Uncertainty analysis in the unavailability assessment using Generalized Stochastic Petri Net with fuzzy parameters

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Abstract
This paper handles the problem of uncertainty in assessment the safety systems using Generalized Stochastic Petri Nets (GSPN). The basic probabilities usually treated to define transition in petri nets (PNs) are substituted by fuzzy number. Experts represent the uncertainties regarding the parameters of the components of systems to assess the influence of this uncertainty on safety system unavailability. The used approach combining Generalized Stochastic Petri Nets (GSPN) and Fuzzy probability revealed how the uncertainty on parameters values of the system affects significantly the performance of system.

Keywords: Fuzzy probability, Imprecision, Uncertainty, Generalized Stochastic Petri Net

INTRODUCTION
The analysis of a safety system allows the identification of its performance and highlights potential problems that may arise. Depending on the obtained results, improvements can be made to the system in order to obtain the best performance possible. Studying the performance of safety systems allows to better understand their evolution and represents an important interest for reliability engineers. The assessment of performance requires the determination of the different constituent parts. With the increasing complexity of safety systems and the importance related to their reliability and the ability to function properly, it is necessary to modelize correctly their functional and dysfunctional behavior and then to assess their overall performance.

Various methods related to reliability study of dynamic systems enable us to identify and assess the combinations of events leading to the occurrence of another event [1]. The aim of the analyst is adopting the most appropriate model to get through quantitative and qualitative analysis of the most reliable results for the studied probabilistic system. Various performance indicators are already commonly used in the field of system reliability. We are interested precisely in the study of the unavailability of system performance. In our paper, we adopt the GSPN to overcome these limitations. In this respect, Many authors [1] [2] [3], have published work-focused approach of Generalized Stochastic Petri Net in reliability engineering. Generalized Stochastic Petri Nets are an important and powerful tool used by reliability analysts for representing the different parameters of the states. In this purpose a special issu of IEC 62551 “Analysis techniques for dependability—Petri net techniques” is published [4]. The Generalized Stochastic Petri Net is used to represent the different modes of components, operating or stopped.

In the present approach, it is assumed that failure and repair are characterized by exponential distributions to model transition firing rate. Furthermore, the system is assumed to be dynamic (its properties change with time).

In the study of the unavailability of the safety system, the probabilities are often considered precise and known. The problem arises when processing imprecise knowledge of probabilities is involved [5]. In our case the fuzzy α-cut analysis based on fuzzy logic and fuzzy set theory are a kind of representation of the imprecision for complex system. [6].

Recent research has tended to focus on Generalized Stochastic Petri Nets (GSPN). Petri Nets enable us to model and analyze the evolution of complex systems. The uncertainty models can be treated more realistically through a combination of Fuzzy probabilities and Petri Nets. Many works are related to the study of epistemic uncertainty in GSPN with several approaches [7] [8] [9].

The present research aims to modelize sophisticated systems using GSPN characterized by ambiguous information. Expending fuzzy reasoning allows us to manipulate the systems’ parameters. Fuzzy logic, based α-cut is applied to model, control and treat the uncertainty parameters’. According to this method, uncertain parameters treated as fuzzy numbers are represented by their membership functions. Section 2 concerns the assessment of the system performance based on the knowledge of the characteristic parameters of the
components using a model based on Generalized Stochastic Petri Net. Section 3 deals with the modeling of the knowledge imprecision of the system’s parameters as fuzzy number. Section 4 represents the integration of imprecise parameters described by fuzzy set in Generalized Stochastic Petri Nets for the assessment of unavailability. The last section concerns the results and a discussion on the application of this approach on the components of safety-instrumented systems (SIS).

**Approaches for the assessment performance**

**Modelling approach**

Various methods (fault tree, event trees …) inherent to reliability study of dynamic systems enable us to identify and assess the combinations of events leading to the occurrence of another event [1].

Among the drawbacks of these methods is that they do not take into consideration the order of the events and time dependency or delays which are essential for the assessment of dynamic systems. Several methods have been proposed to solve these inconveniences like Markov chain, Bayesian Networks and Generalized Stochastic Petri Nets. In our paper, we adopt the GSPN to overcome these limitations.

Petri nets, as means for the simulation of discrete events are widely used in assessing dependability. The purpose of modelling is to evaluate dynamic systems evolving in relation to time. GSPN give the possibility to model a system in which discrete-time variables and discrete events cooperate. This can occur on a stochastic basis. In the present approach, we assume that the failure distributions of individual components of a system are given. Furthermore, the system is assumed to be dynamic.

In this paper, we focus mainly on the reliability of this method to evaluate the performance of safety system. GSPN make it possible to consolidate the qualitative and quantitative aspects of complex systems and to offer an ample knowledge of the interaction of sub-systems with the global system. [10]

**Generalized Stochastic Petri Nets**

Many types of stochastic Petri nets are proposed for analyzing the performance of safety systems. Generalized Stochastic Petri nets were introduced by Ajmone Marsan [11] [12]. The Petri Nets consists of two types of transitions, those with timed transitions characterized by an exponential distribution called stochastic transitions, and those transitions with a zero delay time called immediate transitions. Petri network transitions have involved crossing random times, distributed by an exponential law. This exponential distribution permits to exploit the mathematical properties of Markov process. This concept has largely been developed since the early 1980s [13] to meet the requirements of increasingly complex modeling such as the modeling of production systems. The basic concepts and the main properties are found in work of many books [11].

**GSPN for modeling assessment performance**

The evaluation is linked to the computation of the systems unavailability [14]. In this regard, Generalized Stochastic Petri Nets are important models of all system states, taking into account all the encountered events (failure, maintenance, etc.) and all the studied parameters (failure rate, CCF factor, Rate of repair, etc.).

The Generalized Stochastic Petri Nets have two distinct types of transitions. The first type is the timed transitions whose random exponentially distributed firing delays are linked to transition. The second type is immediate transitions. These have a priority over timed transitions because they fire in no time [15] [16] [17].

A Generalized Stochastic Petri Nets is an eight-tuple \( (P; T; Pr; Post; m_0; V_1; V_2; W) \), where:

- \( P = \{ p_1, p_2, ..., p_k \} \), is a finite set of places, (represented by a circle).
- \( T = \{ t_1, t_2, ..., t_l \} \), is a finite set of transition, (represented by bars).
- \( Pre \), defines weighted arcs between places and transitions.
- \( Post \), defines weighted arcs from transitions to places.
- \( m_0 = (m_{p0}, m_{p1}, ..., m_{pk}) \), \( m_0 \in \mathbb{N}^+ \), \( m_0 \) is the initial marking of place \( p \in P \) and denotes the number of tokens in the place \( p \).
- \( V_1 \subseteq T \) set of timed transition, \( V_1 \neq \emptyset \)
- \( V_2 \subseteq T \) set of immediate transitions \( V_1 \cap V_2 \neq \emptyset \)
- \( V_1 \cup V_2 = T \)
- \( W = (w_1, w_2, ..., w_l) \) is an array whose entry \( w_j \in R^+ \) where:
  - If \( t_i \) is a timed transition, \( w_j \) defines the parameter of the negative exponential probability distribution function of the firing delay transition.
  - If \( t_i \) is an immediate transition, \( w_j \) defines a weight used for the computation of firing probabilities of immediate transitions. Many authors described the importance of using GSPN [13] [12] [18] [19]. The exponential distribution of firing delays is memoryless, which shows that the GSPN systems are isomorphic to continuous time Markov chains (CTMC) [20]. A k-bounded GSPN system, in particular, can be shown to be isomorphic to a finite Markov Chains. The Markov chains...
associated with a given GSPN can be obtained through the following rules:

1) The space state of Markov chains \( S = \{ s_i \} = \{ s_1, s_2, \ldots, s_r \} \) corresponds to the reachability set \( RS(m_0) \) of the PN associated with the GSPN ( \( m_i \leftrightarrow s_j \) )

2) The transition rate from state \( s_i \) (corresponding to marking \( m_i \) ) to state \( s_j \) is obtained as the sum of the firing rates of the transitions that are enabled in \( m_i \) and whose firings generate marking \( m_j \).

In this paper, we assume that the transitions of a GSPN are associated with a marking-independent speed and single-server semantics. Based on these rules, it is possible to devise algorithms for the automatic construction of the infinitesimal generator, also called transition rate matrix status of isomorphic Markov chains, from the GSPN description.

\( Q \) denotes an infinitesimal generator. The matrix \( Q \) is determined from \( w_a \), the firing rate of the transition \( t_a \), and with \( E(m_i) = \{ h : t_h \in E(m_i) \wedge m_i(t_h)m_j \} \) the set of transitions, whose firing brings the GSPN from marking \( m_i \) to a marking \( m_j \). The coefficients \( q_{ij} \) of the matrix \( Q \) are determined by using the following equation:

\[
Q = (q_{ij})_{i,j=1}^{r} \quad (0 \leq i, j \leq k - 1)
\]  

(1)

Where \( Q \) is the transition rate matrix defined by equation 2

\[
q_{ij} = \begin{cases} 
\sum_{t_h \in E(m_i) \wedge m_i(t_h)m_j} w_h & i \neq j \\
-q_i & i = j 
\end{cases}
\]  

(2)

Where

\[
q_i = \sum_{t_h \in E(m_i)} w_h
\]  

(3)

Let \( \pi \) be the steady-state probability distribution of a Generalized Stochastic Petri Nets, \( \pi(m_i, \tau) \) represent the probability that the GSPN is in marking \( m_i \) at time \( \tau \). The Chapman-Kolmogorov equations for the CTMC associated with a GSPN are specified by:

\[
\frac{d\pi(s_j, \tau)}{d\tau} = \sum_{s_j} \pi(s_j, \tau)q_{ij}
\]  

(4)

Using matrix notation, equation 4 becomes

\[
\frac{d\pi(\tau)}{d\tau} = \pi(\tau)Q
\]  

(5)

Whose solution can be formally written as

\[
\pi(\tau) = \pi(0)e^{Q\tau}
\]  

(6)

Where \( \pi(0) \) is the probability of the initial distribution and \( e^{Q\tau} \) is the matrix exponentiation formally defined by equation 7.

\[
e^{Q\tau} = \sum_{k=0}^{\infty} \left( \frac{(Q\tau)^k}{k!} \right)
\]  

(7)

In this paper, we consider only Generalized Stochastic Petri Nets originating from homogeneous and ergodic Continuous Time Markov Chains (CTMC). According to the Markov chain stationary distribution theorem and the Chapman-Kolmogorov equation, the generation of the reachability set \( RS(m_0) \) is obtained by associating each arc with the rate of the corresponding transition. Solving the linear system (equation 8), we obtain the steady-state probability distribution \( \pi \) of a GSPN.

\[
\begin{cases}
\pi(t)Q = 0 \\
\sum_{i} \pi_i(t) = 1
\end{cases}
\]  

(8)

Where \( \pi_i \) refers to the steady-state probability of marking \( m_i \). If the GSPN is ergodic, the steady-state probability distribution on its markings exists and \( \pi = \lim_{\tau \to \infty} \pi(\tau) \) is defined as the limit. Its value can be computed by solving the usual system of linear equations:

\[
\begin{cases}
\pi Q = 0 \\
\sum_{i} \pi_i = 1
\end{cases}
\]  

(9)

Where the \( 0 \) is a row vector of the same size as \( \pi \) and with all its components equal to zero.

The set of transitions with concession at the highest priority level is first found. The probability distribution function of the sojourn time in marking \( m_i \) corresponds to the probability distribution function of the minimum among the firing times of the transitions enabled in the same marking \( m_i \). The probability that given transition \( t_a \in E(m_i) \) fires (first) in marking \( m_i \) is given by the following expression:

\[
P\{t_a \mid m_i \} = \frac{w_a(m_i)}{q_i}
\]  

(10)

With \( w_a \) denoting the firing rate of \( t_a \) and \( E_i(m_i) \) the transitions whose firings transform the net from marking \( m_i \) to marking \( m_j \).

In order to compute the unavailability of the safety system we need to use the \( Post \) function to determine the connections of passages to the failure state (Figure 1). Where \( Post \) denotes the post-incidence function defining weights of arcs from transitions to places.
Let \( F = \{ p_i \} \) be the set of places \( p_i \) with \( i = 1,...,n \) representing places in failure states. We will determine the set of transitions that returns the system to the failure states.

\[
H = \{ t_j \} / \forall t_j \in T \text{ and } p_i \in F, \ \text{Post}(p_i,t_j) > 0
\] (11)

The system unavailability \( U \) is computed by equation 12

\[
U = \sum_{i,j} P\{ t_j / m_i \} = \sum_{i,j} w_i(m_i) / q_i
\] (12)

In dependability studies, the knowledge we have about the component reliability data is generally imperfect [18]. The validity of study results depends on the decision in full or partial account of the imperfection of knowledge used. This requires methods to model and manipulate these imperfections.

MODELING UNCERTAIN DATA

Imprecise Modeling

In the study of unavailability system, probabilities are generally considered as precise and perfectly known. However, in real complex system, defining precise unavailability probability is not easy due to the presence of uncertain parameters. Several sources of uncertainties may be indicated: When a safety system is in low demand, information of the system are weakly credible, or When a system used new components parameters not perfectly known, experts must extrapolate the parameters of the system. In this condition, the parameters are uncertain.

Many other sources can also be provided, poor historical data, environmental condition, less experience of operator collector, technical restraints. All these element present source of uncertainty. This problem of uncertain knowledge of the probability values can be represented in various ways. Many authors treated this approach by different methods. The uncertainty modeled using probabilistic method based on the Monte Carlo sampling led to a powerful tool to the model the uncertain parameters by a uniform distribution, by interval valued probability [21][22], imprecise probabilities [23], fuzzy numbers [24], belief functions [25] or possibility distributions [26][27].

In this research, we opt for fuzzy probabilities that is an interesting representation of uncertainty applied to petri nets. By introducing fuzzy numbers, the confidence of the expert’s valuation is also taken into account. The fuzzy probabilities are a suitable means to model the epistemic and aleatory uncertainty. Moreover, fuzzy sets help to encode the knowledge expressed in a linguistic formulation by an expert.

In this paper, the imprecision of failure rates or repair rate are considered. In this case, experts or designers provide inaccurate estimates of the valued rate of the components. We obtain another simple representation of imprecision by fuzzy number using \( \alpha \)-cut. Monte Carlo Sampling is no longer needed, but the arithmetic operations can be conducted through the theory of fuzzy probability [28][29][30][31].

Fuzzy Modelling

Fuzzy modeling is based on the extension principle of Zadeh’s [32]. When a model’s input parameters are known and the imprecision is defined by fuzzy numbers, its output will be fuzzy number, too, characterized by its Membership Functions (MFs).

Membership Function is a curve defining the degree of participation of an observable element in the set and how each point in the input space is mapped to a membership value between 0 and 1. Different types of MFs (Triangular, rectangular,…) are applied in the analysis of reliability [33]. In this paper, we focus on Triangular Membership Functions (TMF) in modeling uncertain parameters. We opt for (TMF) thanks to their simplicity and understandability [34][35].

The representation of imprecise parameters such as low/high failure rate by MF is used to resolve the fuzzy sets concerning the constituent crisp sets. \( \alpha \)-cuts are essential means in performing arithmetic operations with fuzzy sets. For each variation of \( \alpha \)-cuts between 0 and 1, the minimum and the maximum possible values of the output are determined. The obtained results are used to measure the uncertainty of the output.

This valued is then used to determine the corresponding fuzziness of the output. The obtained membership function obtained is used as a measure of uncertainty. If the output is monotonic with respect to the dependent fuzzy variable/s, the process is rather simple as only two simulations are sufficient for each \( \alpha \)-cut (one for each boundary). Otherwise, optimization must be carried out to determine the minimum and maximum values of the output for each \( \alpha \)-cut.

A fuzzy set \( \tilde{A} \) on a universal \( X \) can be defined by its membership function

\[
\mu_{\tilde{A}} : X \rightarrow [0,1]
\] (13)

Where \( \mu_{\tilde{A}}(x) \) define a membership of element \( x \) in fuzzy set \( \tilde{A} \).
A real value \( \mu_{\tilde{A}}(x) \) in \([0, 1]\) is assigned to each element \( x \in X \) . We represent, for example, the fuzzy set for a triangular fuzzy number, denoted by \( \tilde{A} = \langle a, m, b \rangle \), \( a \leq m \leq b \).

\[
\mu_{\tilde{A}}(x) = \begin{cases} 
0 & \text{if } x \leq a \\
\frac{x-a}{m-a} & \text{if } a \leq x \leq m \\
\frac{b-x}{b-m} & \text{if } m \leq x \leq b \\
0 & \text{if } x \geq b 
\end{cases}
\]  

(14)

The \( \alpha \)-cut of a fuzzy set \( \tilde{A} \), denoted by \( A^\alpha \), is the crisp set composed of all elements \( x \) of universe of discourse for which membership of greater than or equal to \( \alpha \), i.e., \( A^\alpha = \{ x \in X / \tilde{A}(x) \geq \alpha \} \), where \( \alpha \) is a parameter in the range \( 0 \leq \alpha \leq 1 \).

With the introduction of \( \alpha \)-cuts, the confidence interval defined by \( \alpha \)-cuts is written as

\[
\tilde{A} \rightarrow A^\alpha = [A^\alpha_l, A^\alpha_r]
\]  

(15)

Figure 2 depicts a parameter \( \tilde{A} \) represented as a triangular fuzzy number with support of \( A^0 \). The fuzzy set containing all elements with a membership of \( \alpha \) included in \([0,1]\) and above is called the \( \alpha \)-cut of the membership function [36].

![Figure 2: \( \alpha \)-cut of fuzzy number](image)

**Fuzzy performance assessment**

When safety systems are weakly solicited, the feedback is weak and manipulated probabilities may seem inaccurate. The imprecision problems on failure rates and repair rates are also available when working with new components.

In our case, the used databases provide descriptive statistics (min, modal, max) but the actual distribution remains inaccessible [27]. The exploitation (environment) of safety systems is often different from the conditions during the development of the database. The determination of the unavailability of the components is often delicate, sometimes even neglected. Other parameters specific to safety systems are also affected by the knowledge of experts in reliability. The imprecision can be represented simply by discarding hypotheses on the distribution but using fuzzy approach is more interesting [37].

In the same way, the GSPN approach provides a good formalization of the states that the system can take according to the events met (failure, repair...) and the studied parameters (failure rate, repair rate...).

**Fuzzy Assessment of unavailability**

In the literature, exponential distribution represents a good formalism to approximate the failure distributions of systems. However these approximations are estimated from data often characterized by errors and not well represented due to rare events of components, less experience of operator collector, technical restraints or the conditions of environment. This is why the exponential probability distributions of failure rate and repair rate are not enough to define uncertainties. Many researchers suggested fuzzy theory to resolve this ambiguity. Experts define uncertainty and imprecision by fuzzy number in order to facilitate the assessment and interpretation of reliability engineering [38].

In this paper, triangular fuzzy numbers define the input parameters. Applying \( \alpha \)-cuts method leads the problem to an interval calculation. We compose all the bounds of interval and obtain the membership function for each output, which reflects the effect of the uncertainties of the input parameters.

A fuzzy probability assessment is proposed using GSPN to evaluate the unavailability of the safety system. In The proposed approach, the following strategies are applied and shown by figure 3.

1. **Step 1:** GSPN model construction: the system is modeled using GSPN. The set of system states are determined and connected with adequate transition (timed /immediate). All configuration states of the systems must be modeled (worked/failed).

2. **Step 2:** Fuzzy set: According to database, Historical record and reliability expert opinion. Input information are extracted from several sources (failure rates \( \lambda \) and repair rates \( \mu \)) to account for the uncertainty input parameters’ failure rates and repair rates are improved into fuzzy numbers. Each component is defined by membership function (triangular fuzzy number), \( \tilde{\lambda} = \langle \lambda_l, \lambda_m, \lambda_r \rangle \) and \( \tilde{\mu} = \langle \mu_l, \mu_m, \mu_r \rangle \).

3. **Step 3:** Interval values by \( \alpha \)-Cuts: after obtaining membership function to each component, the imprecisions of
the components of the represented system are achieved using $\alpha$-cuts and arithmetic interval operations applied on triangular fuzzy number. For each $\alpha$-level ($\alpha \in [0,1]$), an interval expression value is obtained $\lambda^e$ and $\mu^e$.

Step 4: Fuzzy assessment unavailability: in this step, in order to analyze the system behavior qualitatively as well as quantitatively, the system is modeled with GSPN and temporized transitions are defined by interval values determined in step 3. The run of simulation concludes the unavailability of the system $\hat{U} = \langle a_u, m_u, b_u \rangle$.

Step 5: Decision support: the obtained result $\hat{U}$, represents a decision support for the decision maker to define which technical interventions are needed for the safety system. The decision maker has the responsibility to assess this result.

Figure 3: Strategy of fuzzy probability assessment using GSPN

Fuzzy Generalized Stochastic Petri Nets

As defined in the definition of GSPN, $w_k$ defines the parameter of the negative exponential probability distribution function of the firing delay transition. If transition $t_i$ is a timed transition, the parameters $w_k$ represent the failure rate $\lambda$ or repair rate $\mu$. If transition $t_i$ is an immediate transition, $w_k$ is a weight used for the computation of firing probabilities.

We mentioned in section 2 that our knowledge of the parameters values is imperfect, we model the imprecision of these parameters (failure rate and repair rate) by triangular fuzzy numbers. Each component is defined by membership function (triangular fuzzy number), $\tilde{\lambda} = \langle a_\lambda, m_\lambda, b_\lambda \rangle$ and $\tilde{\mu} = \langle a_\mu, m_\mu, b_\mu \rangle$.

The uncertain parameters $\tilde{w}_k$ is equal to

$$\tilde{w}_k \rightarrow w_k^\alpha = \left[ w_k^{\alpha_i}, w_k^{\alpha_j} \right] \rightarrow w_k^{\alpha_i} \leq w_k^\alpha \leq w_k^{\alpha_j}$$

(16)

Equation 2 becomes

$$\tilde{q}_i \rightarrow q_i^{\alpha} = \left[ q_i^{\alpha_i}, q_i^{\alpha_j} \right] \rightarrow q_i^{\alpha_i} \leq q_i^{\alpha} \leq q_i^{\alpha_j}$$

(17)

Consider that the coefficient $q_{ij}$ of the matrix $Q$ is imprecise. The imprecise probability $\tilde{P}\{t_i / m_j\}$ that gave transition $t_k \in E(m_j)$ fires (first) in marking $m_j$ is given by the following expression:

$$P^{\alpha_i}\{t_i / m_j\} \rightarrow P^{\alpha_j}\{t_i / m_j\} \leq \tilde{P}\{t_i / m_j\} \leq P^{\alpha_i}\{t_i / m_j\}$$

(18)

With
\[
P^{\alpha}{i}_{k} \{t_{k} / m_{j}\} = \min \left\{ \frac{w^{(x)}_{i} \left( m_{j} \right)}{q^{(x)}_{i}} \right\}
\]
\[
P^{\alpha}{k}_{h} \{t_{h} / m_{i}\} = \max \left\{ \frac{w^{(x)}_{k} \left( m_{i} \right)}{q^{(x)}_{k}} \right\}
\]

where \( E(m_{j}) = \{ h : t_{k} \in E(m_{j}) \land m_{j}[t_{k}]m_{j} \} \) is the set of transitions whose firing brings the GSPN from marking \( m_{i} \) to a marking \( m_{j} \).

Our objective is to compute the performance of the system beginning from its imprecise characteristic parameters like the failure and repair rates using the imprecise Petri nets. The unavailability of the system can be calculated by combining the failure probability of all sub-systems that provide the set of safety function.

Equation 19 aims to remove the subdistributivity problem, and thus the inadequate results when the variable is repeated. The lower and upper bounds of the unavailability in the equation 19 are calculated using Equations 12, 16, 17 and 18

\[
U^{(a)} = \left[ U^{(a)}_{l}, U^{(a)}_{u} \right] = \sum_{i \in Fp} P^{(a)}{i}_{k} \{t_{i} / m_{j}\} + \sum_{k \in H} P^{(a)}{k}_{h} \{t_{h} / m_{i}\}
\]

STUDY CASE

The proposed approach is applied in the system given in Figure 4. This Safety Instrumented System (SIS) is devoted to the protection of a part of an offshore production system against overpressure due to its upstream (oil well W1). It is made up of by three layers: Pressure transmitter (PT), logic solver (LS) and final elements (SD, SDV). The pressure sensors are responsible for detecting the pressure rise beyond certain threshold. These three sensors transmit information to a logic solver (LS) which implements a 2oo3 logic. If at least two of the three signals received from sensors confirm the presence of an overpressure in the pipeline, the logic solver controls the opening of solenoid valve SV, which leads to in shutting off hydraulic supply that kept valve SDV open. Then, SDV is closed and reduces the risk of overpressure in the downstream circuit. The undesired event is the inhibition of the SIS which is characterized by the non-closure of the relief valve SDV.

![Figure 4: Safety Instrumented System (SIS)](image-url)

The studied SIS is made up of:

- 2oo3 architecture for the sensor layer (PT1, PT2, PT3)
- 1oo2 architecture for the logic unit (SV and SDV)
- 1oo1 architecture for the actuator (Logic solver)

The reliability block diagram of the SIS is given in Figure 5.
Our aim is to calculate the SIS unavailability starting from its imprecise characteristic parameters, by using the fuzzy GSPN. The SIS unavailability can be computed by combining the failure probability of all sub-systems providing the set of safety function. This approach allows to simplify the model of the SIS.

Petri Nets is a useful means to model any system’s dynamics and visualize various states of a SIS. Therefore, to confirm the application of Petri Nets in modeling a SIS, the 2oo3 system (sensor layer) considered until now was established through Petri Nets module of GRIF software [39]. A demonstration of how this 2oo3 system was modeled is shown in figure 7. After finalizing the design, simulations were carried out.

Petri nets provide a graphical means for the behavioural modelling of (dynamical) systems, and then unavailability analyses can be done by Monte Carlo simulations (i.e. results are obtained statistically from several simulated histories).

The Petri net applied to the case study with 2oo3 architecture is provided in Figure 7 (with the tokens as defined at time t0 = 0). Places (1, 4, and 6) modelled the operating states of the subsystem; Places (2, 5, and 7) modelled the failure state. Place 29 shows that the system 2oo3 worked, however, place 30 shows that the system failed.

Using the fuzzy GSPN method associated to α-cuts as proposed in this paper, the SIS unavailability is determined according to the characteristic parameters of components modeled by fuzzy numbers. The characteristic parameters of the SIS components are given in Table 1. The parameters values of Table 1 are provided by an expert as triangular fuzzy numbers \(<a_i, m_i, b_i>\) as a linguistic expression translating ‘around \(m_i\)’. Parameter \(m_i\) is the model value of the fuzzy number (the most expected value by the expert). Parameter \(a_i\) is the lower bound and parameter \(b_i\) is the upper bound. They define the narrowest interval inside which 100% of the value of the modeled SIS parameter should lie according to the expert knowledge. Note that it is a disjunctive set and not a probability distribution. A description of the failure rate \(\lambda\) and repair rate \(\mu\) as fuzzy numbers of a triangular type is provided in table 1.

<table>
<thead>
<tr>
<th>System Components</th>
<th>(\lambda) (x10^-4/h)</th>
<th>(\mu) (x10^1/h)</th>
</tr>
</thead>
<tbody>
<tr>
<td>PT1</td>
<td>4.3</td>
<td>0.167</td>
</tr>
<tr>
<td>Logic Solver</td>
<td>1.2</td>
<td>0.21</td>
</tr>
<tr>
<td>SDV, SV</td>
<td>7.26</td>
<td>0.833</td>
</tr>
</tbody>
</table>

### Table 1: Parameters of fuzzy number \(\lambda\) and \(\mu\)
The probability of failure of the system of figure. 4, also called unavailability is calculated by the combination of the probability of failure of all the subsystems. The equivalent Petri nets in figure 8 describe the failure mechanism and repair process model of the SIS system.

**Figure 8:** Safety system Generalized Stochastic Petri Nets

The initial marking of the Petri net contains tokens in the places P1, P2, P3, P7, P8, and P14 (Figure 8). This indicates that subsystems pressure transmitter, logic solver and final control are working initially. The token in the place P14 indicates also that the system is working normally. Tokens in places P11, P12, P13 correspond respectively to the partial failure mode of actuator layer, sensor layer and the logic unit layer. The failure of one of these subsystems brings the system to the failure state.

The characteristic parameters of the system components are given in table 2.

**Table 2.** Places, transitions and their firing rates used in the model

<table>
<thead>
<tr>
<th>Places</th>
<th>Interpretation</th>
<th>Transitions</th>
<th>Interpretation</th>
</tr>
</thead>
<tbody>
<tr>
<td>P1</td>
<td>Working state of PT1</td>
<td>T1</td>
<td>Failure rate of PT1</td>
</tr>
<tr>
<td>P2</td>
<td>Working state of PT2</td>
<td>T2</td>
<td>Failure rate of PT2</td>
</tr>
<tr>
<td>P3</td>
<td>Working state of PT3</td>
<td>T3</td>
<td>Failure rate of PT3</td>
</tr>
<tr>
<td>P4</td>
<td>Failed PT1</td>
<td>T4</td>
<td>Repair rate of PT1</td>
</tr>
<tr>
<td>P5</td>
<td>Failed PT2</td>
<td>T5</td>
<td>Repair rate of PT2</td>
</tr>
<tr>
<td>P6</td>
<td>Failed PT3</td>
<td>T6</td>
<td>Repair rate of PT3</td>
</tr>
<tr>
<td>P7</td>
<td>Working state of SV</td>
<td>T7</td>
<td>Failure rate of SV</td>
</tr>
<tr>
<td>P8</td>
<td>Working state of SDV</td>
<td>T8</td>
<td>Failure rate of SDV</td>
</tr>
<tr>
<td>P9</td>
<td>Failed SV</td>
<td>T9</td>
<td>Repair rate of SV</td>
</tr>
<tr>
<td>P10</td>
<td>Failed SDV</td>
<td>T10</td>
<td>Repair rate of SDV</td>
</tr>
<tr>
<td>P11</td>
<td>Failed of actuator layer</td>
<td>T11</td>
<td>Failure rate of LS</td>
</tr>
<tr>
<td>P12</td>
<td>Failed of sensor layer</td>
<td>T12</td>
<td>Repair rate of LS</td>
</tr>
<tr>
<td>P13</td>
<td>Failed of logic unit layer</td>
<td>T13-T20</td>
<td>Immediate transitions</td>
</tr>
<tr>
<td>P14</td>
<td>Working state of LS and SIS</td>
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</table>

Using the fuzzy GSPN method proposed in this paper, we determined the probability of failure of SIS from the distributions of the parameters of components represented by triangular fuzzy numbers. This probability of failure is equal to the unavailability of the system calculated by GSPN from imprecise characteristic parameters. For each α-cut, the upper and lower limits of the Probability of system failure are defined (Figure 9).

Figure 9 shows the system unavailability variation for α-cut=0, α-cut=0.5 and α-cut=1. This probability of failure of the safety system is equal to the asymptotic system unavailability computed by the method of GSPN with imprecise parameters, modeled by value intervals. The distribution of the unavailability varies from $3.39 \times 10^{-3}$ to $1.77 \times 10^{-2}$ for the biggest interval (α-cut=0) until $8.1 \times 10^{-3}$ for α-cut=1. This distribution is obtained using the interval defined by the characteristic parameters. We note that the calculated unavailability for α-cut=0 is included in the interval for α-cut=0.5 similarly for α-cut=1. The monotonic inclusion property of the availability function of the system warrant that intervals of the α-cut of level greater than 0 are strictly included in the support.

**Figure 9:** System Unavailability Variation for α=0, α=0.5, α=1

Using the fuzzy probability and GSPN, we obtained a probability of fuzzy failure in triangular form from the parameters modeled by triangular fuzzy numbers. The support of this fuzzy number varies from $3.39 \times 10^{-3}$ to $1.77 \times 10^{-2}$ which corresponds to a confidence degree of 100 %.( Figure 10)

Figure 10 depicted how the impact of the uncertainty on the failure and repair rates contribute in the performance of system. If a performance classification without uncertainty is preferred, it is necessary to either change the set of components or improve the values of parameters used in the system to reduce its uncertainty.
The fuzzy approach lets the study of the effect of imprecise parameters of several factors on the imprecision of the SIS. We interest in this approach to reveal the unavailability and determine the performance of the safety system, however it can be applied to study the reliability or the availability of all other industrial system

CONCLUSION

In this article, an approach for the assessment of safety systems by fuzzy Generalized Stochastic Petri Nets is proposed. This approach uses fuzzy numbers to represent the uncertainty on the probability of failure of the system components. Failure rate and repair rate are the two imprecise parameters considered in this study. The ambiguity in determining the nature of these failures is due to many factors which makes their quantification more difficult and more uncertain. The proposed approach allows to study the impact of imperfect knowledge of some factors to the imprecision of the system performance. Generalized Stochastic Petri Nets is used to model the functional and dysfunctional state of the system, it allows the study of the reliability and risk of system, it is a powerful tool in addressing problems which are usually fairly difficult to solve by standard approaches.

In this paper, an approach by Fuzzy set implemented in Generalized Stochastic Petri Nets is proposed to assess the performance of the safety system. The main advantage of this method is that it is easy to understand and implement in practice thanks to the availability of computer programs which allow a quantitative and qualitative analysis of the systems modeling. Fuzzy $\alpha$-cut technique based on the extension principle implies that the functional relations of safety systems can be extended to involve fuzzy arguments and can be used to map the dependent variable as a fuzzy set. In simple arithmetic operations, this principle can be used analytically. However, in most practical modeling applications, relations system involve partial differential equations and other complex structures that make the analytical application of the principle difficult. Therefore, Fuzzy $\alpha$-cut is used to carry out the analysis. The fuzzy approach applied in GSPN leads to exact results and guarantees the efficient computation of the smallest final interval of the probability of failure of the system. These variations allow us to have a total knowledge of the evaluation of the system that is important. The proposed method offers many benefits to the decision maker in terms of performance assessment. This method presented for this simple case and the obtained results can be applied for other more complex safety systems. This will help in decision-making and the establishment of a dedicated strategy to reduce both the level of uncertainty and the legal responsibility of the decision maker.

REFERENCES


[38] M. Kumar, S. Yadav et S. Kumar, «A new approach for analysing the fuzzy system reliability using intuitionistic fuzzy number,» *International Journal of Industrial and Systems Engineering*, vol. 8, n° 12, 2011.