Global Stability Analysis with Adaptive Control in Two Prey, One Predator System Involving Infection in Prey with unknown parameters

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Abstract
In this article, we propose and study a three dimensional continuous time prey-predator model where the prey species is separated in to susceptible and infected (SI). It has been assumed that in the interaction of susceptible and infected prey populations, predator predates on infected prey only. In this work, we discussed the problem of controlling chaos of continuous three species prey-predator model with some unknown parameters. That is, we have show that the continuous time three species prey-predator populations can be globally asymptotically stable by using adaptive control inputs with unknown parameters by constructing suitable Lyapunov function. We also estimated the unknown parameters. Finally, numerical examples and analysis of the results are presented.

Keywords: Prey-predator System, Lyapunov function, SI model, Adaptive nonlinear Control, Stability analysis.

Introduction
Differential equations in prey-predator system have played an important role and they will continue to serve as important tools in future research. The study of infection in a prey-predator system has been growing interest because species does not exists alone, they spread disease, competes with the other species for food or space or they predated by other species. To stabilize the system an appropriate control on mathematical model is essential to study the effect of infection on interacting species. Prasenjit Das et.al , Krishnapada Das et.al ([1],[2]) studies the prey predator system with disease in the predator population and discussed the chaos in this system. The nonlinear feedback controls, positive controls prey-predator system has been studied by many authors ( [3-5]). The stabilization of prey-predator system with infection or disease in prey has become great interest ([6-11]). Also the prey-predator models with stage-structure involving parasitic infectious disease has been studied ([12]). The stability analysis of prey-predator with control phenomena is a challenging problem in many ecological systems in which we were studied in many references ([3]-[5],[13]-[14]). The adaptive controls are rapidly growing interest in many fields such as ecological systems, electrical engineering, neural networks and others. If changes arise within the system or external to the system due to the spread of infection, the adaptive control inputs of this system is capable of accommodating the unpredictable environmental changes.

In this paper we have investigated the epidemiological model, namely susceptible-infected prey (SI) and predator model in which how the process of predation influences the epidemics. Here, we consider the case where the predator eats the infected prey only because the infected individuals are less active and are caught more easily, for example, in the reference of Peterson and Page [15], they have indicated wolf attacks on moose are more often successful if the moose is heavily infected by “Echinococcus granulosus”. Our future work is the stochastic effects on these ecosystem perturbed by independent white noises and other related models has been investigated by Mukherjee, Srinivasan et.al ([16] and [17]), where the deterministic model with the effect of random noise of the environment results in a stochastic model in which the parameters of the system oscillate about their mean values.

In this paper we analyse the stability of SI prey and predator model by using adaptive nonlinear control which was not studied earlier. This paper is organized as follows: In section 2: we introduce the basic model and we reduce from 7 parameters to 4 parameters. In section 3: we have presented the plots for the uncontrolled system. In section 4: we have given the problem statement in which the general prey-predator system involves the adaptive control inputs. In section 5: we have proved the asymptotic stability of the controlled system by constructing suitable Lyapunov function and also necessary adaptive control inputs are obtained as nonlinear feedback. In section 6: we have presented the numerical simulations that carried out to investigate the controls and the effect of infection on the populations and also we estimated the unknown parameters. Finally, the last section 7, is devoted to the conclusion and remarks.

The Mathematical Model of the System
In this section, we study the dynamics of the continuous time three species prey-predator populations in which we will use the mathematical tools and biological assumptions for modelling the three species prey-predator system which consists of two preys and one predator.

The Basic Model and Assumptions
In this section, we consider the three species prey-predator system which consists of two preys, namely susceptible prey, infected prey (SI) and one predator in which prey species (SI) is infected. And also here we assume that the predator predates on only infected prey [9]. We assume that, the susceptible prey population grows according to a logistic law involving the only susceptible prey species. The transmission rate among the susceptible prey populations and
infected prey populations follows the simple law of mass action. The disease is spread among the prey population only and that disease is genetically inherited. The infected prey populations do not recover or become immune. The predator population predates only the infected prey and the functional response is of Michaelis-Menten-Holling- type II. Such system can be described by the following set of nonlinear differential equations:

\[ \begin{align*} 
\dot{X}_1 &= rX_1 \left(1 - \frac{X_1}{K}\right) - PX_1X_2 \\
\dot{X}_2 &= PX_1X_2 - f \left(X_2, X_1\right)X_3 \\
\dot{X}_3 &= eff \left(X_2, X_3\right)X_1 - dX_3 
\end{align*} \]  

(1)

where \( X_1(t) \) : the number of the susceptible prey population at time \( t \),
\( X_2(t) \) : the number of the infected prey population at time \( t \),
\( X_3(t) \) : the number of the predator population at time \( t \),
\( r \) : the growth rate of susceptible prey population,
\( K \) : the environmental carrying capacity,
\( P \) : the rate of transmission from susceptible to infected prey population,
\( e \) : the conversion efficiency rate and
\( d \) : the death rate of predator population.

We assume here that the response function are increasing and bounded functions of \( X_2 \) and \( X_3 \). We will consider these response functions as of Holling type II which is given below:

\[ f \left(X_2, X_1\right) = \frac{\gamma X_2}{X_1 + \beta X_2} \]

Moreover, \( \gamma \) is the total attack rate for predator and \( \beta \) is the handling time of predator to prey. Then the system (1) becomes:

\[ \begin{align*} 
\dot{X}_1 &= rX_1 \left(1 - \frac{X_1}{K}\right) - PX_1X_2 \\
\dot{X}_2 &= PX_1X_2 - \frac{\gamma X_2X_1}{X_1 + \beta X_2} \\
\dot{X}_3 &= eff \left(X_2, X_3\right)X_1 - dX_3 
\end{align*} \]  

(2)

\[ \text{Nondimensionalization} \]

Now to reduce the number of the system parameters we will transform the system (2) to the nondimensional form by using the following transformation of the variables:

\[ x_1 = \frac{X_1}{K}, \quad x_2 = \frac{X_2}{K}, \quad x_3 = \frac{X_3}{\gamma \beta K}, \quad t = r \tau \]  

(3)

The modified Michaelis-Menten-Holling prey predator with infected prey dynamics that is, using the transformation (3) the system (2) takes the nondimensional form:

\[ \begin{align*} 
\dot{x}_1 &= x_1 \left(1 - x_1\right) - k x_1x_2 \\
\dot{x}_2 &= k x_1x_2 - b \frac{x_2x_3}{x_1 + x_2} \\
\dot{x}_3 &= c \frac{x_2x_3}{x_1 + x_2} - ax_3 
\end{align*} \]

(4)

where the relations between the nondimensional and dimensional parameters are given by:

\[ \begin{align*} 
k &= \frac{PK}{r}, \quad b = \frac{\gamma}{r}, \quad c = \frac{e}{r \beta}, \quad a = \frac{d}{r} \end{align*} \]  

(5)

The system (4) is more simplicity than (5) for the mathematical study, since the number of system parameters has been reduced from 7 to 4 only.

Now we will analyze the system (4) with the following initial conditions:

\[ x_1(0) > 0, \quad x_2(0) > 0, \quad x_3(0) > 0 \]  

(6)

The conditions (6) represent the conditions of positivity or biologically feasibility of the densities of susceptible prey, infected prey and predator populations respectively.

\[ \text{Numerical Simulations of the System without Control} \]

Now, let us discuss the numerical simulations of the system (4) without control for the fixed parameter values and initial densities, that is, in this section we describe our system (4) without control through simulation works by using the well known MATLAB software and also we discuss their implications to explore the possibility of the chaotic behavior and the effect of the infection to this behavior.

We choose a set of parameters fixed

\[ \{a, b, c\} = \{0.2, 0.9, 0.4\} \]

and varying the disease transmission rate \( k \) of susceptible prey and infected prey.

We choose the initial densities

\[ x_1 = 0.2345, \quad x_2 = 0.2872, \quad x_3 = 0.1652 \].
Figure 1: The variations in the growth rate of the populations without control that approaches the stable equilibrium point 
\((0.5141, 0.5547, 0.5551)\), with
\[ x_1 = 0.2345, \quad x_2 = 0.2872, \quad x_3 = 0.1652, \] 
and with parameters \(a = 0.2, \quad b = 0.9, \quad c = 0.4\) and \(k = 0.8758\).

Figure 1 shows the stable behaviour of susceptible prey \(x_1\), infected prey \(x_2\) and the predator \(x_3\) of the uncontrolled system (4) for the system parameter fixed values which is give above if the transmissible rate \(k = 0.8758\) and approaches the equilibrium point \((0.5141, 0.5547, 0.5551)\).

Figure 2: The three dimensional phase plot of the uncontrolled three species prey-predator system approaches the stable point with parameters \(a = 0.2, \quad b = 0.9, \quad c = 0.4\) and \(k = 0.8758\) and with the initial values \(x_1 = 0.2345, \quad x_2 = 0.2872, \quad x_3 = 0.1652\).

Figure 2 shows the corresponding limit cycle about the equilibrium point \((0.5141, 0.5547, 0.5551)\), which is asymptotically stable for \(k = 0.8758\).

Now, if we increased the transmissible rate to \(k = 1.85\), we observe the chaotic behaviour of the system (4) and the periodic oscillations of the infected prey which is presented in the Figure 5 and Figure 6 shows the corresponding phase plot with chaotic behaviour of the uncontrolled system (4).

Figure 3: The variations in the growth rate of the populations without control that approaches the stable equilibrium point \((0.3002, 0.4661, 0.4666)\), with
\[ x_1 = 0.2345, \quad x_2 = 0.2872, \quad x_3 = 0.1652, \] 
with parameters \(a = 0.2, \quad b = 0.9, \quad c = 0.4\) and \(k = 1.5\).

Figure 4: The three dimensional phase plot of the uncontrolled three species prey-predator system approaches the stable point with parameters \(a = 0.2, \quad b = 0.9, \quad c = 0.4\) and \(k = 1.5\) and with the initial values \(x_1 = 0.2345, \quad x_2 = 0.2872, \quad x_3 = 0.1652\).

Figure 5: Time series graph of the system without control showing unstable behaviour with initial densities
\[ x_1 = 0.2345, \quad x_2 = 0.2872, \quad x_3 = 0.1652, \] 
and with parameters \(a = 0.2, \quad b = 0.9, \quad c = 0.4\) and \(k = 1.85\).
Figure 6: The three-dimensional phase plot of uncontrolled three species prey-predator system in chaotic behaviour for the system parameter \( a = 0.2, \, b = 0.9, \, c = 0.4 \) and \( k = 1.85 \) with the initial values \( x_1 = 0.2345, \, x_2 = 0.2872, \, x_3 = 0.1652 \).

### The Problem Statement of the Basic System with Adaptive Control

**Theorem 1:**
Consider the prey-predator system described by the dynamics

\[
\dot{x} = Ax + f(x) + \alpha_A + u
\]

where \( x \in \mathbb{R}^n \) is the state of the system, \( A \) is the \( n \times n \) matrix of the system parameters, the matrix \( A \) have some unknown parameters, \( f: \mathbb{R}^n \rightarrow \mathbb{R} \) is the nonlinear part of the system, \( u \in \mathbb{R}^n \) is the adaptive feedback controller, \( \alpha_A \) is estimator of unknown parameters.

The global control problem is essentially to find adaptive feedback controller \( u \) and \( \hat{\alpha}_A \), so as to stabilize the error dynamics (7) for all initial conditions \( x(0) \in \mathbb{R}^n \), i.e. \( \lim_{t \to \infty} \| x(t) \| = 0 \) for all initial conditions \( x(0) \in \mathbb{R}^n \).

Lyapunov function methodology is used for establishing the adaptive feedback control of the system (7). By the Lyapunov function methodology, a candidate Lyapunov function is taken as

\[
V(x) = x^T P x + \alpha^T_A P \alpha_A
\]

(8)

where \( P, P_A \) are \( n \times n \) positive definite matrix.

Note that \( V: \mathbb{R}^n \rightarrow \mathbb{R}^n \) is a positive definite function by construction. It is assumed that the parameters of the system (7) are measurable.

If a controller \( u \) and \( \hat{\alpha}_A \) can be found such that

\[
\dot{V}(x) = -x^T Q x - \alpha^T_A Q \alpha_A
\]

(9)

where \( Q, Q_A \) are positive definite matrices, then \( \dot{V} \) is a negative definite function. Hence, by Lyapunov stability theory (Hahn, 1967, [18]), the dynamics (7) is globally asymptotically stable and hence the condition \( \lim_{t \to \infty} \| x(t) \| = 0 \) will be satisfied for all initial conditions \( x(0) \in \mathbb{R}^n \).

Then the states of the system (7) will be globally asymptotically stable.

### Prey–Predator system with Adaptive Nonlinear Controls

In this section we introduced the adaptive control of the three species population that consists of susceptible prey, infected prey and predator. To study the adaptive control of the three species system using nonlinear feedback control approach, we start by assuming that the system (4) can be written in the following suitable form

\[
\begin{align*}
\dot{x}_1 &= x_1(1 - x_1) - k x_1 x_2 + u_1 \\
\dot{x}_2 &= k x_1 x_2 - b \frac{x_2 x_3}{x_3 + x_2} + u_2 \\
\dot{x}_3 &= c \frac{x_2 x_3}{x_3 + x_2} - a x_3 + u_3
\end{align*}
\]

(10)

where \( u_1, u_2, u_3 \) are adaptive non linear feedback controllers, which is the function of the state variables and which will be suitably choice to make the trajectory of the whole system (4). As long as these feedbacks stabilize the system (10) converge to zero as the time \( t \) goes to infinity.

That means that, this gives the system (10) \( \lim_{t \to \infty} \| x(t) \| = 0 \).

**Theorem 2**

The modified Michaelis-Menton-Holling prey-predator with infected prey dynamics (4) is asymptotically stable with the adaptive nonlinear control

\[
\begin{align*}
\dot{x}_1 &= -2x_1 + x_1^2 + \hat{k} x_1 x_2 \\
\dot{x}_2 &= -x_2 + \hat{b} \frac{x_2 x_3}{x_3 + x_2} - \hat{k} x_1 x_2 \\
\dot{x}_3 &= -x_3 + \hat{c} \frac{x_2 x_3}{x_3 + x_2} - a x_3 + u_3
\end{align*}
\]

(11)

with the parameter estimator

\[
\begin{align*}
\dot{\hat{a}} &= -x_1 + e_a \\
\dot{\hat{b}} &= -x_2 + \frac{x_2 x_3}{x_3 + x_2} + e_b \\
\dot{\hat{c}} &= e_c + \frac{x_2 x_3}{x_3 + x_2} + e_c \\
\dot{\hat{k}} &= x_1 x_2 (x_2 - x_1) + e_k
\end{align*}
\]

(12)

**Proof**

The candidate Lyapunov function is taken as

\[
V(x_1, x_2, x_3) = \frac{1}{2} x_1^2 + \frac{1}{2} x_2^2 + \frac{1}{2} x_3^2 + \frac{1}{2} e_a^2 + \frac{1}{2} e_b^2 + \frac{1}{2} e_c^2 + \frac{1}{2} e_k^2
\]

(13)

Consider the unknown estimator

\[
\begin{align*}
e_a &= k - \hat{k} \\
e_b &= b - \hat{b} \\
e_c &= c - \hat{c}
\end{align*}
\]

(14)
The derivative of the unknown estimator is
\[
\hat{e}_a = -\hat{k}, \quad \hat{e}_b = -\hat{b}, \quad \hat{e}_c = -\hat{c},
\]
(15)
The derivative of (13) along (15)
\[
\dot{V} = x_i \dot{x}_1 + x_j \dot{x}_2 + x_k \dot{x}_3 +
\]
\[
e_a \left(-\hat{k}\right) + e_a \left(-\hat{a}\right) + e_b \left(-\hat{b}\right) + e_c \left(-\hat{c}\right)
\]
\[
= x_1 \left(x_1 (1-x_1) - k x_1 x_2 + u_1 \right) + x_2 \left(k x_1 x_2 - b x_2 x_3 / x_2 + x_3 + u_2 \right)
\]
\[
+ x_3 \left(c x_3 / x_3 + x_3 - a x_3 + u_3 \right)
\]
(16)
\[
+ e_a \left(-\hat{k}\right) + e_a \left(-\hat{a}\right) + e_b \left(-\hat{b}\right) + e_c \left(-\hat{c}\right)
\]
By using (11) in the above derivative equation, we get
\[
\dot{V} = -x_1^2 - x_2^2 - x_3^2 + e_a \left(-\hat{a}\right)
\]
\[
+ e_b \left(-\hat{b}\right)
\]
\[
+ e_c \left(-\hat{c}\right)
\]
\[
+ e_k \left(-\hat{k}\right)
\]
(17)
Applying the dynamics of the unknown estimators which is from (12) to the above equation, we get
\[
\dot{V} = -x_1^2 - x_2^2 - x_3^2 + e_a^2 - e_b^2 - e_c^2 - e_k^2
\]
(18)
which is a negative function. Hence by theorem 1, the system (10) is globally asymptotically stable.

**Numerical Simulation with Control and Estimation of the Unknown Parameters**

Now, let us discuss the numerical solutions of the system (10) with nonlinear feedback controls for the fixed parameter values and initial densities, that is, in this section we describe our system (10) with control through numerical simulation works by using the well known MATLAB software and also we have estimated the unknown parameters and presented their figures.

Now, we choose a set of parameters fixed \{a, b, c\} = \{0.2, 0.9, 0.4\} and varying the disease transmission rate \(k\) of susceptible prey and infected prey. We choose the initial densities
\[
x_1 = 0.2345, \quad x_2 = 0.2872, \quad x_3 = 0.1652
\]
and the parameter estimator values
\[
\hat{a} = 0.736563; \quad \hat{b} = 0.16872; \quad \hat{c} = 0.121; \quad \hat{k} = 0.2324
\]
Figure 7, shows the stable behaviour of susceptible prey \(x_1\), infected prey \(x_2\) and the predator \(x_3\) of the controlled system (10) with the parameter estimators
\[
\hat{a} = 0.736563; \quad \hat{b} = 0.16872; \quad \hat{c} = 0.121; \quad \hat{k} = 0.2324
\]
and the initial densities
\[
x_1 = 0.2345, \quad x_2 = 0.2872, \quad x_3 = 0.1652
\]
if the transmissible rate \(0.8<k<1.9\).

Figure 7: The stable behaviour of the system population with control approaches the equilibrium point \((0.1390, 0.1573, 0.1292)\)

Figure 8: The phase portrait approaches stable point of the controlled system with the initial conditions
\[
x_1 = 0.2345, \quad x_2 = 0.2872, \quad x_3 = 0.1652
\]
Figure 8 shows the corresponding phase plot. of control parameters are
\[
\hat{a} = 0.736563; \quad \hat{b} = 0.16872; \quad \hat{c} = 0.121; \quad \hat{k} = 0.2324
\]
and the estimated parameter values are
\[
a = 0.1999; \quad b = 0.8999; \quad c = 0.4 \quad and \quad k = 1.8499.
\]
The figure 9 proves our theoretical work, that is, by Lyapunov stability theory (Hahn, 1967, [18]), the dynamics (10), the system with control (10) is globally asymptotically stable and hence the condition \(\lim_{t \to +\infty} \|x(t)\| = 0\) will be satisfied for all initial conditions \(x(0) \in R^n\).

That is, the chaotic behaviour of the system (4) which is shown in figure 6 are controlled by the adaptive control
inputs of the system (10) for the same parameter values and initial densities which is shown in figure 8

Figure 9: The stable behaviour of the system population (10) with control approaches the point (0,0,0) with
\[ x_1 = 0.2345, \quad x_2 = 0.2872, \quad x_3 = 0.1652, \]

Figure 10 (a), (b), (c) and (d) shows the dynamical estimators of the unknown parameters \( a, b, c \) and \( k \) for the same initial densities and parameter estimators. Therefore the estimated parameter values are

Finally, we conclude that the continuous three species prey-predator model with some unknown parameters can be globally stabilized for arbitrary values of the system parameters and initial densities. Further all the densities of the susceptible prey, infected prey and the predator are converge to the equilibrium states.

Conclusion
In this paper, we discussed the problem of controlling chaos of continuous three species prey-predator model with some unknown parameters. The adaptive control problem of three species prey-predator model with infected prey is studied. The chaotic behaviour of the system is investigated and the unknown parameters of the model were estimated. We observe the dynamic behaviour of the infected prey species on varying the transmissible rate \( k \) and keeping the other parameters fixed. If \( k > 1.7 \) then, we observed that the system becomes unstable. We also observed that the prey-predator system is stable when the disease transmissible rate lies between 0.7 and 1.7, that is, \( 0.7 < k < 1.7 \). The asymptotic stability of the controlled system is proved by
constructing the suitable Lyapunov function. The necessary adaptive control inputs for this asymptotic stability is obtained as nonlinear feedback. We easily understand the stability of the system between without control and with control, for example, in Figure 1 the system without control approaches the stable equilibrium point (0.5141, 0.5547, 0.5551) and in Figure 7, for the parameter values and initial densities, the system with control approach stable equilibrium point (0.1390, 0.1573, 0.1292), that is, the system with control approaches the stability which is faster than the system without control. Numerical simulations were done to observe the effect of the infection and controls on the three species prey-predator model and diagrams were presented which are supporting our results.

References


