Optimal Recovery of the Second Derivatives of Analytic Functions from their Values at a Regular Polygon Vertices

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Abstract
This paper tells about the problem of the linear best method of approximation of the second derivative of bounded analytic functions from information about their values at a finite number of points. That problem is solved for the case when analytic functions are defined in the circle and nodes form a regular polygon. At the beginning of the article the author finds the error of the best approximation method. Then he calculates the coefficients of the linear best recovery method. At the end of the article the author greatly simplifies the expressions for calculating the coefficients of the linear best method.

Keywords: optimal recovery, error of the best method, linear best method, extremal function.

INTRODUCTION
In this paper we consider the problem of optimal recovery of a linear functional from information about the values of a finite number of linear functionals on a certain set lying in a linear space. This class of theoretical problems is used to solve applied problems of calibration of sensing elements of inertial navigation systems. Similar inertial systems are realized in sea, air and space navigation tools. Below we propose the solution of the following problem of optimal recovery.

Let’s start with the fact that we denote the unit circle by $K = \{z: |z| < 1\}$ and denote the set of analytic functions in $K$ by $B^1(K) = \{f(z): |f(z)| \leq 1, \; z \in K\}$. We introduce the following set of nodes:

$$z_0 = 0, \; z_1 = R, \; z_2 = R e^{\frac{2\pi i}{n}}, \ldots, \; z_n = R e^{\frac{2\pi i (n-1)}{n}} (n \geq 3; \; 0 < R < 1)$$

Recall some of the results of the work [1].

The error of approximation by S-method of the complex function $S(t_1, \ldots, t_n)$ with $n$ complex variables called the following expression:

$$r_n(S) = \inf_{S} \sup_{f(z) \in B^1(K)} \left|f''(0) - S(f(z_1), \ldots, f(z_n))\right|$$

The method $S_0$ is called the best method of approximation, if $r_n(S_0) = \inf_{S} r_n(S)$. According to [1], there exists a linear best method $S_0 = \sum_{k=1}^{n} c_k f(z_k)$, and error of the best method is determined by the next formula:

$$r_n(S_0) = r_n(z_0, z_1, \ldots, z_n) = \sup_{f(z) \in B^1(K)} |f''(0)| (1)$$

We note that this type of problem have been studied in many articles (see [1],[2],[3]).

We use some results from the article [4]. If $\omega(\zeta) − \varphi(\zeta)$ boundary value (I′ – boundary) of function $\omega(z)$ which is meromorphic in $K$, that performed the next duality relation:

$$\sup_{f \in B^1(K)} \int \frac{f(\zeta)\omega(\zeta)\, d\zeta}{\Gamma(\zeta)} = \min_{\varphi \in H_1} \int |\omega(\zeta) − \varphi(\zeta)|\, d\zeta, \quad (2)$$

In this duality relation $H_1$ – Hardy class. In addition, if the functions $f^*(z) \in B^1(K)$ and $\varphi^*(z) \in H_1$ are extreme functions, then on the boundary $\Gamma$ the following relationship is determined:

$$f^*(\zeta)[\omega(\zeta) − \varphi^*(\zeta)]\, d\zeta = e^{i\delta}|\omega(\zeta) − \varphi^*(\zeta)|\, d\zeta \quad (3)$$

In this relationship $\delta$ is a real number.

FINDING OF ERROR OF THE BEST METHOD
We denote as $B(z) = \prod_{k=1}^{n} \frac{z - z_k}{1 - z_k z}$ a finite Blaschke product. At first we verify that $B'(0) = B''(0) = 0$. Indeed because

$$LnB(z) = Ln(z - z_1) + \cdots + Ln(z - z_n) - Ln(1 - z_1 z) - \cdots - Ln(1 - z_n z),$$

then

$$\frac{B'(0)}{B(0)} = -\frac{1}{R^2} (\overline{z}_1 + \cdots + \overline{z}_n) + (\overline{z}_1 + \cdots + \overline{z}_n) = 0$$

From this equation we see that $B'(0) = 0$. Further, since

$$z_1^2 + \cdots + z_n^2 = R^2 \left(1 + e^{\frac{4\pi i}{n}} + \cdots + \left(e^{\frac{4\pi i}{n}}\right)^{n-1}\right) = R^2 \frac{1 - e^{4\pi i}}{1 - e^{\frac{4\pi i}{n}}} = 0$$

and
\[
(LnB(z))'' = -\frac{1}{(z - z_1)^2} - \frac{1}{(z - z_n)^2} + \frac{z_1^2}{(1 - z_1)^2} + \frac{z_n^2}{(1 - z_n)^2} + \ldots
\]
then
\[
\frac{B''(0)B(0) - (B'(0))^2}{B^2(0)} = -\frac{1}{z_1^2} - \frac{1}{z_n^2} + z_1^2 + \ldots + z_n^2 = 0
\]
which implies that \(B''(0) = 0\). We introduce the following notation:
\[
A = \{f(z): f(z_1) = \ldots = f(z_n) = 0, f(z) \in B^1(K)\}
\]
If \(f(z) \in A\), then \(f(z) = B(z)g(z) (g(z) \in B^1(K))\). Consequently
\[
r_2(z_1, z_2, \ldots, z_n) = |B(0)| \sup_{g(z) \in B^1(K)} |g''(0)| = 2|B(0)|
\]
where
\[
r_2 = 2|B(0)|
\]
From this equation also follows that the extremal function \(f^*(z)\) of problem (1) has the form \(f^*(z) = e^{i\delta}z^2B(z)\), where \(\delta \in R\).

**CALCULATION OF THE COEFFICIENTS OF THE LINEAR BEST METHOD**

In the studied case, the linear best method is unique. We prove this assertion. If \(\sum_{k=1}^n c_kf(z_k)\) - linear best method, then

\[
\sup_{f \in B^1(K)} \int_{\Gamma} \omega(\xi)f(\xi)d\xi = 2|B(0)|
\]
where \(\omega(\xi) = \frac{1}{2\pi}\left(\frac{2}{\xi^2} - \sum_{k=1}^n \frac{c_k}{\xi - z_k}\right)\).

Assume that the method \(\sum_{k=1}^n c_kf(z_k)\) also is the best linear method. Then will be performed the duality relation (see (2)), in which
\[
\omega_1(\xi) = \frac{1}{2\pi}\left(\frac{2}{\xi^2} - \sum_{k=1}^n \frac{c_k}{\xi - z_k}\right).
\]

In this case, for extreme functions \(f^*(\xi)\) and \(\varphi_1^*(\xi)\) (here \(\varphi_1^*(\xi)\) is the extreme function in the right side of equation (2) with the corresponding function \(\omega_1(\xi)\)) the following relation is satisfied:

\[
f^*(\xi)(\omega_1(\xi) - \varphi_1^*(\xi))d\xi = e^{i\delta_1}\omega_1(\xi) - \varphi_1^*(\xi)ds
\]
In this equation \(\delta_1 \in R\). In this case the functions \(R(z) = f^*(\xi)(\omega(z) - \varphi(z))\) and \(R_1(z)\) have no zeros in \(K\) (here \(R_1(z) = f^*(\xi)(\omega(z) - \varphi(z))\), see [2], p.45).

The function \(P(z) = e^{i\xi_0(\xi_1 - \varphi_1(\xi_0))}(\alpha = \delta - \delta_1)\) is analytic in \(K\) and takes positive values at the boundary \(\Gamma\). Consequently, \(P(z) = c, c > 0\). Because \(P(0) = e^{i\alpha}\), then \(P(z) = 1\). This implies that \(c_1 = \overline{c_1}, \ldots, c_n = \overline{c_n}\). It follows that the linear best method is unique.

Substitute the function \(g(z) = f\left(e^{i\pi/2}z\right) \in B^1(K)\) into the inequality \(|f''(0) - \sum_{k=1}^n c_kf(z_k)| \leq 2|B(0)|\):

\[
\left|e^{i\pi/2}f''(0) - \sum_{k=1}^n c_kf\left(e^{i\pi/2}z_k\right)\right| \leq 2|B(0)|
\]
which corresponds to the following inequality

\[
\left|f''(0) - \sum_{k=1}^n c_ke^{-i\pi/2}f\left(e^{i\pi/2}z_k\right)\right| \leq 2|B(0)|
\]
It follows the formula for the coefficient \(c_k\): \(c_k = c_ke^{-i\pi/2}/(k-1), k = 1, \ldots, n\).

Next, calculate the coefficient \(c_1\). To do this, we use the following integral

\[
l = \frac{1}{\pi} \int_{\Gamma} \frac{B(0)}{z^2B(z)} f(z)dz, \text{ where } f(z) \in B^1(K)\]

We estimate this integral modulo:

\[
|l| \leq \frac{1}{\pi} \int_{\Gamma} |B(0)||f(z)||dz| \leq 2|B(0)|
\]
Now we calculate this integral. We introduce the following notation \(Q(z) = \frac{1}{z^2B(z)}\). We expand this function into Laurent series:

\[
Q(z) = \frac{c_3}{z^3} + \frac{c_2}{z^2} + \frac{c_1}{z} + \psi(z)
\]
In the expansion function \(\psi(z)\) is analytic in the circle. Hence we find the coefficient \(c_{-3} = \lim_{z \to \infty} z^3Q(z) = \frac{1}{B(0)}\). Now we find the coefficient \(c_{-2}\):

\[
c_{-2} = \lim_{z \to \infty} \frac{B'(z)}{B^2(z)} = 0
\]
Finally, we calculate the coefficient \(c_{-1}\):

\[
c_{-1} = \left|\lim_{z \to 0} \frac{B'(z)B^2(z) - 2B(z)(B'(z))^2}{B^4(z)}\right|
\]
We apply the obtained values of the coefficients for finding the integral \(l\):

\[
l = \frac{1}{2\pi i} \int_{\Gamma} \left(\frac{2}{z^3} - \sum_{k=1}^n \frac{c_k}{(z - z_k)}\right) f(z)dz = f''(0) - \sum_{k=1}^n c_kf(z_k)
\]
where \(c_k = -2B(0)\) res \(Q(z)\). Hence we obtain an estimate \(|f''(0) - \sum_{k=1}^n c_kf(z_k)| \leq 2|B(0)|\) for any function \(f(z) \in B^1(K)\). From this estimate can be concluded that the method \(\sum_{k=1}^n c_kf(z_k)\) is the best linear method. We calculate the coefficient \(c_1\):

\[
c_1 = \frac{2B(0)(R^2 - 1)}{R^3} \prod_{j=1}^n \frac{R^2 - z_j}{R^3 - z_j}
\]
In the paper [4] is proved that \(\prod_{j=1}^n (R - z_j) = R^{n-1}n\)
and \( \prod_{j=2}^{n}(1 - z_j R) = \frac{1 - R^{2n}}{1 - R^2} \). Substituting these expressions into the obtained formulas, we find the final formulas for the coefficients:

\[
c_1 = \frac{2(1 - R^{2n})}{R^{2n}}, \quad c_k = c_1 e^{-\frac{i\pi (k-1)}{n}}, \quad (k = 1, \ldots, n).
\]

REFERENCES


