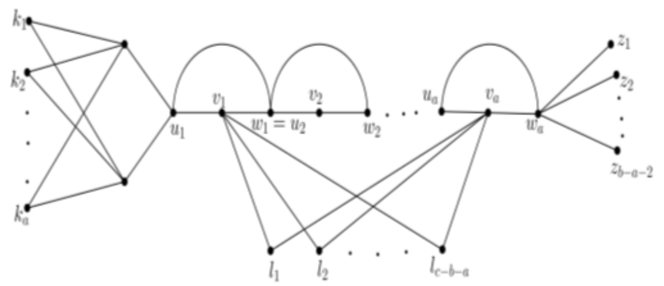

Figure 5: The graph G in Theorem 13

Let $Z = \{z_1, z_2, \ldots, z_{b-a-2}\}$ be the extreme vertices of G. It is clear that M is not a monophonic set of G. $Z_1 = Z \cup \{x, y, v_1, v_2, \ldots, v_a\}$ is a monophonic set of G so that $m(G) = b$ and $f_m(G) = 0$. Next, we show that $m_r(G) = c$. Let M be any monophonic set of G. Then $Z_1 \subseteq M$. It is clear that Z_1 is not a retrained monophonic set of G. For $1 \leq i \leq a$, let $M_i = \{k_i\}$. We observe that every restrained monophonic set of G must contain a vertex from M_i and each $l_j(1 \leq j \leq c-b-a)$ so that $m_r(G) \geq b + a + c - b - a = c$. Now, $W = Z_1 \cup \{l_1, l_2, \ldots, l_{c-b-a}\} \cup \{k_1, k_2, \ldots, k_a\}$ is a restrained monophonic set of G so that

$m_r(G) \leq b + c - a - b + a = c$. Thus $m_r(G) = c$. Next, we show that $f_{m_r}(G) = a$. Since every restrained monophonic set contains $Z_1 \cup \{l_1, l_2, \ldots, l_{c-b-a}\}$,it follows from theorem 9, that $f_{m_r}(G) \leq m_r(G) - (b + c - b - a) = a$. Now, since $m_r(G) = c$ and every restrained monophonic set contains $Z_1 \cup \{l_1, l_2, \ldots, l_{c-b-a}\}$. It is easily seen that every restrained monophonic set M is of the form $Z_1 \cup \{l_1, l_2, \ldots, l_{c-b-a}\} \cup \{p_1, p_2, \ldots, p_a\}$, where $p_i \in M_i (1 \leq i \leq a)$. Let T be any proper subset of M with $|T| < a$.Then there exists p_j such that $p_j \in T$. Let e_j be the vertex of M_j distinct from p_j . Then $W = (M - \{p_j\}) \cup \{e_j\}$ is a restrained monophonic set properly containing T. Thus M is not the unique restrained monophonic set containing T so that T is not a forcing subset of M. This is true for all restrained monophonic sets containing G so that $f_{m_r}(G) = a$.

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