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# A Remark on Relationship between Lucas Triangle and Zeckendorf Triangle

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**Abstract**

This article surfaces some of the characteristics of what is referred to as a zeckendorf triangle which is composed of Lucas number multiples of the Lucas sequence. It ascends Lucas number re- lated to the Fibonacci numbers, pascal array.

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**Keywords:** Lucas numbers, Fibonacci numbers, Recurrence relations, Kronecker delta, Zeckendorf representations.

## 1 Introduction

In [1] Charles K.,Cook ., A.G.Shannon studied Generalization of well known identities the Fibonacci and Lucas Sequences respectively by

$$F_{n+2} = F_{n+1} + F_n$$

and

$$L_n = F_{n-1} + F_{n+1}, \forall n \geq 0.$$

The aim of this paper is primarily to collect and relate a number of known Lucas-related triangle in the literature, the most famous of which is the appearance of the Lucas numbers along the diagonals of the pascal triangle [1],[2],[3],[4]. This research article aim at finding out the sequences which arise from diagonal and row sums and the partial central column sums of this triangle. To generalize a result which connects the Fibonacci and Lucas numbers, namely $L_n = F_{n-1} + F_{n+1}, n \geq 1$. Arising out of this is a pascal-type array which is related to the Fibonacci numbers in the same way that the pascal array is related to the Lucas numbers.

## 2 The Zeckendorf Triangle

Rather than re-label the triangle as a Lucas triangle because it has Lucas numbers along its left and right edges, it might be appropriate to refer to it as the "Zeckendorf triangle". It appears below in a left-corrected form because this makes it more obvious that the columns are Lucas number multiples of the numbers in the Lucas sequence than when the triangle is presented in an isosceles format.

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | | | | | | | | | | |
| 3 | 3 | | | | | | | | | |
| 4 | 3 | 4 | | | | | | | | |
| 7 | 6 | 4 | 7 | | | | | | | |
| 11 | 9 | 8 | 7 | 11 | | | | | | |
| 18 | 15 | 12 | 14 | 11 | 18 | | | | | |
| 29 | 24 | 20 | 21 | 22 | 18 | 29 | | | | |
| 47 | 39 | 32 | 35 | 33 | 36 | 29 | 47 | | | |
| 76 | 63 | 52 | 56 | 55 | 54 | 58 | 47 | 76 | | |
| 123 | 102 | 84 | 91 | 88 | 90 | 87 | 94 | 76 | 123 | |
| 199 | 165 | 136 | 147 | 143 | 144 | 145 | 141 | 152 | 123 | 199 |

The column sequences are actually particular cases of the generalized Lucas and Fibonacci sequences $\{L_{m,n}\}$ which satisfy the Lucas partial recurrence relation [3],

$$L_{m,n} = L_{m,n-1} + L_{m,n-2}, m \geq 0, n>2 \tag{1}$$

We now label the sequences of diagonal, row and partial column sums by $\{d_n\}, \{r_n\}, \{c_n\}$ respectively. we observe in turn that,

| n | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $\{d_n\}$ | 1 | 3 | 7 | 10 | 21 | 31 | 59 | 90 | 160 | 250 | 428 |
| $\{r_n\}$ | 1 | 6 | 11 | 24 | 46 | 88 | 163 | 298 | 537 | 958 | 1694 |
| $\{c_n\}$ | 3 | 12 | 28 | 77 | 198 | 522 | 1363 | 3572 | 9348 | 24477 | 64078 |

In this, the $\{c_n\}$ has been formed from the central column of the original isosceles from of the triangle in[4], namely

$$\{c_n\} = \{z_{1,1}, z_{3,2}\} = \{1,3\} \tag{2}$$

in which the $\{z_{i,j}\}$ are the elements of the isosceles form of the Zeckendorf triangle. The Sequences $\{F_n\}$ and $\{L_n\}$ are Fibonacci and Lucas Sequences that are related by the equations,

$$5F_n = L_{n+1} + L_{n-1} \tag{3}$$

$$5L_n = 5F_{n+1} + 5F_{n-1} \tag{4}$$

Then, from the triangle, it can observed that we get the recurrence relations,

$$d_{2n+j} + d_{2n+j+1} + \delta_{1,j} L_{n+j+1} = d_{2n+j+2} \tag{5}$$

in which $\{\delta_{i,j}\}$ is the Kronecker delta and $\{L_n\}$ are the Lucas numbers;Similarly,

$$r_n + r_{n+1} + L_{n+2} = r_{n+2} \tag{6}$$

$$d_{2n-1} + d_{2n} + L_{n+1} = d_{2n+1} \tag{7}$$

$$d_{2n} + d_{2n+1} = d_{2n+2} \tag{8}$$

and

$$c_n + c_{n+1} + S_{n+1} = c_{n+2} \tag{9}$$

where $\{S_n\}$ is the layer susceptibility series for square lattices [2]:

$$\{S_n\} = \{1,13,37,163,247,643,1687,4413,11557,30253,\ldots\ldots\} \tag{10}$$

**Theorem 1:** If $5F_n = L_{n+1} + L_{n-1}$ then $5F_n + 5F_{n+1} + 5F_{n+2} = 2(L_{n+1} + L_{n+3})$.

**Proof :**

$$5F_n + 5F_{n+1} + 5F_{n+2} = L_{n+1} + L_{n-1} + L_{n+2} + L_n + L_{n+3} + L_{n+1}$$

$$= L_{n-1} + L_n + 2L_{n+1} + L_{n+2} + L_{n+3}$$

$$= 2L_{n+1} + L_{n+3} + L_{n+3}$$

$$= 2(L_{n+1} + L_{n+3})$$

**Theorem 2:** If $5L_n = 5F_{n+1} + 5F_{n-1}$ then $L_n + L_{n+1} + L_{n+2} = 2(F_{n+1} + F_{n+3})$.

**Proof :**

$$5L_n + 5L_{n+1} + 5L_{n+2} = 5F_{n+1} + 5F_{n-1} + 5F_{n+2} + 5F_n + 5F_{n+3} + 5F_{n+1}$$

$$= 5F_{n+1} + 5F_{n+1} + 5F_{n+1} + 5F_{n+2} + 5F_{n+3}$$

$$= 10F_{n+1} + 5F_{n+3} + 5F_{n+3}$$

$$= 2(F_{n+1} + F_{n+3})$$

**Theorem 3:** If $L_n, L_{n+1}, L_{n+2}$ are any three Lucas numbers then the following inequalities are holds:

$$a)\, 2(L_n^4 + L_{n+1}^4 + L_{n+2}^4) > 9(L_n L_{n+1} L_{n+2})^{\frac{4}{3}}$$

$$b)\, (L_n^2 + L_{n+1}^2 + L_{n+2}^2) > \sqrt{6}(L_n L_{n+1} L_{n+2})^{\frac{2}{3}}$$

$$c)\, 2(L_n^4 + L_{n+1}^4 + L_{n+2}^4)\left(\frac{1}{L_n^2} + \frac{1}{L_{n+1}^2} + \frac{1}{L_{n+2}^2}\right)^2 > 81$$

$$d)\, 2(L_n^2 + L_{n+1}^2 + L_{n+2}^2)\left(\frac{1}{L_n^2} + \frac{1}{L_{n+1}^2} + \frac{1}{L_{n+2}^2}\right) > 18$$

$$e)\, (L_n L_{n+1} L_{n+2})^2 (L_n^2 + L_{n+1}^2 + L_{n+2}^2)^3 = (L_n^2 L_{n+1}^2 + L_{n+1}^2 L_{n+2}^2 + L_{n+2}^2 L_n^2)^3$$

**Proof :**

a) To solve the inequalities, use the identity,

$$(L_n^2 + L_{n+1}^2 + L_{n+2}^2)^2 = 2(L_n^4 + L_{n+1}^4 + L_{n+2}^4) \qquad (11)$$

To prove this result we replace LHS with $(L_n^2 + L_{n+1}^2 + L_{n+2}^2)$ and then take squares on both sides,

$$(L_n^2 + L_{n+1}^2 + L_{n+2}^2) > 3(L_n L_{n+1} L_{n+2})^{\frac{2}{3}}$$

$$2(L_n^4 + L_{n+1}^4 + L_{n+2}^4) > 9(L_n L_{n+1} L_{n+2})^{\frac{4}{3}}$$

b) By AM-GM inequality for three numbers, with $L_n^4, L_{n+1}^4 and L_{n+2}^4$

$$\frac{L_n^4 + L_{n+1}^4 + L_{n+2}^4}{3} > \left(L_n^4 L_{n+1}^4 L_{n+2}^4\right)^{\frac{1}{3}}$$

Multiply bothsides by 6 and using once again the identity(11),

$$\left(L_n^2 + L_{n+1}^2 + L_{n+2}^2\right)^2 > 6\left(L_n^4 L_{n+1}^4 L_{n+2}^4\right)^{\frac{1}{3}}$$

$$\left(L_n^2 + L_{n+1}^2 + L_{n+2}^2\right) > \sqrt{6}\left(L_n L_{n+1} L_{n+2}\right)^{\frac{2}{3}}$$

c) Using AM-HM inequality with $L_n^2, L_{n+1}^2 and L_{n+2}^2$

$$\frac{L_n^2 + L_{n+1}^2 + L_{n+2}^2}{3} > \frac{3}{\left(\frac{1}{L_n^2} + \frac{1}{L_{n+1}^2} + \frac{1}{L_{n+2}^2}\right)}$$

$$2\left(L_n^4 + L_{n+1}^4 + L_{n+2}^4\right)\left(\frac{1}{L_n^2} + \frac{1}{L_{n+1}^2} + \frac{1}{L_{n+2}^2}\right)^2 > 81$$

d) Take square root and multiply 2 on both sides,

$$2\left(L_n^2 + L_{n+1}^2 + L_{n+2}^2\right)\left(\frac{1}{L_n^2} + \frac{1}{L_{n+1}^2} + \frac{1}{L_{n+2}^2}\right) > 18$$

e) using AM-GM-HM inequality with $L_n^2, L_{n+1}^2 and L_{n+2}^2$

$$L_n^2 + L_{n+1}^2 + L_{n+2}^2 = \frac{L_n^2 L_{n+1}^2 + L_{n+1}^2 L_{n+2}^2 + L_{n+2}^2 L_n^2}{\left(L_n^2 L_{n+1}^2 L_{n+2}^2\right)^{\frac{1}{3}}}$$

$$\left(L_n L_{n+1} L_{n+2}\right)^{\frac{2}{3}}\left(L_n^2 + L_{n+1}^2 + L_{n+2}^2\right) = L_n^2 L_{n+1}^2 + L_{n+1}^2 L_{n+2}^2 + L_{n+2}^2 L_n^2$$

$$\left(L_n L_{n+1} L_{n+2}\right)^2\left(L_n^2 + L_{n+1}^2 + L_{n+2}^2\right)^3 = \left(L_n^2 L_{n+1}^2 + L_{n+1}^2 L_{n+2}^2 + L_{n+2}^2 L_n^2\right)^3$$

**Theorem 4:** If $L_n^2, L_{n+1}^2 and L_{n+2}^2$ are any three Lucas numbers then

$$\left(L_n^2 L_{n+1}^2 + L_{n+1}^2 L_{n+2}^2 + L_{n+2}^2 L_n^2\right)\left(L_n^2 + L_{n+1}^2 + L_{n+2}^2\right) >$$

$$3\left(L_n^2L_{n+1}^2 + L_{n+1}^2L_{n+2}^2 + L_{n+2}^2L_n^2\right)\left(L_nL_{n+1}L_{n+2}\right)^{\frac{2}{3}} > 9\left(L_nL_{n+1}L_{n+2}\right)^2$$

**Proof :**
using the relation AM > GM >HM,

$$\frac{\left(L_n^2 + L_{n+1}^2 + L_{n+1}^2\right)}{3} > \left(L_nL_{n+1}L_{n+2}\right)^{\frac{2}{3}} > \frac{3\left(L_nL_{n+1}L_{n+2}\right)^2}{\left(L_n^2L_{n+1}^2 + L_{n+1}^2L_{n+2}^2 + L_{n+2}^2L_n^2\right)}$$

$$\left(L_n^2L_{n+1}^2 + L_{n+1}^2L_{n+2}^2 + L_{n+2}^2L_n^2\right)\left(L_n^2 + L_{n+1}^2 + L_{n+2}^2\right) >$$

$$3\left(L_n^2L_{n+1}^2 + L_{n+1}^2L_{n+2}^2 + L_{n+2}^2L_n^2\right)\left(L_nL_{n+1}L_{n+2}\right)^{\frac{2}{3}} > 9\left(L_nL_{n+1}L_{n+2}\right)^2$$

**Theorem** 5: If $F_1,F_2,F_3,\ldots,F_n$ are Fibonacci numbers and $L_1,L_2,L_3,\ldots,L_n$ are Lucas numbers then

$$\sqrt{F_1^2 + F_2^2 + \ldots + F_n^2} + \sqrt{L_1^2 + L_2^2 + \ldots + L_n^2} \geq \sqrt{\left(F_1 + L_1\right)^2 + \left(F_2 + L_2\right)^2 + \ldots + \left(F_n + L_n\right)^2}.$$

**Proof :**
using Minkowski-inequality,

$$\left[\sum_{k=1}^{n}\left(F_k + L_k\right)^2\right]^{\frac{1}{2}} \leq \left[\sum_{k=1}^{n}F_k^2\right]^{\frac{1}{2}} + \left[\sum_{k=1}^{n}L_k^2\right]^{\frac{1}{2}}$$

$$\sum_{k=1}^{n}F_k^2 = \left(F_1^2 + F_2^2 + \ldots. + F_n^2\right)^{\frac{1}{2}}$$

$$\sum_{k=1}^{n}L_k^2 = \left(L_1^2 + L_2^2 + \ldots. + L_n^2\right)^{\frac{1}{2}}$$

$$\left[\left(F_1 + L_1\right)^2 + \left(F_2 + L_2\right)^2 + \ldots. + \left(F_n + L_n\right)^2\right]^{\frac{1}{2}} \leq \left(F_1^2 + F_2^2 + \ldots. + F_n^2\right)^{\frac{1}{2}} + \left(L_1^2 + L_2^2 + \ldots. + L_n^2\right)^{\frac{1}{2}}$$

$$\sqrt{F_1^2 + F_2^2 + \ldots + F_n^2} + \sqrt{L_1^2 + L_2^2 + \ldots + L_n^2} \geq \sqrt{\left(F_1 + L_1\right)^2 + \left(F_2 + L_2\right)^2 + \ldots + \left(F_n + L_n\right)^2}$$

**Theorem 6:** If $F_1,F_2,F_3,\ldots,F_n$ are Fibonacci numbers and $L_1,L_2,L_3,\ldots,L_n$ are Lucas numbers then $F_1L_1 + F_2L_2 + \ldots.. + F_nL_n \geq \frac{\left(F_1 + F_2 + \ldots. + F_n\right)\left(L_1 + L_2 + \ldots. + L_n\right)}{n}$.

**Proof :**
using Cauchy-Schwarz inequality,

$$\sum_{k=1}^{n} F_k = (F_1 + F_2 + .... + F_n)$$

$$\sum_{k=1}^{n} L_k = (L_1 + L_2 + .... + L_n)$$

$$n\sum_{k=1}^{n} F_k L_k = n(F_1L_1 + F_2L_2 + .... + F_nL_n)$$

$$(F_1 + F_2 + .... + F_n)\ (L_1 + L_2 + .... + L_n) \le n(F_1L_1 + F_2L_2 + .... + F_nL_n)$$

$$F_1L_1 + F_2L_2 + .......... ... + F_nL_n \ge \frac{(F_1 + F_2 + .... + F_n)(L_1 + L_2 + .... + L_n)}{n}$$

**Theorem 7:** If $L_n^2 = 4(-1)^n + 5F_n^2 then\ L_n^2 + L_{n+1}^2 + L_{n+2}^2 = 5\left[F_n^2 + F_{n+1}^2 + F_{n+2}^2\right] + 14(-1)^n.$

**Proof :**

$$L_n^2 + L_{n+1}^2 + L_{n+2}^2 = 4(-1)^n + 5F_n^2 + 4(-1)^{n+1} + 5F_{n+1}^2 + 4(-1)^{n+2} + 5F_{n+2}^2$$

$$= 4(-1)^n + 5\left[F_n^2 + 5F_{n+1}^2 + 5F_{n+2}^2\right]$$

$$= 4(-1)^n + 5\left[F_{n+1}^2\left(F_{n-1}^2 + F_{n+3}^2\right) + F_{n+2}^2F_n^2 - (-1)^n\right]$$

$$= 5\left[F_n^2 + (-1)^n + F_{n+2}^2 + (-1)^n + F_{n+1}^2 + (-1)^n\right] - (-1)^n$$

$$= 5\left[F_n^2 + F_{n+1}^2 + F_{n+2}^2\right] + 14(-1)^n.$$

## 3 Some observations on Diagonal, Row and Partial Column Sums

$$r_{2n} \equiv 0 (\text{mod } 2) \qquad (12)$$

$$16r_{2n-1} - r_{2n} \equiv 0 (\text{mod } 2) \qquad (13)$$

$$r_{n+2} - 2r_{n+1} + r_n \equiv 0 (\text{mod } 2) \qquad (14)$$

$$c_{n+2} - 2c_{n+1} + c_n - 3 \equiv 0 (\text{mod } 5) \qquad (15)$$

$$c_{2n+1} + c_{2n+2} + c_{2n} - 2 \equiv 0 (\text{mod } 5) \qquad (16)$$

$$r_{2n} c_{2n} \equiv 0 (\text{mod } 2) \qquad (17)$$

$$r_{n+1} c_n \equiv 0 (\text{mod } 2) \qquad (18)$$

$$c_{n+1} d_{2n+3} \equiv 0 (\text{mod } 4) \qquad (19)$$

$$2 d_{2n-1} + 3 r_{2n} \equiv 0 (\text{mod } 2) \qquad (20)$$

$$r_{n+1} d_{n+1} \equiv 0 (\text{mod } 2) \qquad (21)$$

$$d_{2n} r_{2n} c_{2n} \equiv 0 (\text{mod } 3) \qquad (22)$$

$$d_{2n+1} + d_{2n+2} + d_{2n} \equiv 0 (\text{mod } 3) \qquad (23)$$

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