Generalized semi regular closed sets in bitopological spaces

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Abstract

In this paper, we introduce a new type of closed sets in bitopological space \((X, \tau_1, \tau_2)\), used it to construct new types of normality, and introduce new forms of continuous function between bitopological spaces. Finally, we proved that the our new normality properties are preserved under some types of continuous functions between bitopological spaces.

INTRODUCTION AND PRELIMINARIES

The concepts of regular closed, generalized closed (briefly, \(g\)-closed), semiopen, regular generalized closed (briefly, \(rg\)-closed), and generalized semiclosed (briefly, \(gs\)-closed) sets have been introduced and investigated in [1-5]. The concepts of semiopen sets and regular open sets have been extended to bitopological spaces [6] called \(ij\)-semiopen and \(ij\)-regular open respectively. The mild normality and almost normality have been introduced in [7]. A weak form of normal spaces has been introduced in [8] called mildly normal spaces. In [9], the author used the semiopen sets to define seminormal spaces, recently, in [10] the author have continued the study of further properties of prenormal spaces and also defined and investigated mildly s-normal (resp. almost s-normal) spaces which are generalization of both mildly normal (resp. almost normal) spaces and s-normal spaces. The concept of generalised semiregular closed (briefly, \(gsr\)-closed) sets has been introduced in [11]. The concept of binormal spaces has been introduced in [12]. In [13,14] extended the concepts of \(g\)-closed, \(gs\)-closed and \(rg\)-closed sets, mildly normal and almost normal spaces to bitopological spaces. In this paper, we extend the concept of \(gsr\)-closed sets to bitopological spaces \((X, \tau_1, \tau_2)\) called \(ij-\\gsr\)-closed sets. Also, we construct a new types of normality in bitopological spaces based on \(ij\)-semiopen sets called semibinormal, almostsemibinormal and mildly semibinormal. We use the class of \(ij-\\gsr\)-closed sets to characterize these types of normality and construct new types of continuous functions. We prove that the introduced binormality properties are preserved under some types of continuous functions. Throughout this paper, the following abbreviations will be adopted: Let \(A\) be a subset of a bitopological space \((X, \tau_1, \tau_2)\), the interior (resp. closure) of \(A\) with respect to topology \(\tau_i\) \((i=1,2)\) will be denoted by \(\text{int}^i(A)\) (resp. \(\text{cl}^i(A)\) ). We denote the set of all closed sets with respect to the topology \(\tau_i\) by \(i-C(X)\).

In what follows, let \(i, j \in \{1,2\}\) and \(i \neq j\).

Definition 1.1 [6]. A subset \(A\) of a bitopological space \((X, \tau_1, \tau_2)\) is said to be

1. \(ij\)-semi open if \(A \subseteq \text{cl}^i(\text{int}^j(A))\)
2. \(ij\)-regular open if \(A = \text{int}^i(\text{cl}^j(A))\)

The complement of \(ij\)-semi-open (resp. \(ij\)-regular open) set is called \(ij\)-semi-closed (resp. \(ij\)-regular closed) set. We denote the set of all \(ij\)-semi-open (resp. \(ij\)-semi-closed, \(ij\)-regular open and \(ij\)-regular closed) sets by \(ij-O^i(X)\) (resp. \(ij-C^i(X)\), \(ij-O^i(X)\) and \(ij-C^i(X)\)).

Definition 1.2 [6]. For any bitopological space \((X, \tau_1, \tau_2)\) and \(A \subseteq X\), \(ij\)-semi-interior (resp. \(ij\)-semi-closure) of \(A\) is denoted by \(ij-\text{int}^i(A)\) (resp. \(ij-\text{cl}^i(A)\) ) and defined as

\[
ij-\text{int}^i(A) = \bigcup\{F \subseteq X : F \in ij-O^i(X), F \subseteq A\}
\]
\[
ij-\text{cl}^i(A) = \bigcap\{F \subseteq X : F \in ij-C^i(X), F \supseteq A\}
\]
Definition 1.3. A subset $A$ of a bitopological space $(X, \tau_1, \tau_2)$ is said to be

1. $ij$ -generalized closed [14] (briefly, $ij$ - $g$ -closed) if $A \subseteq U$, $U \in \tau_i \Rightarrow cl^i(A) \subseteq U$.
2. $ij$ -regular generalized closed [13] (briefly, $ij$ - $rg$ -closed) if $A \subseteq U$, $U \in ij-O^R(X) \Rightarrow cl^i(A) \subseteq U$.
3. $ij$ -generalized semi-closed [14] (briefly, $ij$ - $gsr$ -closed) if $A \subseteq U$, $U \in \tau_i \Rightarrow ji-cl^i(A) \subseteq U$.

Definition 1.4. A subset $A$ of a bitopological space $(X, \tau_1, \tau_2)$ is said to be $ij$ -generalized semi-regular closed (briefly, $ij$ - $gsr$ -closed) if $A \subseteq U$, $U \in ij-O^R(X) \Rightarrow ji-cl^i(A) \subseteq U$.

The complement of $ij$ - $g$ -closed (resp. $ij$ - $rg$ -closed, $ij$ - $gs$ -closed and $ij$ - $gsr$ -closed) set is called $ij$ - $g$ -open (resp. $ij$ - $rg$ -open, $ij$ - $gs$ -open and $ij$ - $gsr$ -open) set and defined in the following lemma. Definition 1.4 is a particular case of Definition 8 from Noiri [15]. From Proposition 4 in [15], we obtain the following lemma.

Lemma 1.1. A subset $A$ of a bitopological space $(X, \tau_1, \tau_2)$ is:

1. $ij$ - $g$ -open iff $A \supseteq F$, $F \in i-C(X) \Rightarrow j-int(A) \supseteq F$
2. $ij$ - $rg$ -open iff $A \supseteq F$, $F \in ij-C^R(X) \Rightarrow j-int(A) \supseteq F$
3. $ij$ - $gs$ -open iff $A \supseteq F$, $F \in i-C(X) \Rightarrow ji-int^i(A) \supseteq F$
4. $ij$ - $gsr$ -open iff $A \supseteq F$, $F \in ij-C^{gs}(X) \Rightarrow ji-int^i(A) \supseteq F$

We denote the set of all $ij$ - $g$ -closed (resp. $ij$ - $g$ -open, $ij$ - $rg$ -closed, $ij$ - $rg$ -open, $ij$ - $gs$ -closed, $ij$ - $gs$ -open $ij$ - $gsr$ -closed and $ij$ - $gsr$ -open) sets by $ij-C^s(X)$ (resp. $ij-O^s(X)$, $ij-C^{gs}(X)$, $ij-O^{gsr}(X)$, $ij-C^{gsr}(X)$). The arbitrary union of $ij$ - $gsr$ -closed sets is an $ij$ - $gsr$ -closed set. But the intersection of two of $ij$ - $gsr$ -closed sets need not be an $ij$ - $gsr$ -closed set as shown by the following example.

Example 1.1. Let $X = \{a, b, c, d\}$, $\tau_1 = \{x, \phi, \{a\}, \{b\}, \{a, b\}, \{c, d\}\}$, $\tau_2 = \{x, \phi, \{a\}, \{b\}, \{a, b\}, \{c\}, \{d\}\}$.
We have $\{a, c\} \cap \{a, d\} = \{a\} \not\in 21 - C^{gsr}(X)$ but $\{a, c\} \cap \{a, d\} = \{a\} \not\in 21 - C^{gsr}(X)$.

Proposition 1.1. The following diagram shows the relationship between the above different types of closed sets.

![Diagram showing relationships between closed sets](image)

Where none of these implications is reversible as shown by the following example.

Example 1.2. Let $X = \{a, b, c, d\}$, $\tau_1 = \{x, \phi, \{a\}, \{b\}, \{a, b\}\}$ and $\tau_2 = \{x, \phi, \{a\}, \{b\}, \{c\}, \{d\}\}$.
(Arrows 1, 5) $\{d\} \in 12 - C^s(X) \cap 21 - C^s(X)$ but $\{d\} \not\in 2 - C(X)$
(Arrows 2, 6) $\{a\} \in 12 - C^{gs}(X) \cap 12 - C^{gsr}(X)$ but $\{a\} \not\in 12 - C^{gs}(X)$
(Arrow 3) $\{a, d\} \in 12 - C^{gsr}(X)$ but $\{a, d\} \not\in 12 - C^s(X)$
(Arrow 4)
Example 1.3. In Example 1.2. Let \( X = \{a, b, c, d, e\} \),
\[ \tau_1 = \{X, \phi, \{a\}, \{b\}, \{a, b, c\}\}, \]
\[ \tau_2 = \{X, \phi, \{a, e\}, \{a, e, d\}\}, \]
\[ \{a, b, c\} \in 21 - O^{gr}(X) \text{ but } \{a, b, c\} \not\in 21 - C^{gr}(X). \]

(Arrow 7) In Example 1.3. \[ \{b, e\} \in 21 - O^{gr}(X) \text{ but } \{b, e\} \not\in 21 - C^{gr}(X) \]

Remark 1.1. For any bitopological space \((X, \tau_1, \tau_2)\) we note that:

1. The classes \( ij - C^g(X) \) and \( ji - C^s(X) \) are independent
2. The classes \( ij - C^{gr}(X) \) and \( ji - C^{gs}(X) \) are independent

The following example investigate the previous remark.

Example 1.4. Let \((X, \tau_1, \tau_2)\) as in Example 1.3.

1. \( \{b, c, d\} \in 21 - C^{gr}(X) \text{ but } \{b, c, d\} \not\in 21 - C^{gs}(X) \text{ also } \{a, e\} \in 21 - C^{gs}(X) \text{ but } \{a, e\} \not\in 21 - C^{gr}(X) \)
2. \( \{c, e\} \in 12 - C^{gs}(X) \text{ but } \{c, e\} \not\in 21 - C^{gr}(X) \text{ also } \{b\} \in 21 - C^{s}(X) \text{ but } \{b\} \not\in 12 - C^{g}(X) \)

Theorem 1.1. For any bitopological space \((X, \tau_1, \tau_2)\), \( A \subseteq X \), the following are holds:

1. If \( A \in ij - C^{gs}(X) \cap \tau_i \) then \( A \in j - C(X) \).
2. If \( A \in ij - C^{gs}(X) \cap \tau_i \) then \( A \in ji - C^{s}(X) \).
3. If \( A \in ij - C^{gr}(X) \) and \( \tau_i = ij - O^{r}(X) \) then \( A \in ij - C^{s}(X) \).
4. If \( A \in ij - C^{es}(X) \) and \( \tau_i = ij - O^{r}(X) \) then \( A \in ij - C^{s}(X) \).
5. If \( A \in ij - C^{g}(X) \) and \( j - C(X) = ji - C^{s}(X) \) then \( A \in ij - C^{gr}(X) \).
6. If \( A \in ij - C^{s}(X) \) and \( j - C(X) = ji - C^{s}(X) \) then \( A \in ij - C^{gs}(X) \).

Proof: obvious.

Theorem 1.2. For any bitopological space \((X, \tau_1, \tau_2)\). If \( A \in ij - O^{gr}(X) \) and \( A \subseteq B \subseteq ji - cl^i(A) \), then \( B \in ij - C^{gr}(X) \).

Proof: Let \( B \subseteq U \). \( U \in ij - O^r(X) \). Since \( A \subseteq B \) and \( ij - C^{gr}(X) \), then \( ji - cl^i(A) \subseteq U \). Since \( B \subseteq ji - cl^i(A) \), then we have \( ji - cl^i(B) \subseteq ji - cl^i(A) \subseteq U \). Consequently \( B \in ij - C^{gr}(X) \).

Theorem 1.3. Let \((X_1, \tau_1, \tau_2)\) and \((X_2, \tau'_1, \tau'_2)\) be two bitopological spaces. If \( A \in ij - O^{gr}(X_1) \) and \( B \in i^* j^* - O^{gr}(X_2) \), then \( A \times B \subseteq i^* \times i^* \times j^* - O^{gr}(X_1 \times X_2) \).

Proof: Let \( A \in ij - O^{gr}(X_1) \) and \( B \in i^* j^* - O^{gr}(X_2) \). Let \( W = A \times B \subseteq X_1 \times X_2 \). Let \( F = F_1 \times F_2 \subseteq W \), \( F_1 \subseteq A \), \( F_2 \subseteq B \) and \( F_1 \subseteq ji - int^i(A) \) and \( F_2 \subseteq ji^* j^* - int^i(B) \). Then, there are \( F_1 \subseteq ij - C^r(X_1) \) and \( F_2 \subseteq i^* j^* - C^r(X_2) \). Hence \( F_1 \subseteq ij - int^i(A) \times i^* j^* - int^i(B) \). Therefore \( A \times B \subseteq i^* \times i^* \times j^* - O^{gr}(X_1 \times X_2) \).

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Some types of \( ij \)-near continuous functions

In this section we introduce two types of continuous functions between bitopological spaces and study their properties.

**Definition 2.1** [14]. A function \( f : (X, \tau_1, \tau_2) \to (Y, \mu_1, \mu_2) \) is called:

1. \( ij \)-semi-continuous if \( \forall V \in i - C(Y), f^{-1}(V) \in ij - C'(X) \),
2. \( ij - g \)-continuous if \( \forall V \in j - C(Y), f^{-1}(V) \in ij - C^g(X) \),
3. \( ij - gs \)-continuous if \( \forall V \in j - C(Y), f^{-1}(V) \in ij - C^{gs}(X) \),
4. \( i \)-continuous if \( \forall V \in i - C(Y), f^{-1}(V) \in i - C(X) \).

**Definition 2.2.** A function \( f : (X, \tau_1, \tau_2) \to (Y, \mu_1, \mu_2) \) is called:

1. \( ij - semi - gs r \)-continuous if \( \forall V \in ji - C^g(Y), f^{-1}(V) \in ij - C^{gsr}(X) \),
2. \( ij - semi - gs \)-continuous if \( \forall V \in ji - C^s(Y), f^{-1}(V) \in ij - C^{gs}(X) \)

**Theorem 2.1.** The relationship between the previous concepts of continuity of functions between bitopological spaces are stated in the following diagram

![Diagram 2.1](Diagram2.1)

**Proof:** Straightforward.

In Diagram 2.1, the arrows are not reversible as one may see the following examples:

**Example 2.1** Let \( X = \{a, b, c, d\}, Y = \{u, v, w\} \), \( \tau_1 = \{x, \phi, \{a\}, \{a, d\}\} \) and \( \tau_2 = \{x, \phi, \{a, b\}, \{c, d\}\} \), \( \mu_1 = \{y, \phi, \{v\}, \{v, w\}\} \) and \( \mu_2 = \{y, \phi, \{v\}, \{v, u\}\} \) let .

- \( f : (X, \tau_1, \tau_2) \to (Y, \mu_1, \mu_2) \)

**Example 2.2.** Let \( X = \{a, b, c\}, Y = \{u, v, w\} \), \( \tau_1 = \{X, \phi, \{a\}, \{a, b\}\} \), \( \tau_2 = \{X, \phi, \{c\}, \{a, c\}\} \), \( \mu_1 = \{Y, \phi, \{u\}, \{v, w\}\} \) and \( \mu_2 = \{Y, \phi, \{v\}, \{v, u\}\} \). We have \( f \) is \( 12 - g s r \)-continuous and \( 12 - semi - G S R \)-continuous but it is not \( 12 - semi - g s \)-continuous.

Since there exist \( \{u\} \in 21 - C^s(Y) \), such that \( f^{-1} \{u\} = \{a\} \notin 12 - C^{gsr}(X) \).
Remark 2.1 For any function $f : (X, \tau_1, \tau_2) \rightarrow (Y, \mu_4, \mu_5)$ we note that:

1. $ij - g$-continuous and $ij - semi - gs$-continuous are independent.
2. $ij - semi$-continuous and $ij - g$-continuous are independent.
3. $ij - gs$-continuous and $ij - semi - gsr$-continuous are independent.

The following example justifies the previous remark. Example 2.3 $[i, ii]$ [resp. $iii, iv]$ and 2.4 $[v, vi]$ investigate Remark 2.1 (1) [resp. (2) and (3)].

Example 2.3. Let $f : (X, \tau_1, \tau_2) \rightarrow (Y, \mu_4, \mu_5)$ as in Example 2.1.

(i) If $f$ is defined by $f(a) = f(b) = u, f(c) = v$ and $f(d) = w.$ We have $f$ is $12 - semi - gs$-continuous, but it is not $12 - g$-continuous. Since there exist $\{w\} \in 2 - C(Y)$ such that $f^{-1}(\{w\}) = \{c, d\} \notin 12 - C^s(X).$

(ii) If $f$ is defined by $f(a) = f(c) = v, f(b) = u$ and $f(d) = w.$ We have $f$ is $12 - semi - gs$-continuous, but it is not $12 - g$-continuous. Since there exist $\{w\} \in 2 - C(Y)$ such that $f^{-1}(\{w\}) = \{d\} \notin 12 - C^s(X).$

(iii) If $f$ is defined by $f(a) = f(b) = u, f(c) = v$ and $f(d) = w.$ We have $f$ is $21 - semi$-continuous, but it is not $12 - g$-continuous. Since there exist $\{w\} \in 2 - C(Y)$ such that $f^{-1}(\{w\}) = \{d\} \notin 12 - C^s(X).$

(iv) If $f$ is defined by $f(a) = f(c) = v, f(b) = w$ and $f(d) = u.$ We have $f$ is $12 - g$-continuous, but it is not $21 - semi$-continuous. Since there exist $\{v\} \in 2 - C(Y)$ such that $f^{-1}(\{v\}) = \{a, c\} \notin 21 - C^s(X).$

(v) If $f$ is defined by $f(a) = f(c) = w, f(b) = v$ and $f(d) = u.$ We have $f$ is $12 - semi - gsr$-continuous, but it is not $12 - g$-continuous. Since there exist $\{w\} \in 2 - C(Y)$ such that $f^{-1}(\{w\}) = \{a, c\} \notin 12 - C^s(X).$

(vi) If $f$ is defined by $f(a) = f(d) = u, f(b) = w$ and $f(c) = v.$ We have $f$ is $12 - gs$-continuous, but it is not $12 - semi - gsr$-continuous. Since there exist $\{u\} \in 21 - SC(Y)$ such that $f^{-1}(\{u\}) = \{a, d\} \notin 12 - C^s(X).$

Definition 2.3. A function $f : (X, \tau_1, \tau_2) \rightarrow (Y, \mu_4, \mu_5)$ is called:

1. $ij - R$-map if $\forall V \in ij - O^R(Y), f^{-1} \in ij - O^R(X).$
2. $ij - semi$ irresolute if $\forall V \in ij - C^i(Y), f^{-1}(V) \in ij - C^i(X).$
3. $ij - r$-closed if $\forall G \in ij - C^R(X), f(G) \in ij - C^R(Y).$
4. $ij - semi - gs$-closed if $\forall G \in ij - C^s(X), f(G) \in ji - C^{gs}(Y).$
5. $ij - semi - rgs$-closed if $\forall G \in ij - C^R(X), f(G) \in ji - C^{gs}(Y).$

Lemma 2.1. For any surjection function $f : (X, \tau_1, \tau_2) \rightarrow (Y, \mu_4, \mu_5)$ the following are equivalent.

(a) $f$ is $ij - semi - gs$-closed function.
(b) for any $B \subseteq Y, U \in ij - O^s(X)$ such that $f^{-1}(B) \subseteq U,$ there exist $V \in ji - O^s(Y)$ such that $B \subseteq V$ and $f^{-1}(V) \subseteq U.$

Proof. Necessity. Let such that $B \subseteq Y, U \in ij - O^s(X)$ such that $f^{-1}(B) \subseteq U.$ Since $f$ is $ij - semi - gs$-closed function, then $f(U) \subseteq ji - O^s(Y)$ such that $f^{-1}(B) \subseteq U.$ Since $f^{-1}(B) \subseteq U,$ then $B = f\left(f^{-1}(B)\right) \subseteq f(U) \subseteq V$ and $f^{-1}(V) = f^{-1}(U) \subseteq U.$ Sufficiency, Let $G \subseteq ij - O^s(X)$ such that $F \subseteq C(Y),$ then $G \subseteq f^{-1}(F), F \subseteq Y.$ This implies that there exist $V \subseteq ji - O^{gs}(Y)$ such that $F \subseteq V$ and $f^{-1}(V) \subseteq G.$ Since $V \subseteq ji - O^{gs}(Y), F \subseteq j - C(Y)$ and $F \subseteq V.$ Consequently, $ij - int(V) \subseteq F$ Since $V \subseteq f(G)$ then $F \subseteq ji - int(V) \subseteq ij - int(f(G)).$ This implies that $f(G) \subseteq ji - O^{gs}(Y).$ Therefore, $f$ is $ij - semi - gs$-closed function.

Lemma 2.2. For any surjection function $f : (X, \tau_1, \tau_2) \rightarrow (Y, \mu_4, \mu_5)$ the following are equivalent.
(a) $f$ is $ij$-semi-rgs-closed function.

(b) For any $B \subseteq Y, U \in ij-O^{\ast}(X)$ such that $f^{-1}(B) \subseteq U$, there exist $V \in ji-O^{\ast r}(Y)$ such that and $B \subseteq V$ and $f^{-1}(V) \subseteq U$.

**Proof:** Similar to Lemma 2.1

**Theorem 2.2.** Let $f : (X, \tau_1, \tau_2) \rightarrow (Y, \mu_1, \mu_2)$ is $ij$-semi-gsr-continuous (resp. $ij$-semi-gsr-continuous) function and $g : (Y, \mu_1, \mu_2) \rightarrow (Z, \eta_1, \eta_2)$ is $ij$-semi-irresolute function, then $gof : (X, \tau_1, \tau_2) \rightarrow (Z, \eta_1, \eta_2)$, is $ij$-semi-gs-continuous (resp. $ij$-semi-gsr-continuous).

**Proof:** Let $V \in ij-C^{s}(Z)$, since $g$ is $ij$-semi-irresolute, then $g^{-1}(V) \in ij-C^{s}(Y)$. Since $f$ is $ij$-semi-gs-continuous, then $f^{-1}(g^{-1}(V)) = (gof)^{-1}(V)$. Consequently, $gof$ is $ij$-semi-gs-continuous.

**Some types of normality in bitopological spaces**

In this section, we introduce three concepts of normality in bitopological spaces namely semibinormal, mild semibinormal, and almost semibinormal. We give a new characterization of these types of binormality by $ij$-gsr-open sets.

**Definition 3.1** [12]. A bitopological space $(X, \tau_1, \tau_2)$ is said to be semibinormal if given disjoint subsets $A, B, A \in i-C(X)$ and $B \in j-C(X)$, there are disjoint subsets $U, V$ such that $U \in \tau_1 \wedge V \in \tau_2 \wedge A \subseteq U \wedge B \subseteq V$.

**Definition 3.2.** A bitopological space $(X, \tau_1, \tau_2)$ is said to be semibinormal if given disjoint subsets $A, B, A \in i-C(X)$ and $B \in j-C(X)$, there are disjoint subsets $U, V$ such that $U \in ji-O^{\ast}(X) \wedge V \in ji-O^{\ast r}(X) \wedge A \subseteq U$ and $B \subseteq V$.

**Theorem 3.1.** For any bitopological space $(X, \tau_1, \tau_2)$, the following statements are equivalent:

(a) $X$ is semibinormal;

(b) for any disjoint sets $A \in i-C(X)$ and $B \in j-C(X)$, there exist $U \in ji-O^{\ast}(X) \wedge V \in ji-O^{\ast r}(X)$ and $U \cap V = \emptyset$ such that $A \subseteq U$ and $B \subseteq V$.

(c) for any $A \in i-C(X), G \in \tau_j$ and $G \supseteq A$, there exists $U \in ji-O^{\ast r}(X)$ such that $A \subseteq U \subseteq ji-\text{cl}^i(U) \subseteq G$.

**Proof:**

(a) $\Rightarrow$ (b). Let $A \in i-C(X)$ and $B \in j-C(X)$, and $A \cap B = \emptyset$.

Since $X$ is semibinormal, then there exist $U \in ji-O^{\ast}(X), V \in ji-O^{\ast r}(X)$ and $U \cap V = \emptyset$ such that $A \subseteq U$ and $B \subseteq V$, this follows that, there exist $U \in ji-O^{\ast r}(X), V \in ji-O^{\ast r}(X)$ and $U \cap V = \emptyset$ such that $A \subseteq U$ and $B \subseteq V$.

(b) $\Rightarrow$ (c). Let $A \in i-C(X), G \in \tau_j$, and $G \supseteq A$. Then, $A \in i-C(X), X \setminus G \in j-C(X), (X \setminus G) \cap A = \emptyset$.

Then, there exist $U \in ji-O^{\ast r}(X), V \in ji-O^{\ast r}(X)$ and $U \cap V = \emptyset$ such that $A \subseteq U$ and $X \setminus G \subseteq V$. Since $V \in ji-O^{\ast r}(X), X \setminus G \in ji-RC(X)$ and $X \setminus G \subseteq V$, then by using Lemma 1.1 (4) we have $ij-\text{cl}^i(V) \supseteq X \setminus G \cap V = \emptyset$ implies $U \cap ij-\text{cl}^i(V) = \emptyset$. Consequently, $A \subseteq U \subseteq X \setminus ij-\text{cl}^i(V) \subseteq G$.

(c) $\Rightarrow$ (a). Let $A \in i-C(X), B \in j-C(X)$ and $A \cap B = \emptyset$. Then, $A \in i-C(X), X \setminus B \in \tau_j$ and $A \subseteq X \setminus B$.

Consequently, there exist $G \in ij-O^{\ast r}(X)$ such that $A \subseteq G \subseteq ij-\text{cl}^r(G) \subseteq X \setminus B$. Since $A \subseteq G, A \in ij-C^r(X)$ and $G \in ij-O^{\ast r}(X)$ then, by using Lemma 1.1 (4) we have $ij-\text{int}^r(G)$. This follows that $B \subseteq X \setminus (ij-\text{cl}^r(G)) = (ij-\text{cl}^r(G)) \cap G = ij-\text{int}^r(G) \subseteq ji-O^{\ast}(X), ij-\text{cl}^r(G) \subseteq ij-O^{\ast}(X)$ and $
\[ j_i - \text{int}^r(G) \cap i_j - \text{int}^r(G^r) = \phi. \] Put \( U = \text{int}^r(cl(\text{int}^r(G))) \) and \( V = \text{int}^r(cl(\text{int}^r(G^r))) \). Then \( U, V \) are disjoint, \( U \subseteq i_j - O^r(X) \) and \( V \subseteq j_i - O^r(X) \) such that \( U \supseteq A \) and \( V \supseteq B \).

**Definition 3.3.** A space \( (X, \tau_1, \tau_2) \) is said to be almost semibinormal if given disjoint subsets \( A \) and \( B \), \( A \in i - C(X), B \in j_i - C^r(X) \), there are disjoint subsets \( U \) and \( V \) such that \( U \subseteq i_j - O^r(X), V \subseteq j_i - O^r(X), A \subseteq U \) and \( B \subseteq V \).

**Theorem 3.2.** For any bitopological space \( (X, \tau_1, \tau_2) \), the following statements are equivalent:

(a) \( X \) is almost semibinormal;

(b) for each disjoint sets \( A \in i - C(X) \) and \( B \in j_i - C^r(X) \) there are disjoint subsets \( U \subseteq i_j - O^r(X) \) and \( V \subseteq j_i - O^r(X) \) such that \( A \subseteq U \) and \( V \subseteq B \);

(c) for each disjoint sets \( A \in i - C(X) \) and \( B \in j_i - C^r(X) \) there are disjoint subsets \( U \subseteq i_j - O^r(X) \) and \( V \subseteq j_i - O^r(X) \) such that \( A \subseteq U \) and \( V \subseteq B \);

(d) for each \( A \in i - C(X) \) and \( K \subseteq j_i - O^r(X) \), and \( K \supseteq A \) there exists \( U \subseteq i_j - O^r(X) \) such that \( A \subseteq U \subseteq j_i - cl^r(U) \subseteq K \).

**Proof:** It is obvious that \((a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (d) \). Let \( A \in \tau_1^r \) and \( K \subseteq j_i - O^r(X) \) and \( K \supseteq A \). This implies that \( A \subseteq \tau_1^r \) and \( X \subseteq K \subseteq j_i - C^r(X) \). Then, there exists \( U \subseteq i_j - O^r(X) \) such that \( A \subseteq U \subseteq X \subseteq K \). Then, by Lemma 1.1 (4), we have \( \text{int}^r(V) = X \cap K \subseteq V \subseteq K \). Consequently, \( A \subseteq U \subseteq i_j - \text{int}^r(U) \subseteq X \subseteq K \).

Therefore, \( A \subseteq U \subseteq j_i - \text{int}^r(U) \subseteq K \).

\[(d) \Rightarrow (c) \] Let \( A \subseteq i - C(X), B \subseteq j_i - C^r(X) \) and \( A \cap B = \phi \). This implies that \( A \subseteq i - C(X) \), \( X \cap B = j_i - C^r(X) \) and \( A \subseteq X \subseteq B \). Consequently, there exists \( U \subseteq i_j - O^r(X) \) such that \( A \subseteq U \subseteq i_j - \text{int}^r(U) \subseteq X \subseteq B \). Since \( A \subseteq U \subseteq i_j - \text{int}^r(U) \subseteq X \subseteq B \), then, by using Lemma 1.1 (4), we have \( A \subseteq U \subseteq j_i - \text{int}^r(U) \). This follows that \( B \subseteq X \subseteq j_i - \text{int}^r(U) = i_j - \text{int}^r(U) \). Put \( G = \text{int}^r(cl(\text{int}^r(U))) \) and \( H = \text{int}^r(cl(\text{int}^r(U^c))) \). Then, \( G, H \) are disjoint, \( G \subseteq j_i - O^r(X) \) and \( H \subseteq j_i - O^r(X) \).

**Definition 3.4.** A bitopological space \( (X, \tau_1, \tau_2) \) is said to be mildly semibinormal if given disjoint subsets \( A \subseteq i_j - C^r(X) \) and \( B \subseteq j_i - C^r(X) \), there are disjoint subsets \( U \subseteq i_j - O^r(X) \) and \( V \subseteq j_i - O^r(X) \) such that \( A \subseteq U \) and \( V \subseteq B \).

**Theorem 3.3.** For any bitopological space \( (X, \tau_1, \tau_2) \), the following statements are equivalent:

(a) \( X \) is mildly semibinormal;

(b) for any \( A \subseteq i_j - C^r(X) \) and \( B \subseteq j_i - C^r(X) \) and \( A \cap B = \phi \) there are \( U \subseteq i_j - O^r(X) \), \( V \subseteq j_i - O^r(X) \) and \( U \cap V = \phi \) such that \( A \subseteq U \) and \( V \subseteq B \);

(c) for any \( A \subseteq i_j - C^r(X) \) and \( B \subseteq j_i - C^r(X) \) and \( A \cap B = \phi \) there are \( U \subseteq i_j - O^r(X), V \subseteq j_i - O^r(X), \) and \( U \cap V = \phi \) such that \( A \subseteq U \) and \( V \subseteq B \);

(d) for any \( A \subseteq i_j - C^r(X) \), \( K \subseteq j_i - O^r(X) \) and \( A \subseteq K \) there exists \( U \subseteq i_j - O^r(X) \), such that \( A \subseteq U \subseteq i_j - cl^r(U) \subseteq K \).
(e) for any $A \in ij - C^g(X)$, $K \supseteq ji - O^s(X)$ and $A \subseteq K$ there exists $U \in ij - O^{ss}(X)$, such that $A \subseteq U \subseteq ij - c^l(U) \subseteq K$.

**Proof:** Similar to that of Theorem 3.2

**Preservation theorems**

In this section, we prove that the three types of binormality properties are preserved under some types of function between bitopological spaces.

**Theorem 4.1.** If $f : (X, \tau_1, \tau_2) \rightarrow (Y, \mu_1, \mu_2)$ is $ij - semi - gs$ -closed, $i$ - continuous, surjection and $X$ is semibinormal then $Y$ is also semibinormal.

**Proof:** Let $A \in i - C(Y), B \in j - C(Y)$ and $A \cap B = \phi$. Since $f$ is surjection $i$-continuous, then $f^{-1}(A) \subseteq i - C(X), f^{-1}(B) \subseteq j - C(X)$ and $f^{-1}(A) \cap f^{-1}(B) = f^{-1}(A \cap B) = \phi$. Since $X$ is semibinormal, there exist $U \in ji - O^s(X), V \in ij - O^s(X)$, and $U \cap V = \phi$, such that $f^{-1}(A) \subseteq U$ and $f^{-1}(B) \subseteq V$. Since $f$ is $ij - semi - gs$ -closed, by Lemma 2.1, there exist $G \in i - O^{ss}(Y)$ and $H \in ji - O^{ss}(Y)$ such that $A \subseteq G, B \subseteq H, f^{-1}(G) \subseteq U$ and $f^{-1}(H) \subseteq V$. Since $U$ and $V$ are disjoint, $G$ and $H$ are disjoint. Since $G \in ij - O^{ss}(Y)$ and $H \in ji - O^{ss}(Y)$, by Lemma 1.1 (3), then we have $A \subseteq ji - int^i(G), B \subseteq ji - int^i(H)$ and so $ji - int^i(G) \cap ji - int^i(H) = \phi$. Consequently, $Y$ is also semibinormal.

**Theorem 4.2.** If $f : (X, \tau_1, \tau_2) \rightarrow (Y, \mu_1, \mu_2)$ is $ij - semi - rgs$ -closed, $ij - R$ - map, surjection and $X$ is mildly semibinormal then $Y$ is also mildly semibinormal.

**Proof:** Let $A \in ij - C^g(Y), B \in ji - C^g(Y)$ and $A \cap B = \phi$. Since $f$ is surjection $ij - R$ - map, then $f^{-1}(A) \subseteq ij - C^g(X), f^{-1}(B) \subseteq ji - C^g(X)$ and $f^{-1}(A) \cap f^{-1}(B) = \phi$. Since $X$ is mildly semibinormal, then there exist $U \in ji - O^s(X), V \in ij - O^s(X)$ and $U \cap V = \phi$, such that $f^{-1}(A) \subseteq U$ and $f^{-1}(B) \subseteq V$. Since $f$ is $ij - semi - rgs$ - closed, by Lemma 2.1, there exist $G \in i - O^{ss}(Y)$ and $H \in ji - O^{ss}(Y)$ such that $A \subseteq G, B \subseteq H, f^{-1}(G) \subseteq U$ and $f^{-1}(H) \subseteq V$. Since $U$ and $V$ are disjoint, $G$ and $H$ are disjoint. Since $G \in ij - O^{ss}(Y)$ and $H \in ji - O^{ss}(Y)$, by Lemma 1.1 (4) then we have $A \subseteq ji - int^i(G), B \subseteq ji - int^i(H)$ and so $ji - int^i(G) \cap ji - int^i(H) = \phi$. Consequently, $Y$ is also mildly semibinormal.

**Theorem 4.3.** If $f : (X, \tau_1, \tau_2) \rightarrow (Y, \mu_1, \mu_2)$ is $ij - semi - rgs$ -closed, $ij - R$ - map, $i$ - continuous, surjection and $X$ is almost semibinormal, then $Y$ is almost semibinormal.

**Proof:** Let $A \in i - C(Y), B \in ji - C^g(Y)$ and. Since $f$ is $A \cap B = \phi$. Since $f$ is $ij - R$ - map, then, $f^{-1}(B) \subseteq ji - C^g(X)$. Since $f$ is $i$ - continuous, then $f^{-1}(A) \subseteq i - C(X)$ and we have $f^{-1}(A) \cap f^{-1}(B) = \phi$. Since $X$ is almost semibinormal, then there exist $U \in ji - O^s(X), V \in ij - O^s(X)$ and $U \cap V = \phi$, such that $f^{-1}(A) \subseteq U$ and $f^{-1}(B) \subseteq V$. Since $f$ is $ij - semi - rgs$ - closed, by Lemma 2.1, there exist $G \in ij - O^{ss}(Y)$ and $H \in ji - O^{ss}(Y)$ such that $A \subseteq G, B \subseteq H, f^{-1}(G) \subseteq U$ and $f^{-1}(H) \subseteq V$. Since $U$ and $V$ are disjoint, $G$ and $H$ are disjoint. Since $G \in ij - O^{ss}(Y)$ and $H \in ji - O^{ss}(Y)$, by Lemma 1.1 (4) then we have $A \subseteq ji - int^i(G), B \subseteq ji - int^i(H)$ and so $ji - int^i(G) \cap ji - int^i(H) = \phi$. Consequently, $Y$ is also almost semibinormal.

**Theorem 4.4.** If $f : (X, \tau_1, \tau_2) \rightarrow (Y, \mu_1, \mu_2)$ is $ij - semi - rgs$ -continuous $i$ - closed, injection and $Y$ is semibinormal, then $X$ is also semibinormal.
Proof: Let $A \in i-C(X), B \in j-C(X)$ and $A \cap B = \emptyset$. Since $f$ is iclosed injection, then $f(A) \in i-C(Y), f(B) \in j-C(Y)$ and $f(A) \cap f(B) = \emptyset$. By semi-normality of $Y$, there exist $U \in j_i-O'(Y), V \in i_j-O'(Y)$ and $U \cap V = \emptyset$ such that $f(A) \subseteq U$ and $f(B) \subseteq V$. Since $f$ is $ij-semi-grs$-continuous, $f^{-1}(U) \in i-j-O''(X)$ and $f^{-1}(V) \in j-i-O''(X)$ such that $A \subseteq f^{-1}(U), B \subseteq f^{-1}(V)$ and $f^{-1}(U) \cap f^{-1}(V) = \emptyset$. By Theorem 2.1 (b), therefore, $X$ is semi-normal.

Theorem 4.5. If $f : (X, \tau_1, \tau_2) \rightarrow (Y, \mu_1, \mu_2)$ is $ij-semi-grs$-continuous $ij-grc$-preserving, injection and $Y$ is mildly semi-normal then $X$ is also mildly semi-normal.

Proof: Let $A \in ij-C(X), B \in ij-C(X)$, and $A \cap B = \emptyset$. Since $f$ is $ij-grc$-preserving injection, then $f(A) \in ij-C(X), f(B) \in j-i-C(X)$ and $f(A) \cap f(B) = \emptyset$. By mildly semi-normality of $Y$, there exist $U \in j_i-O'(Y), V \in i_j-O'(Y)$ and $U \cap V = \emptyset$ such that $f(A) \subseteq U$ and $f(B) \subseteq V$. Since $f$ is $ij-semi-grs$-continuous, $f^{-1}(U) \in ij-GrSO(X)$ and $f^{-1}(V) \in j-i-O''(X)$ such that $A \subseteq f^{-1}(U), B \subseteq f^{-1}(V)$ and $f^{-1}(U) \cap f^{-1}(V) = \emptyset$. By Theorem 2.3 (c), therefore, $X$ is mildly semi-normal.

Theorem 4.6. If $f : (X, \tau_1, \tau_2) \rightarrow (Y, \mu_1, \mu_2)$ is $ij-semi-grs$-continuous $ij-grc$-preserving, $i$-closed injection and $Y$ is almost semi-normal then $X$ is also almost semi-normal.

Proof: Let $A \in i-C(Y), B \in i-C(X)$, and $A \cap B = \emptyset$. Since $f$ is $ij-grc$-preserving and $i$-closed injection, then $f(A) \in i-C(X), f(B) \in ij-C(X)$ and $f(A) \cap f(B) = \emptyset$. By almost semi-normality of $Y$, there exist $U \in j_i-O'(Y), V \in i_j-O'(Y)$ and $U \cap V = \emptyset$ such that $f(A) \subseteq U$ and $f(B) \subseteq V$. Since $f$ is $ij-semi-grs$-continuous, $f^{-1}(U) \in ij-o''(X)$ and $f^{-1}(V) \in j-i-o''(X)$ such that $A \subseteq f^{-1}(U), B \subseteq f^{-1}(V)$ and $f^{-1}(U) \cap f^{-1}(V) = \emptyset$. By Theorem 2.2 (c), therefore, $X$ is almost semi-normal.

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