Abstract: In this paper, we have considered the explicit representation of the Pál type \((0; 0, 1)\) interpolation when function values are prescribed on the zeros of Laguerre Polynomials \(L^{(0)}_n(x)\) and Hermite data is prescribed on the zeros of the derivative of Laguerre Polynomials \((L^{(0)}_n)'(x)\), \(\alpha > -1\) and vice-versa.

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INTRODUCTION

In 1975, L.G. P’al [4] introduced the following interpolation process. Let

\[-\infty < x_{n,n} < \cdots < x_{1,n} < \infty\]

be a system of distinct real points which are zeros of \(W_n(x)\), i.e.,

\[W_n(x) = \prod_{i=1}^{n} (x - x_{i,n}).\]

The roots \(y_{i,n}(i = 1, 2, \ldots, n - 1)\) of \(W'_n(x)\) are interscaled between the roots of \(W_n(x)\), i.e.,

\[-\infty < x_{n,n} < y_{n-1,n} < x_{n-1,n} < \cdots < y_{1,n} < x_{1,n} < +\infty.\]

Pál proved that for given arbitrary numbers \((\alpha_{i,n})_{i=1}^{n}\) and \((\beta_{i,n})_{i=1}^{n-1}\), there exists a unique polynomial of degree \(2n-1\) satisfying the conditions:

\[R_n(x_{i,n}) = \alpha_{i,n}, \quad i = 1, 2, \ldots, n; \quad R'_n(y_{i,n}) = \beta_{i,n-1}, \quad i = 1, 2, \ldots, n - 1,\]

and an initial condition \(R_n(a) = 0\), where \(a\) is a given point, different from the nodal points (1). Szilí [11] was the first to apply this method on infinite interval by taking the mixed of the Hermite polynomial \(H_n(x)\) and its derivative \(H'_n(x)\). Later I. Joo ([5],[6]) sharpened his results by improving the estimates of fundamental polynomials. Srivastava and Mathur [9] studied the problem of \((0;0,1)\) - interpolation on the mixed zeros of \(H_n(x)\) and its derivative. In another paper, Lenard [7] showed that a modified Pál type interpolation on Laguerre abscissas when \((x_{i,n})_{i=1}^{n}\) and \((y_{i,n})_{i=1}^{n}\) as the zeros of the Laguerre polynomials \(L^{(0)}_n(x)\) and \(L^{(-1)}_n(x)\), respectively and \(x_0 = 0\) is regular and there exists a polynomial \(R_m(x)\) of degree \(2n + k\) satisfying the following conditions:

\[R_m(x_i) = y_i, \quad R'_m(x_i^n) = y'_i (i = 1, 2, \ldots, n),\]

with Hermite type boundary conditions:

\[R'_m(x_0) = y'_0 (j = 0, 1, \ldots, k),\]

where \(y_i\), \(y'_i\) and \(y'_0(j)\) are arbitrary real numbers.

In this paper, we have considered \(\{x_{k,n}\}_{k=1}^{n}\) and \(\{y_{k,n}\}_{k=1}^{n-1}\) as the zeros of Laguerre Polynomial \(L^{(0)}_n(x)\) and its derivative \((L^{(0)}_n)'(x)\) respectively which are interscaled as:

\[0 < x_1 < y_1 < x_2 < \cdots < y_{n-1} < x_n < \infty.\]
Then for an arbitrarily given set of real numbers:
\[ \alpha_k, k = 1(1)n; \beta_k, k = 1(1)n - 1; \gamma_k, k = 1(1)n - 1, \]
we seek to determine a polynomial \( R_n(x) \) of minimal possible degree \( \leq 3n - 3 \) such that:
\[
\begin{align*}
R_n(x_k) &= \alpha_k, & k &= 1(1)n, \\
R_n(y_k) &= \beta_k, & k &= 1(1)n - 1, \\
R'_n(y_k) &= \gamma_k, & k &= 1(1)n - 1.
\end{align*}
\]

The theorem when \( \{x_k\}_{k=1}^{n} \) and \( \{y_k\}_{k=1}^{n-1} \) are interchanged i.e., when function values are prescribed on the zeros of \( (L_n^{(\alpha)})'(x), \alpha > -1 \) and Hermite data is prescribed on the zeros of \( L_n^{(\alpha)}(x) \) has also been dealt with.

**Preliminaries**

The differential equation of Laguerre Polynomial \( L_n^{(\alpha)}(x) \) is given by
\[
x(y) + (\alpha + 1 - x)(\frac{dy}{dx}) + n y = 0.
\]
where \( n \) is a positive integer and \( \alpha > -1 \). The recurrence relations between Laguerre polynomial and its derivative are as follows:
\[
(n + 1)L_n^{(\alpha)}(x) = (2n + 1 - x)L_n^{(\alpha)}(x) - nL_{n-1}^{(\alpha)}(x).
\]

Let the fundamental polynomials of Lagrange interpolation on the nodes \( x_k \) and \( y_k \) be denoted by
\[
l_k(x) = \frac{L_n^{(\alpha)}(x)}{(x - x_k)(L_n^{(\alpha)})'(x_k)}, \quad k = 1(1)n
\]
and
\[
l_k'(x) = \frac{(L_n^{(\alpha)})'(x)}{(x - y_k)(L_n^{(\alpha)})'(y_k)}, \quad k = 1(1)n - 1,
\]
respectively.

**Explicit Representation of the Interpolatory Polynomial**

Let \( (2n - 1) \) points in \( (0, \infty) \) be given by (2). Then to the prescribed numbers \( \{\alpha_k\}_{k=1}^{n}, \{\beta_k\}_{k=1}^{n-1} \) and \( \{\gamma_k\}_{k=1}^{n-1} \), there exists a unique polynomial \( R_n(x) \) of degree \( \leq 3n - 3 \) satisfying the conditions (4).

The polynomial \( R_n(x) \) is explicitly given by:
\[
R_n(x) = \sum_{k=1}^{n} \alpha_k A_k(x) + \sum_{k=1}^{n-1} \beta_k B_k(x) + \sum_{k=1}^{n-1} \gamma_k C_k(x),
\]
where \( \{A_k(x)\}_{k=1}^{n}, \{B_k(x)\}_{k=1}^{n-1} \) and \( \{C_k(x)\}_{k=1}^{n-1} \) are uniquely determined polynomials each of degree \( \leq 3n - 3 \), satisfying the following conditions:

For \( k = 1, \ldots, n \),
\[
A_k(x_j) = \delta_{jk}, \quad j = 1, 2, \ldots, n
\]
\[
A_k(y_j) = 0, \quad j = 1, 2, \ldots, n - 1
\]

For \( k = 1, \ldots, n - 1, \)
\[
B_k(x_j) = 0, \quad j = 1, 2, \ldots, n
\]
\[
B_k(y_j) = \delta_{jk}, \quad j = 1, 2, \ldots, n - 1
\]

For \( k = 1, \ldots, n - 1, \)
\[
C_k(x_j) = 0, \quad j = 1, 2, \ldots, n
\]
\[
C_k(y_j) = 0, \quad j = 1, 2, \ldots, n - 1
\]

The explicit representation of the \( A_k(x), k = 1, 2, \ldots, n \); \( B_k(x), k = 1, 2, \ldots, n - 1 \) and \( C_k(x), k = 1, 2, \ldots, n - 1 \) are given in the following theorems.

**Theorem 1.** The fundamental polynomials \( \{C_k(x)\}_{k=1}^{n-1} \) satisfying the conditions (13) can be explicitly represented as:
\[
C_k(x) = -\frac{y_k L_n^{(\alpha)}(x)(L_n^{(\alpha)})'(x)}{n [L_n^{(\alpha)}(y_k)]^2} l_k^*(x), \quad k = 1, 2, \ldots, n - 1
\]
where \( l_k^*(x) \) are given by (9).

**Proof.** For \( k = 1, 2, \ldots, n - 1 \), consider
\[
C_k(x) = c_k L_n^{(\alpha)}(x)(L_n^{(\alpha)})'(x) l_k^*(x),
\]
where \( c_k \) are constants. Then obviously \( C_k(x) \) are polynomials of degree \( \leq 3n - 3 \) with \( C_k(x) = 0, j = 1, 2, \ldots, n \) and \( C_k(y_j) = 0, j = 1, 2, \ldots, n - 1 \). On differentiating (23) with respect to \( x \) and substituting \( x = y_j \), we get
\[
C_k'(y_j) = c_k [L_n^{(\alpha)}(y_j)(L_n^{(\alpha)})'(y_j) l_k^*(y_j)]
\]
Theorem 2. The fundamental polynomials \( \{ B_k(x) \}_{k=1}^n \) satisfying the conditions (12) can be explicitly represented as:

\[
B_k(x) = \frac{L_n^{(a)}(x)(l_k^a(x))^2}{L_n^{(a)}(y_k)} + \frac{(y_k - 2)L_n^{(a)}(x)(L_n^{(a)}(y)l_k^a(x))}{n[L_n^{(a)}(y_k)]^2},
\]

where \( l_k^a(x) \) are given by (9).

Proof. For \( k = 1, 2, \ldots, n-1 \), consider

\[
B_k(x) = b_{k1} L_n^{(a)}(x)(l_k^a(x))^2 + b_{k2} C_k(x),
\]

where \( b_{k1} \) and \( b_{k2} \) are constants. Then obviously \( B_k(x) \) are polynomials of degree \( \leq 3n - 3 \) with \( B_k(y_j) = 0, j = 1, 2, \ldots, n \). Also, due to (12), equation (19) implies that for \( j \neq k \), \( B_k(y_j) = 0 \) and for \( j = k \), \( B_k(y_k) = 1 \), which leads to

\[
b_{k1} = \frac{1}{L_n^{(a)}(y_k)}. \tag{20}
\]

On differentiating (19) with respect to \( x \) and substituting \( x = y_j \), we get

\[
(B_k)'(y_j) = 2b_{k1} L_n^{(a)}(y_j)l_k^a(y_j)(l_k^a)'(y_j) + b_{k2} C_k'(y_j),
\]

which owing to third condition of (12) and (13) for \( j \neq k \), \( (B_k)'(y_j) = 0 \). For \( j = k \), \( (B_k)'(y_k) = 1 \), \( (C_k)'(y_k) = 1 \) and using (20), we have

\[
b_{k2} = -2l_k^a(y_k). \tag{21}
\]

Now, on differentiating (9) with respect to \( x \) and substituting \( x = y_k \), we get

\[
(l_k^a)'(y_k) = \frac{(L_n^{(a)})''(y_k)}{2(L_n^{(a)})'(y_k)}. \tag{22}
\]

and on differentiating (5) with respect to \( x \) and substituting \( x = y_k \), we get

\[
\frac{(L_n^{(a)})''(y_k)}{(L_n^{(a)})'(y_k)} = y_k - 2. \tag{23}
\]

which completes the proof. \( \square \)

Theorem 3. The fundamental polynomials \( \{ A_k(x) \}_{k=1}^n \) satisfying the conditions (11) can be explicitly represented as:

\[
A_k(x) = \frac{[(L_n^{(a)})'(x)]^2 l_k(x)}{[(L_n^{(a)})'(y_k)]^2}, \quad k = 1, 2, \ldots, n, \tag{24}
\]

where \( l_k(x) \) are given by (8).

Proof. For \( k = 1, 2, \ldots, n \) let \( A_k(x) \) be defined as

\[
A_k(x) = a_k [(L_n^{(a)})'(x)]^2 l_k(x), \tag{25}
\]

where \( a_k \) are constants. Obviously, \( A_k(x) \) are polynomials of degree \( \leq 3n - 3 \) with \( A_k(y_j) = 0, j = 1, 2, \ldots, n \) and on differentiating (25) with respect to \( x \) and substituting \( x = y_j \), we get \( A_k(y_j) = 0 \). Also, due to (11), equation (25) implies that for \( j \neq k \), \( A_k(x_j) = 0 \) and for \( j = k \), \( A_k(x_k) = 1 \), which leads to

\[
a_k = \frac{1}{[(L_n^{(a)})'(x_k)]^2}. \tag{26}
\]

Substituting \( a_k \) in (25), we get the required result. \( \square \)

EXPLICIT REPRESENTATION OF THE INTERPOLATORY POLYNOMIAL WHEN NODES ARE INTECHANGEDS

Let \( (2n - 1) \) points in \((0, \infty)\) be given by (2) then to the prescribed numbers \( \{ \alpha_k^* \}_{k=1}^n, \{ \beta_k^* \}_{k=1}^n \) and \( \{ \gamma_k^* \}_{k=1}^{n-1} \), there exists a unique polynomial \( R_n^*(x) \) of degree \( \leq 3n - 2 \) satisfying the conditions

\[
\begin{cases}
R_n^*(x_k) = \alpha_k^*, & k = 1(1)n. \\
R_n^*(y_k) = \beta_k^*, & k = 1(1)n - 1. \\
(R_n^*)'(y_k) = \gamma_k^*, & k = 1(1)n - 1.
\end{cases} \tag{27}
\]
The polynomial $R_n^*(x)$ can be explicitly represented as:

$$R_n^*(x) = \sum_{k=1}^{n} \alpha_k^* A_k(x) + \sum_{k=1}^{n} \beta_k^* B_k(x) + \sum_{k=1}^{n-1} \gamma_k^* C_k^*(x),$$

(28)

where $\{A_k^*(x)\}_{k=1}^{n}$, $\{B_k^*(x)\}_{k=1}^{n}$, and $\{C_k^*(x)\}_{k=1}^{n-1}$ are uniquely determined polynomials each of degree $\leq 3n - 2$ and can be explicitly represented as

$$C_k^*(x) = \frac{[L_n^{(\alpha)}(x)]^2}{[L_n^{(\alpha)}(y_k)]^2} l_k^*(x), \quad k = 1, 2, \ldots, n - 1,$$

(29)

$$B_k^*(x) = \frac{L_n^{(\alpha)}(x) L_n^{(\alpha)'}(x) l_k(x)}{[L_n^{(\alpha)'}(x_k)]^2}, \quad k = 1, 2, \ldots, n,$$

(30)

$$A_k^*(x) = \frac{[L_n^{(\alpha)'}(x)]^2}{[L_n^{(\alpha)'}(x_k)]^2} + \frac{2(1 + \alpha - x_k) L_n^{(\alpha)}(x) L_n^{(\alpha)'}(x) l_k(x)}{x_k [L_n^{(\alpha)'}(x_k)]^2}, \quad k = 1, 2, \ldots, n,$$

(31)

where $l_k(x)$ and $l_k^*(x)$ are given by (8) and (9) respectively.

REFERENCES


