Existence Results of Fractional Order Semilinear Integro-Differential Equations with Infinite Delay

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Abstract
In this paper, we prove the existence and uniqueness of mild solutions for the fractional order semilinear integro-differential equations with infinite delay in \( \alpha \)-norm in Banach space. The results are obtained by using analytic semigroup theory, probability density function and fixed point theorems.

Keywords: Semilinear fractional integro differential equations; Caputo fractional derivative; mild solution; Infinite delay; fixed point theorems; analytic semigroup; probability density function.

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Introduction
Fractional calculus is generalization of the ordinary differentiation and integration to arbitrary non-integer order. The subject is as old as the differential calculus since, starting from some speculations of G.W. Leibnitz (1697) and L. Eular (1730), it has been progressing up to nowadays. The theory of differential equations of fractional order have been proved to be good tools in the investigation of many phenomena in engineering, physics, electrodynamics of complex medium and other fields, see [1,2,3] and the monographs of Kilbas, Pondlubny [4,5].

Differential delay equations have been used in modeling scientific phenomena for many years. Often, it has been assumed that the delay is either a fixed point constant or is given as an integral in which case it is called distributed delay; see for instance the books by Lunel [6], Hino et al. [7], Kolmanovskii and Myshkis [8], Lakmikantham et al. [9] and Wu [10] and the papers [11,12].

The starting point of this work is the works in the papers [13,14,15,16]. Especially the authors of [13] investigated the existence results for semilinear fractional order integro-differential equations with nonlocal condition in Banach space \( X \):

\[
D^q x(t) + Ax(t) = f\left(t, x(t), \int_0^t a(t,s)x(s)ds\right), \quad t \in [0,T],
\]

\[x(0) = g(x) + x_0\]  

by using Banach contraction principle and Sadovskii’s fixed point theorem. Motivated by the above mentioned works, the main purpose of this work is to discuss the following fractional semilinear integro-differential systems with infinite delay:

\[
D^q x(t) = f(t,x(t), \int_0^t a(t,s)x(s)ds), \quad t \in [0,T],
\]

\[x(t) = \phi(t) \in B, \quad t \in (-\infty,0]\]

(2)

where \( D^q \) is the Caputo fractional derivative of order \( 0 < q < 1 \), \( A \) is a generator of an analytic semigroup \( \{S(t)\}_{t \geq 0} \) of uniformly bounded linear operators on Banach Space \( X \), \( f : J \times X \times X \to X \) is defined later. \( D = \{(t,s) \in [0,T] \times [0,T]: t \geq s\} \), \( \phi \in B \) where \( B \) is called abstract phase space to be defined in Section 2. For any function \( x \) defined on \( (-\infty,T] \) and any \( t \in J \), we denote \( x_t \) the element of \( B \) defined by:

\[
x_t(\theta) = x(t + \theta), \quad \theta \in (-\infty,0).
\]

Here \( x_t \) represents the history of the state up to present time \( t \).

In the present paper we deal with an infinite delay. Note that in this case the phase space \( B \) plays a crucial role in the study of both qualitative and quantitative aspects of theory of functional equations. see [12].

The rest of the paper is organized as follows. In Section 2, we give some preliminaries. In Section 3, we study the existence and uniqueness of mild solution for the problem (2). At last an example is given to demonstrate the applicability of results in Section 4.

Preliminaries
Let \( (X, || . ||) \) be a real Banach Space. Let \( L(X) \) be the Banach Space of all linear bounded operators on \( X \). \( L^1(J, X) \) the space of \( X \)-valued Bochner integrable functions on \( J \) with the norm:

\[
|| y ||_{L^1} = \int_0^T || y(t) || dt.
\]

\( L^\infty(J, R) \) is the Banach Space of essentially bounded functions normed by:

\[
|| y ||_{L^\infty} = \inf \{d > 0 : |y(t)| \leq d \quad a.e. \quad t \in J\}
\]
Let $A : D(A) \to X$ be the infinitesimal generator of an analytic semigroup of uniform linear bounded operators on $X$. Let $\rho(A)$, then it is possible to define the fractional power $A^\alpha$, for $0 < \alpha \leq 1$, $X_0 \to X_0$ and the embedding is compact whenever the resolvent operator $A$ is compact. Also for every $0 < \alpha \leq 1$, there exist a positive constant $M_\alpha$ such that:
\[
\left\| A^\alpha S(t) \right\| \leq \frac{M_\alpha}{t^\alpha}, \quad 0 < t \leq T.
\]

Let $C(J, X_\alpha)$ be the Banach Space of all continuous functions from $J$ into $X_\alpha$ with the norm:
\[
\left\| X \right\|_J = \sup_{t \in J} \left\| X(t) \right\|.
\]

In this paper, we will employ an axiomatic definition for the abstract phase space $B$ which is similar to those introduced by Hale and Kato [12].

**Definition 2.1.** $B$ will be a linear space of functions mapping $(-\infty, T]$ into $X$ endowed with a seminorm $\left\| \cdot \right\|_B$, and satisfies the following axioms:

(A1): If $x : (-\infty, T] \to X$ is continuous on $J$ and $x_0 \in B$, then the $x_0$ in $B$ and $x_0$ is continuous in $t \in J$ and $\left\| x(t) \right\| \leq C \left\| x_0 \right\|$, where $C \geq 0$ is a constant.

(A2): There exist a continuous function $C_1(t) > 0$ and a locally bounded function $C_2(t) \geq 0$ such that:
\[
\left\| x(t) \right\|_B \leq C_1(t) \sup_{s \in [0,t]} \left\| x(s) \right\| + C_2(t) \left\| x_0 \right\|_B,
\]
for $t \in [0,T]$ and $x$ as in (A1).

(A3): The space $B$ is complete.

**Remark 2.2.** Condition (3) in (A1) is equivalent $\left\| \phi(0) \right\| \leq C\left\| \phi \right\|_B$ for all $\phi \in B$.

**Definition 2.3.** [17]: a continuous function $x : (-\infty, T] \to X_\alpha$ is said to be mild solution of system (2) if $x$ satisfies:
\[
\begin{cases}
\phi(t), & t \in (-\infty,0),

X(t) = \begin{cases}
Q(t)\phi(0) + \int_0^t (t-s)^{\alpha-1} R(t-s)f(s,x(s),x_\alpha(s))ds, & t \in J,
\end{cases}
\end{cases}
\]

where
\[
(Q(t)) = \int_0^\infty \xi_q(\sigma)S(t^\alpha \sigma)d\sigma, \quad R(t) = q \int_0^\infty \sigma^{-q-1} \xi_q(\sigma)S(t^\alpha \sigma)d\sigma
\]
and $\xi_q$ is a probability density function defined on $(0, \infty)$ such that
\[
\xi_q(\sigma) = \frac{1}{\alpha q} (\sigma^{-\alpha q} - \sigma^{-\alpha q-\frac{1}{q}}) \geq 0,
\]
where
\[
\Theta_q(\sigma) = \pi \sum_{n=1}^{\infty} (-1)^{n-1} \frac{\Gamma(nq+1)}{n!} \sin(nq\sigma), \sigma \in (0, \infty).
\]

**Remark 2.4.** Note that $\{S(t)\}_{t \geq 0}$ is a uniformly bounded semigroup i.e. there exists a constant $M \geq 1$ such that $\left\| S(t) \right\|$ for all $t \in [0,T]$.

**Lemma 2.5.** [17]. The above defined operators $Q$ and $R$ have the following properties.

(i) For any fixed $t \geq 0$, $Q$ and $R$ are linear and bounded operators, i.e. for any $x \in X$,
\[
\left\| Q(t)x \right\| \leq M, \quad \left\| R(t)x \right\| \leq \frac{qM}{\Gamma(1+q)} \left\| x \right\|.
\]

(ii) $\{Q(t)\}_{t \geq 0}$ are strongly continuous.

(iii) For every $t > 0$, $Q(t)$ and $R(t)$ are also compact operators.

(iv) For any $x \in X$, $\alpha, \beta \in (0,1)$, we have
\[
\left\| A^\alpha R(t)x \right\| \leq \frac{qM\Gamma(2-\alpha)}{(1+q)(1-\alpha)} t^{-\alpha q}, \quad 0 < t \leq T.
\]

**Remark 2.6.** According to [17], a direct calculation gives that for $v \in [0,1]$,
\[
\int_0^\infty \sigma^v \xi_q(\sigma) d\sigma = \int_0^\infty \sigma^{-qv} \Theta_q(\sigma) d\sigma = \frac{\Gamma(1+v)}{\Gamma(1+qv)}.
\]

By Lemma 2.5(i), for fixed $t \geq 0$ and any $x \in X_\alpha$, we have
\[
\left\| Q(t)x \right\|_{\alpha} \leq M \left\| x \right\|_{\alpha}, \quad \left\| R(t)x \right\|_{\alpha} \leq \frac{qM}{\Gamma(1+q)} \left\| x \right\|_{\alpha}.
\]

**Lemma 2.7.** (Schauder’s fixed point theorem) [18]. If $K$ is a closed, bounded and convex subset of a Banach Space $X$ and $F : K \to K$ is completely continuous, then $F$ has a fixed point in $K$.

**Existence Result**

In order to establish our result, we make following assumptions for $\alpha \in (0,1)$:

(H1) $a : D = \{(t,s) \in [0,T] \times [0,T] : t \geq s \} \times B \to X_\alpha$ is continuous and there exists a constant $\mu_1 > 0$ such that for all $(t,s) \in D, x, y \in B$,
\[
(i) \left\| a(t,s,x) - a(t,s,y)ds \right\|_{\alpha} \leq \mu_1 \left\| x - y \right\|_B,
\]
\[
(ii) \left\| a(t,s,x)ds \right\|_{\alpha} \leq \mu_2 \left(1 + \left\| x \right\|_B \right).
\]

(H2) $f : J \times B \times X_\alpha \to X_\alpha$ is continuous and there exist positive constant $\mu_2$ such that for each $(t, x, y) \in J \times B \times X_\alpha$, $i = 1,2$
\[
\left\| f(t, x, y) \right\|_{\alpha} \leq \mu_2 \left( \left\| x_1 - x_2 \right\|_{\alpha} \right),
\]
\[
\left\| f(t, x, y) \right\|_{\alpha} \leq \mu_2 \left( \left\| y_1 - y_2 \right\|_{\alpha} \right).
\]
(H3) The function $f : J \times B \times X \to X$, $\phi(t,x) \to f(t,\phi,x)$ is continuous with respect to $\phi$ for a.e. $t \in J$ and is strongly measurable with respect to $t$ for any $(\phi,x) \in B \times X$. For each positive number $r > 0$, there exists a function $\alpha_r \in C(J,\mathbb{R}_+)$ with $\sup_{t \in J} \alpha_r(t) < +\infty$ such that

$$\sup \{ ||f(t,\phi,x)||_B \mid \phi \mid \leq r \} \leq \alpha_r(t), \quad \text{for } t \in J$$

and

$$\lim_{r \to \infty} \inf_{t \in J} \frac{\alpha(r)}{r} = \beta < \infty$$

Our first existence result for the problem (2) is based on the Banach fixed point theorem.

**Theorem 3.1.** Assume that the condition (H1) – (H2) and the Lemma 2.7 are satisfied. Further, if $\phi \in X_\alpha$ and

$$\frac{\ell \mu \Gamma^{3-q}}{\Gamma^{2-q}} (1+\mu_1) < 1,$$

where

$$\ell = \int_{-\infty}^0 h(t) dt < +\infty$$

for any $\alpha > 0$ and $h : (-\infty,0) \to (0,+)$. Then the system (2) has a unique solution on the interval $(-\infty,T)$.

Proof. Let $\Omega$ be a set defined by

$$\Omega = \{ x : (-\infty,T) \to X_\alpha \mid x\}$$

Now we transform the problem (2) into a fixed point problem. Define a mapping $\Phi$ from $\Omega$ into itself by

$$\Phi(x)(t) = \begin{cases} x(0), & t \in (-\infty,0] \\ 0(t) + \int_0^t (t-s)^{q-1} R(t-s)f(s,x(s),x'(s)) ds, & t \in J. \end{cases}$$

Clearly, fixed points of the operator $\Phi$ are mild solutions of the problem (2).

For $\phi \in B$, we will define the function $y(\cdot) : (-\infty,T) \to X_\alpha$ by

$$y(t) = \begin{cases} \phi(t), & t \in (-\infty,0] \\ Q(t)\phi(t), & t \in J \end{cases}$$

then $y(t) \in \Omega$. For each $t \in J$ we have $z \in C(J,X_\alpha)$ with $z(0) = 0$ and function $z(t)$ satisfies

$$z(t) = \int_0^t (t-s)^{q-1} R(t-s)f(s,z(s),z'(s)) ds.$$

As we can decompose $x(t) = y(t) + z(t)$, for $t \in J$. Let $Z_0 = \{ z \in \Omega : z(0) = 0 \}$. For any $z \in Z_0$, we have

$$\| z \|_{Z_0} = \sup_{t \in J} \| z(t) \|_B + \| z_0 \|_B = \sup_{t \in J} \| z(t) \|_B.$$

Thus $(Z_0, \| . \|_{Z_0})$ is a Banach space. We define the operator $\Phi : Z_0 \to Z_0$ by

$$\Phi(z_0)(t) = \begin{cases} 0, & t \in (-\infty,0] \\ \int_0^t (t-s)^{q-1} R(t-s)f(s,y(s),y'(s)) ds, & t \in J. \end{cases}$$

Obviously, the operator $\Phi$ has a fixed point is equivalent to $\Phi$ has one, so in turns to prove that $\Phi$ has a fixed point. Let $r > 0$ and consider the set

$$B_r = \{ z \in Z_0 : \| z \|_{Z_0} \leq r \}$$

then $B_r$ is uniformly bounded.

Now we need following lemma.

**Lemma 3.2.** Assume $x \in X$ then for $t \in J$, $x(t) \in B$. Moreover

$$\ell \| f(x) \|_B \leq \| x(t) \|_B \leq \| x(0) \|_B + \ell \sup_{s \in [0,T]} \| f(s) \|_B.$$

Then for any $z \in B$, we have

$$\| y(t) + z(t) \|_B \leq r'.$$

By assumption (H3) it is easy to obtain

$$\lim_{r \to \infty} \inf \frac{r'}{r} = \ell.$$

Proof. Using (H1) – (H2) and (10) we have

$$\| z(t) + y(t) \|_B \leq \ell \sup_{s \in [0,T]} \| z(s) \|_B + \ell \sup_{s \in [0,T]} \| f(s) \|_B.$$

Now we shall show that the operator $\tilde{\Phi}$ is a contraction map on $Z_0$. In fact for each $z, z_0 \in Z_0, t \in J$, we have

$$\| (\tilde{\Phi}z)(t) - (\tilde{\Phi}z_0)(t) \|_B \leq \frac{MT^q}{\Gamma(q+1)} \| z - z_0 \|_B + \ell \sup_{s \in [0,T]} \| f(s) \|_B.$$

Thus, taking supremum over $t$, we have

$$\| (\tilde{\Phi}z)(t) - (\tilde{\Phi}z_0)(t) \|_B \leq \frac{MT^q}{\Gamma(q+1)} \mu_2 (1+\mu_1) \sup_{s \in [0,T]} \| f(s) \|_B.$$

From (6), we see $\Phi$ is a contraction. Therefore the system has a unique mild solution on the interval $(-\infty,T)$. Now, we give another existence result for the system (2) by means of Schauder’s fixed point theorem.
Theorem 3.3. Suppose that the assumption (H1) – (H3) are satisfied. Then the system (2) has at least one mild solution on J, provided
\[
\frac{MT^q}{\Gamma(1+q)} \beta \ell(1+\mu_1) < 1.
\]  
(13)
Proof. Let \( \Phi \colon Z_0 \to Z_0 \) be defined as in (9). Now we will prove that \( \Phi \) has a fixed point by using lemma 2.7. We proceed in following steps.

Step 1. \( \Phi(B_r) \subseteq B_r \) for some \( r > 0 \).

We claim that there exists a positive integer \( r \), such that \( \Phi(B_r) \subseteq B_r \). If it is not true, then for each positive number \( r \), there exists a function \( z^r(.) \in B_r \), but \( \Phi(z^r) \not\in B_r \).

That is \(|(\Phi z^r)(t)| > 0 \) for some \( t \in J \). However, on the other hand, we have from (H1) – (H3) and 11.

\[
r < \langle \Phi z^r(t) \rangle \alpha \leq \frac{\alpha}{\Gamma(1+q)} \int_0^1 (1-s)^{\alpha-1} \left| f\left( s, y_s + z_s, \int_0^s a(s, \tau, y_\tau + z_\tau) d\tau \right) \right| ds
\]  
(14)
where \( r^* = 1 + (1 + \mu_1) r \) and \( r^* = \ell(r + M \phi(0)) + ||\phi||_B \).

Dividing both sided of (14) by \( r \), and taking \( r \to \infty \), we have

\[
1 \leq \frac{MT^q}{\Gamma(1+q)} \lim_{r \to \infty} \left( \sup_{t \in J} \frac{\alpha_{r^*}(t)}{r^*} \right) \frac{r^*}{r} = \frac{MT^q}{\Gamma(1+q)} \beta \ell(1+\mu_1).
\]
That is

\[
\frac{MT^q}{\Gamma(1+q)} \beta \ell(1+\mu_1) \geq 1.
\]
This contradicts (13). Hence for some positive number \( r \Phi(B_r) \subseteq B_r \).

Step 2. \( \Phi \colon Z_0 \to Z_0 \) is continuous.

Let \( \{ z^k \}_{k \in \mathbb{N}} \) be a sequence such that \( z^k \to z \) in \( B \) as \( k \to \infty \).

Then there exists a number \( r > 0 \) such that \( |z^k(t)| \leq r \) for all \( k \) and a.e. \( t \in J \), so \( z^k \in B_r \) and \( z \in B_r \).

By hypothesis (H3) we have for a.e. \( (t, s) \in D \), and since

\[
(1-s)^{\alpha-1} \left| f\left( s, y_s + z_s, \int_0^s a(s, \tau, y_\tau + z_\tau) d\tau \right) \right| ds \leq 2(t-s)^{\alpha-1} \alpha_{r^*}(s),
\]

where the function \( 2(t-s)^{\alpha-1} \alpha_{r^*}(s) \) is integrable since

\[
2 \int_0^1 (t-s)^{\alpha-1} \alpha_{r^*}(s) ds \leq 2 \sup_{t \in J} \alpha_{r^*}(t) < \infty.
\]
We have by the Lebesgue Dominated Convergence Theorem,

\[
\|\Phi z^k - \Phi z\|_B \
\]

\[
\leq |R(t-s)||\int_0^1 (1-s)^{\alpha-1} \left| f\left( s, y_s + z_s, \int_0^s a(s, \tau, y_\tau + z_\tau) d\tau \right) \right| ds
\]

\[
\leq \frac{Mq}{\Gamma(1+q)} \int_0^1 (1-s)^{\alpha-1} \left| f\left( s, y_s + z_s, \int_0^s a(s, \tau, y_\tau + z_\tau) d\tau \right) \right| ds
\]

\[
\to 0 \quad \text{as} \quad k \to 0,
\]

which proves that \( \Phi \) is continuous.

Step 3. \( \Phi \) maps \( B \) into an equicontinuous family.

Let \( z \in B \) and \( r_1, r_2 \in J \), then if \( 0 < r_1 < r_2 \leq T \), in view of (H3) and 11 we have

\[
\|((\Phi z)(r_1) - (\Phi z)(r_2))\|_\alpha \leq \int_0^{r_1} [(r_1-s)^{\alpha-1} R(r_1-s) - (r_2-s)^{\alpha-1} R(r_2-s)]
\]

\[
(x) R\left( s, y_s + z_s, \int_0^s a(s, \tau, y_\tau + z_\tau) d\tau \right) ds
\]

\[
+ \int_{r_1}^{r_2} [(r_1-s)^{\alpha-1} R(r_1-s) - (r_2-s)^{\alpha-1} R(r_2-s)]
\]

\[
(x) R\left( s, y_s + z_s, \int_0^s a(s, \tau, y_\tau + z_\tau) d\tau \right) ds
\]

\[
\to 0 \quad \text{as} \quad k \to 0.
\]
\[
\begin{align*}
&\leq \int_0^{t_1-\varepsilon} \| (r_1-s)^{q-1} R(r_1-s) - (r_2-s)^{q-1} R(r_2-s) \|_{\mathcal{A}^*}(s) \, ds \\
&+ \int_{t_1-\varepsilon}^{t_2} \| (r_1-s)^{q-1} R(r_1-s) - (r_2-s)^{q-1} R(r_2-s) \|_{\mathcal{A}^*}(s) \, ds \\
&+ \int_{t_1}^{r_2} \| (r_2-s)^{q-1} R(r_2-s) \|_{\mathcal{A}^*}(s) \, ds,
\end{align*}
\]

The right hand side is independent of \( z \in B \), and tends to zero as \( r_1 - r_2 \to 0 \) with \( \varepsilon \) sufficiently small, since the compactness of \( R(t) \) for \( t > 0 \) implies the continuity in the uniform operator topology. Thus \( \Phi \) maps \( B \) into an equicontinuous family. The equicontinuities for the cases \( r_1 < r_2 \leq 0 \) and \( r_1 < 0 < r_2 \) are obvious.

**Step 4.** \( \Phi \) maps \( B \) into precompact set in \( X \).

Let \( 0 < t \leq T \) be fixed, for \( \varepsilon \in (0, t) \) and \( \forall \varepsilon > 0 \), define the operator \( \Phi_{\varepsilon_1}^{t} \) on \( B \) by the formula

\[
(\Phi_{\varepsilon_1}^{t} z)(t) = \int_0^t \xi_{q}(\sigma) R((t-s)^q) \phi(t) d\sigma + \int_0^t \xi_{q}(\sigma) R((t-s)^q) \xi_{q}(\sigma) R((t-s)^q) f\left(s,z, t \int_0^t a(t,\tau,z_\tau) d\tau \right) ds d\sigma.
\]

where \( z \in B \), then the compactness of \( R(\varepsilon^q_1) \) \( (\varepsilon^q_1 > 0) \), we obtain that the set \( V_{\varepsilon_1} = \{(\Phi_{\varepsilon_1}^{t} z)(t) : z \in B \} \) is relatively compact in \( X \) for all \( \varepsilon \in (0,t) \) and \( \varepsilon_1 > 0 \).

Moreover, for each \( z \in B \), we have that

\[
\| (\Phi z)(t) - (\Phi_{\varepsilon_1}^{t} z)(t) \|_{\mathcal{A}}
\]

\[
\leq \int_0^t \xi_{q}(\sigma) R((t-s)^q) \phi(t) d\sigma + \int_0^t \xi_{q}(\sigma) R((t-s)^q) f\left(s,z, t \int_0^t a(t,\tau,z_\tau) d\tau \right) ds d\sigma
\]

\[
- \int_0^t \xi_{q}(\sigma) R((t-s)^q) f\left(s,z, t \int_0^t a(t,\tau,z_\tau) d\tau \right) ds d\sigma
\]

\[
\leq q \int_0^t \xi_{q}(\sigma) R((t-s)^q) d\sigma f\left(s,z, t \int_0^t a(t,\tau,z_\tau) d\tau \right) ds d\sigma
\]

\[
+ \int_0^t (t-s)^q \xi_{q}(\sigma) R((t-s)^q) f\left(s,z, t \int_0^t a(t,\tau,z_\tau) d\tau \right) ds d\sigma
\]

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An Example

In this section we give an example to illustrate the above results. Consider the following integrodifferential equation.

\[
\frac{\partial^q}{\partial t^q} v(t,\eta) = \frac{\partial^2}{\partial \eta^2} v(t,\eta) + \int_0^t (t-s)^{-q} \sin(n(s,\eta))v(s,\eta)(|s|+\eta(0+\eta))dsds \\
+ \int_0^t (t-s)^{-q/2}\sin(n(s,\eta))\int_0^s \cos(\tau,\eta)d\tau ds,
\]

\[
v(\theta,\eta) = v_0(\theta,\eta) - \infty < \theta < 0,
\]

(15)

where \(\frac{\partial^q}{\partial t^q}\) is Caputo partial fractional derivative of order 0< q < 1, and \(t, \eta \in [0,1], \gamma_1: (-\infty,0] \rightarrow \mathbb{R}, v_0: (-\infty,0] \times [0,1] \rightarrow \mathbb{R}\) are continuous functions and

\[
\int_{-\infty}^0 |\gamma_1(\theta)| d\theta < \infty.
\]

Set \(X = L^2([0,1],\mathbb{R})\) with the norm \(\|\cdot\|_{L^2}\) and define \(A: X \rightarrow X\) by \(Au = v^0\) with the domain

\[
D(A) = H^2((0,1)) \cap H^1(0,1)).
\]

Then \(A\) generates a compact, analytic semigroup \(\{S(t)\}_{t \geq 0}\) in \(X\) of uniformly bounded, linear operators such that \(\|S(t)\| \leq 1\).

Let the phase space \(B = C((-\infty,0]X), the space of bounded, uniformly continuous functions endowed with the norm

\[
\|\psi\|_B = \sup_{-\infty<\theta<0} |\psi(\theta)|, \forall \psi \in B,
\]

Let \(h(s) = e^{2s}, s < 0\) then \(\ell = \int_{-\infty}^0 h(s)ds = \frac{1}{2}\).

Set

\[
x(t)(\eta) = v(t,\eta) \\
\phi(\theta)(\eta) = v_0(\theta,\eta), \theta \in (-\infty,0]
\]

\[
\mathcal{F}(\psi,\chi)(\eta) = \int_{-\infty}^0 \gamma_1(0)\sin(\psi(\theta)(\eta))d\theta + \frac{s^2}{2}\sin(\chi(\tau)(\eta))d\tau.
\]

Therefore \(||(\overline{Q}z)(t) - \overline{Q}z(t)|| \rightarrow 0\) as \(\varepsilon, \varepsilon_1 \rightarrow 0^+\)

and there are relatively compact sets arbitrarily close to set \(V(t)\) is relatively compact in \(X\).

Thus by Arzela-Ascoli theorem \(\overline{Q}\) is a compact operator and by Schauder fixed point theorem there exists a fixed point for \(\overline{Q}\) on \(B_0\).

Hence \(x(t) = y(t) + z(t), t \in (-\infty, T]\) is a fixed point of the operator \(\overline{Q}\) which is a mild solution of the problem (2). The proof is now completed.

An Example
\[
\| f(t, x(t), \eta) \| \leq t \| \psi \| \int_{-\infty}^{0} |\gamma_1(\theta)| d\theta + \frac{t^3}{2} \| x(0) \|
\]
\[
= \mu_1(t) \| \psi \|_B + \mu_2(t) \| x(t) \|
\]
where
\[
\mu_1(t) = t \int_{0}^{\infty} |\gamma_1(\theta)| d\theta, \quad \mu_2(t) = \frac{t^3}{2}.
\]
Then (15) has a mild solution by Theorem 3.3. Further we can impose some suitable conditions on above defined functions to verify the assumptions of Theorem 3.3. For example, if we put
\[
\gamma_1(\theta) = e^\theta, q = \frac{1}{2}, \beta = \frac{1}{8}
\]
Thus we see
\[
\frac{MT^q}{\Gamma(1+q)} \beta/(1+\mu_1) = \frac{1}{4\sqrt{\pi}} < 1.
\]

References


