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Existence results for mixed fractional integrodifferential equations in Banach spaces with impulsive and nonlocal conditions

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Abstract

In this manuscript, we shall establish the existence and uniqueness of mild solutions for a class of impulsive fractional integrodifferential equations in Banach spaces. We will derive the results are obtained by using Banach contraction principle and Krasnoselskii's fixed point theorem.

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1. Introduction

The purpose of this paper is to prove the existence and uniqueness of mild solutions for impulsive fractional functional integrodifferential equations of the form

$$D^\alpha x(t) = Ax(t) + f\Big(t, x_t, \int_0^t h_1(t,s,x_s)ds, \int_0^t h_2(t,s,x_s)ds\Big), \quad t \in I = [0,T],\ t \neq t_k, \quad k = 1,2,\ldots,m, \tag{1.1}$$

$$\Delta x|_{t=t_k} = I_k(x(t_k^-)), \quad k = 1,2,\ldots,m, \tag{1.2}$$

$$x_0 = \phi + g(x),\ t \in [-r,0], \tag{1.3}$$

where $T > 0$, D^{α} is Caputo fractional derivative of order $0 < \alpha < 1$, $A : D(A) \subset X \to X$ is the bounded linear operator of an α-resolvent family $\{S_{\alpha}(t) : t \geq 0\}$ defined on a Banach space X, $h_1 : J \times J \times D \to X$, $h_2 : J \times J \times D \to X$ and $f : J \times D \times X \times X \to X$ are given functions, where $D = \{\psi : [-r, 0] \to X$ such that ψ is continuous everywhere except for a finite number of points s at which $\psi(s^-)$ and $\psi(s^+)$ exists and $\psi(s^-) = \psi(s)\}$, $\phi \in D(0 < r < \infty)$, $0 =< t_0 < t_1 < \cdots < t_k < \cdots < t_m < t_{m+1} = T$, $\Delta x|_{t=t_k} = I_k(x(t_k^-))$, $x(t_k^+) = \lim_{h\to 0^+} x(t_k + h)$ and $x(t_k^-) = \lim_{h\to 0^-} x(t_k + h)$ represent the right and left limits of $x(t)$ at $t = t_k$ respectively.

For any continuous function x defined on the interval $[-r, T] - \{t_1, t_2, \ldots, t_m\}$ and any $t \in J$. We denote by x_t be the element of D defined by

$$x_t(\theta) = x(t + \theta),\ \theta \in [-r, 0].$$

Here $x_t(\cdot)$ represents the history of the time $t - r$, upto the present time t. For $\psi \in D$, then $\|\psi\|_D = \sup\{|\psi(\theta)| : \theta \in [-r, 0]\}$.

Fractional order semilinear equations are abstract formulations for many problems arising in engineering and physics. The potential applications of fractional calculus are in diffusion process, electrical science, electrochemistry, viscoelasticity, control science, electro magnetic theory and several more. In fact, such models can be considered as an efficient alternative to the classical nonlinear differential models to simulate many complex processes. In the recent years, there has been a significant development in ordinary and partial differential equations involving fractional derivatives, see the monographs of Kilbas et al. [21], Lakshmikantham et al. [23], Miller and Ross [27], Podlubny [30], Michalski [22] and Tarasov [31] and the papers of [12, 13, 14, 26, 19, 28, 10, 33, 34, 35, 36, 37, 38, 39].

Differential equations with impulsive conditions constitute an important field of research due to their numerous applications in ecology, medicine biology, electrical engineering, and other areas of science. There has been a significant development in impulsive theory especially in the area of impulsive differential equations with fixed moments, see for instance the monographs by Lakshmikantham et al. [24], Bainov et al. [4], Samoilenko et al. [17] and the papers of [8, 11]. Nowadays, many authors [7, 15, 16, 28, 18, 32] have been studied the existence results combined with fractional derivative and impulsive conditions.

In [18], Xiao-Bao Shu et al. studied the existence of mild solutions for impulsive fractional differential equations of the form

$$\begin{aligned} D_t^{\alpha}x(t) &= Ax(t) + f\big(t, x(t)\big), \quad t \in I = [0, T],\ t \neq t_k, \quad k = 1, 2, \ldots, m, \\ x(0) &= x_0 \in X, \\ \Delta x|_{t=t_k} &= I_k(x(t_k^-)), \quad k = 1, 2, \ldots, m, \end{aligned}$$

where $0 < \alpha < 1$, A is a sectorial operator on a Banach space X, D^{α} is the Caputo fractional derivative and by using Banach contraction principle and Leray-Schauder's Alternative fixed point theorem.

Very recently, Archana Chauhan et al. [2] extended the results of [18] into the following impulsive fractional order semilinear evolution equations with nonlocal conditions of the form

$$\begin{aligned} \frac{d^\alpha}{dt^\alpha}x(t) + Ax(t) &= f\Big(t, x(t), x(a_1(t)), \ldots, x(a_m(t))\Big), \\ & \qquad t \in J = [0, T],\ t \neq t_i, \quad i = 1, 2, \ldots, p, \\ x(0) + g(x) &= x_0, \\ \Delta x(t_i) &= I_i(x(t_i^-)), \quad i = 1, 2, \ldots, p, \end{aligned}$$

where $\frac{d^\alpha}{dt^\alpha}$ is Caputo fractional derivative of order $0 < \alpha < 1$, $-A$ generates α-resolvent family $\{S_\alpha(t) : t \geq 0\}$ of bounded linear operators in X and by using Banach contraction principle and Krasnoselskii's fixed point theorem.

Motivated by the above mentioned works [2, 6, 18, 33, 34, 36, 39], we consider the problem $(1.1) - (1.3)$ to study the existence and uniqueness of a mild solution using the solution operator and fixed-point theorems. The rest of this paper is organized as follows: In Section 2, we present some necessary definitions and preliminary results that will be used to prove our main results. The proof of our main results are given in Section 3.

2. Preliminaries

In this section, we mention some definitions and properties required for establishing our results. Let X be a complex Banach space with its norm denoted as $\|\cdot\|_X$, and $L(X)$ represents the Banach space of all bounded linear operators from X into X, and the corresponding norm is denoted by $\|\cdot\|_{L(X)}$. Let $C(J, X)$ denote the space of all continuous functions from J into X with supremum norm denoted by $\|\cdot\|_{C(J,X)}$. In addition, $B_r(x, X)$ represents the closed ball in X with the center at x and the radius r.

A two parameter function of the Mittag-Leffler type is defined by the series expansion

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)} = \frac{1}{2\pi i}\int_{Ha} \frac{\mu^{\alpha-\beta}e^{\mu}}{\mu^\alpha - z}\, d\mu,\ \alpha, \beta > 0,\ z \in C,$$

where Ha is a Hankel path, i.e. a contour which starts and ends at $-\infty$ and encircles the disc $|\mu| \leq |z|^{\frac{1}{\alpha}}$ contour clockwise. For short, $E_\alpha(z) = E_{\alpha,1}(z)$. It is an entire function which provides a simple generalization of the exponent function: $E_1(z) = e^z$ and the cosine function: $E_2(-z^2) = cos(z)$, and plays an important role in the theory of fractional differential equations. The most interesting properties of the Mittag-Leffler functions are associated with their Laplace integral

$$\int_0^\infty e^{-\lambda t} t^{\beta-1} E_{\alpha,\beta}(wt^\alpha)dt = \frac{\lambda^{\alpha-\beta}}{\lambda^\alpha - w},\ \text{Re}\,\lambda > w^{\frac{1}{\alpha}},\ w > 0,$$

see [30] for more details.

Definition 2.1. [2] Caputo derivative of order α for a function $f : [0,\infty) \to R$ is defined as

$$\frac{d^\alpha}{dt^\alpha} f(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s)ds,$$

for $n-1 < \alpha < n, n \in N$. If $0 < \alpha \leq 1$, then

$$\frac{d^\alpha}{dt^\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} f^{(1)}(s)ds.$$

The Laplace transform of the Caputo derivative of order $\alpha > 0$ is given as

$$L\{D_t^\alpha f(t) : \lambda\} = \lambda^\alpha \widehat{f}(\lambda) - \sum_{k=0}^{n-1} \lambda^{\alpha-k-1} f^{(k)}(0); \; n-1 < \alpha \leq n.$$

Definition 2.2. [1] Let A be a closed and linear operator with domain $D(A)$ defined on a Banach space X and $\alpha > 0$. Let $\rho(A)$ be the resolvent set of A. We call A the generator of an α-resolvent family if there exists $w \geq 0$ and a strongly continuous function $S_\alpha : R_+ \to L(X)$ such that $\{\lambda^\alpha : Re\,\lambda > w\} \subset \rho(A)$ and

$$(\lambda^\alpha I - A)^{-1} x = \int_0^\infty e^{-\lambda t} S_\alpha(t) x dt, \; Re\,\lambda > w, \; x \in X.$$

In this case, $S_\alpha(t)$ is called the α-resolvent family generated by A.

Definition 2.3. [3] Let A be a closed and linear operator with domain $D(A)$ defined on a Banach space X and $\alpha > 0$. Let $\rho(A)$ be the resolvent set of A. We call A the generator of an α-resolvent family if there exists $w \geq 0$ and a strongly continuous function $S_\alpha : R_+ \to L(X)$ such that $\{\lambda^\alpha : Re\,\lambda > w\} \subset \rho(A)$ and

$$(\lambda^\alpha I - A)^{-1} x = \int_0^\infty e^{-\lambda t} S_\alpha(t) x dt, \; Re\,\lambda > w, \; x \in X.$$

In this case, $S_\alpha(t)$ is called the solution operator generated by A.

The concept of the solution operator is closely related to the concept of a resolvent family ([29], Chapter 1). For more details on α-resolvent family and solution operators, we refer to [29, 25] and the references therein.

3. Existence Results

In this section, we present and prove the existence of mild solutions for the system $(1.1)-(1.3)$. In order to prove the existence results, we need the following results which is taken from [5, 18]. If $\alpha \in (0,1)$ and $A \in A^{\alpha}(\theta_0, w_0)$, then for any $x \in X$ and $t > 0$, we have

$$\|S_\alpha(t)\| \le Me^{wt},\ \|T_\alpha(t)\| \le Ce^{wt}(1+t^{\alpha-1}),\ t>0,\ w>w_0.$$

Let

$$\widetilde{M}_S := \sup_{0\le t\le T} \|S_\alpha(t)\|_{L(X)},$$
$$\widetilde{M}_T := \sup_{0\le t\le T} Ce^{\omega t}(1+t^{1-\alpha}),$$

where $L(X)$ is the Banach space of bounded linear operators from X into X equipped with its natural topology. So, we have

$$\|\S_\alpha(t)\|_{L(X)} \le \widetilde{M}_S, \quad \|T_\alpha(t)\|_{L(X)} \le t^{1-\alpha}\widetilde{M}_T. \tag{3.1}$$

Let us consider the set functions

$$PC([-r,T],X) = \{x : [-r,T] \to X : x \in C((t_k, t_{k+1}], X),\ k = 0,1,2,\ldots,m$$

and there exist $x(t_k^-)$ and $x(t_k^+)$, $k = 1,2,\ldots,m$ with $x(t_k^-) = x(t_k),\ x_0 = \phi + g(x)\}$. Endowed with the norm

$$\|x\|_{PC} = \sup_{t\in[-r,T]} \|x(t)\|_X,$$

the space $(PC([-r,T],X), \|\cdot\|_{PC})$ is a Banach space.

Lemma 3.1. [2, 18] If f satisfies the uniform Holder condition with the exponent $\beta \in (0,1]$ and A is a sectorial operator, then the unique solution of the Cauchy problem

$$D^\alpha x(t) = Ax(t) + f\Big(t, x_t, \int_0^t h_1(t,s,x_s)ds, \int_0^T h_2(t,s,x_s)ds\Big),$$
$$t > t_0,\ t_0 \in R,\ 0<\alpha<1$$
$$x_0 = \phi + g(x),\ t\in[-r,0]$$

is given by

$$x(t) = S_\alpha(t-t_0)(x(t_0^+)) + \int_{t_0}^t T_\alpha(t-s) f\Big(s, x_s, \int_0^t h_1(s,\tau,x_\tau)d\tau, \int_0^T h_2(s,\tau,x_\tau)d\tau\Big)ds,$$

where

$$S_\alpha(t) = E_{\alpha,1}(At^\alpha) = \frac{1}{2\pi i}\int_{\widehat{B_r}} e^{\lambda t}\frac{\lambda^{\alpha-1}}{\lambda^\alpha - A}d\lambda,\ T_\alpha(t) = t^{\alpha-1}E_{\alpha,\alpha}(At^\alpha)$$
$$= \frac{1}{2\pi i}\int_{\widehat{B_r}} e^{\lambda t}\frac{1}{\lambda^\alpha - A}d\lambda,$$

$\widehat{B_r}$ denotes the Bronwich path, $T_\alpha(t)$ is called the α-resolvent family, $S_\alpha(t)$ is the solution operator, generated by A.

Now, we define the mild solution of a system $(1.1)-(1.3)$.

Definition 3.2. A function $x(\cdot) \in PC$ is called a mild solution of the system $(1.1)-(1.3)$ if $x_0 = \phi + g(x)$ on $[-r, 0]$; $\Delta x|_{t=t_k} = I_k(x(t_k^-)), k = 1, 2, \ldots, m$ and satisfies the following integral equation

$$x(t) = \begin{cases} S_\alpha(t)[\phi(0) + gx(0)] \\ \qquad + \int_0^t T_\alpha(t-s) f\left(s, x_s, \int_0^s h_1(s, \tau, x_\tau)d\tau, \int_0^T h_2(s, \tau, x_\tau)d\tau\right) ds, \ t \in (0, t_1]; \\ S_\alpha(t-t_1)(x(t_1^-) + I_1(x(t_1^-))) \\ \qquad + \int_{t_1}^t T_\alpha(t-s) f\left(s, x_s, \int_0^s h_1(s, \tau, x_\tau)d\tau, \int_0^T h_2(s, \tau, x_\tau)d\tau\right) ds, \ t \in (t_1, t_2]; \\ \vdots \\ S_\alpha(t-t_m)(x(t_m^-) + I_1(x(t_m^-))) \\ \qquad + \int_{t_m}^t T_\alpha(t-s) f\left(s, x_s, \int_0^s h_1(s, \tau, x_\tau)d\tau, \int_0^T h_2(s, \tau, x_\tau)d\tau\right) ds, \ t \in (t_m, T]. \end{cases}$$

From Lemma (3.1) we can verify this definition. Now we need the following assumptions:

H₁ $f : J \times D \times X \times X \to X$ is continuous and there exist functions $L \in L^1(J, R^+)$ such that

$$\|f(t, x_t, u_1, u_2) - f(t, y_t, v_1, v_2)\|_X \le L\Big[\|x-y\| + \|u_1 - v_1\| + \|u_2 - v_2\|\Big],$$
$$\text{for } x, y \in PC, \ u_i, v_i \in X, i = 1, 2.$$

H₂ $h_1 : J \times J \times D \to X$ is continuous and there exists a constant $\Lambda_1 > 0$ such that for all $(t, s) \in J \times J$

$$\left\| \int_0^t [h(t, s, x_s) - h(t, s, y_s)]ds \right\|_X \le \Lambda_1 \|x - y\|_{PC}.$$

H₃ $h_2 : J \times J \times D \to X$ is continuous and there exists a constant $\Lambda_2 > 0$ such that for all $(t, s) \in J \times J$

$$\left\| \int_0^t [h_2(t, s, x_s) - h_2(t, s, y_s)]ds \right\|_X \le \Lambda_2 \|x - y\|_{PC}.$$

H₃ The function $I_k : X \to X$ are continuous and there exists $\Psi_k > 0$ such that

$$\|I_k(x) - I_k(y)\|_X \le \Psi_k \|x - y\|, \ x, y \in X, \ k = 1, 2, \ldots, m.$$

Theorem 3.3. Assume that $(H_1) - (H_3)$ are satisfied and

$$\Big[\widetilde{M}_S(\Psi_i + 1) + \frac{1}{\alpha}\widetilde{M}_T T^\alpha L(1 + \Lambda_1 + \Lambda_2)\Big] < 1.$$

Then the impulsive differential system (1.1) − (1.3) has a unique mild solution $x \in PC([-r, T], X)$.

Proof. We define the operator $N : PC([-r, T], X) \to PC([-r, T], X)$ by

$$\Upsilon x(t) = \begin{cases} S_\alpha(t)[\phi(0) + g(0)] \\ \quad + \int_0^t T_\alpha(t-s) f\left(s, x_s, \int_0^s h_1(s,\tau,x_\tau)d\tau, \int_0^T h_2(s,\tau,x_\tau)d\tau\right) ds, \ t \in (0, t_1]; \\ S_\alpha(t - t_1)(x(t_1^-) + I_1(x(t_1^-))) \\ \quad + \int_{t_1}^t T_\alpha(t-s) f\left(s, x_s, \int_0^s h_1(s,\tau,x_\tau)d\tau, \int_0^T h_2(s,\tau,x_\tau)d\tau\right) ds, \ t \in (t_1, t_2]; \\ \vdots \\ S_\alpha(t - t_m)(x(t_m^-) + I_1(x(t_m^-))) \\ \quad + \int_{t_m}^t T_\alpha(t-s) f\left(s, x_s, \int_0^s h_1(s,\tau,x_\tau)d\tau, \int_0^T h_2(s,\tau,x_\tau)d\tau\right) ds, \ t \in (t_m, T]. \end{cases}$$

Note that Υ is well defined on $PC([-r, T], X)$. Let us take $t \in (0, t_1]$ and $x, y \in PC([-r, T], X)$. From the equation (3.1) and the hypothesis $(H_1) - (H_2)$, we have

$$\begin{aligned} \|(\Upsilon_1 x)(t) - (\Upsilon_2 y)(t)\|_X &\le \widetilde{M}_T \frac{1}{\alpha} T^\alpha L\Big[\|x_t - y_t\|_D + \Lambda_1\|x_t - y_t\|_D + +\Lambda_2\|x_t - y_t\|_D\Big] \\ &\le \widetilde{M}_T \frac{1}{\alpha} T^\alpha L\Big[\|x - y\|_{PC} + \Lambda_1\|x - y\|_{PC} + \Lambda_2\|x - y\|_{PC}\Big] \\ &\le \widetilde{M}_T \frac{1}{\alpha} T^\alpha L(1 + \Lambda_1 + \Lambda_2)\|x - y\|_{PC}. \end{aligned}$$

For $t \in (t_1, t_2]$, and by using (3.1), $(H_1) - (H_3)$, we have

$$\begin{aligned} &\|(\Upsilon_1 x)(t) - (\Upsilon_2 y)(t)\|_X \\ &\le \widetilde{M}_S(1 + \Psi_1)\|x - y\|_{PC} \\ &\quad + \int_0^t (t-s)^{\alpha-1}\widetilde{M}_T L\Big[\|x - y\|_{PC} + \Lambda_1\|x - y\|_{PC} + \Lambda_2\|x - y\|_{PC}\Big]ds \\ &\le \Big[\widetilde{M}_S(1 + \Psi_1) + \frac{1}{\alpha}\widetilde{M}_T T^\alpha L(1 + \Lambda_1 + \Lambda_2)\Big]\|x - y\|_{PC}. \end{aligned}$$

Similarly, for $t \in (t_i, t_{i+1}]$

$$\|(\Upsilon_1 x)(t) - (\Upsilon_2 y)(t)\|_X \le \Big[\widetilde{M}_S(1 + \Psi_i) + \frac{1}{\alpha}\widetilde{M}_T T^\alpha L(1 + \Lambda_1 + \Lambda_2)\Big]\|x - y\|_{PC}.$$

and for $t \in (t_m, T]$

$$\|(\Upsilon_1 x)(t) - (\Upsilon_2 y)(t)\|_X \le \Big[\widetilde{M}_S(1 + \Psi_m) + \frac{1}{\alpha}\widetilde{M}_T T^\alpha L(1 + \Lambda_1 + \Lambda_2)\Big]\|x - y\|_{PC}.$$

Thus, for all $t \in [0, T]$, we have

$$\|(\Upsilon_1 x) - (\Upsilon_2 y)\|_{PC} \leq \max_{1\leq i\leq m}\Big[\widetilde{M}_S(1+\Psi_i) + \frac{1}{\alpha}\widetilde{M}_T T^\alpha L(1+\Lambda_1+\Lambda_2)\Big]\|x-y\|_{PC}.$$

Since $\max_{1\leq i\leq m}\Big[\widetilde{M}_S(\Psi_i+1)+\frac{1}{\alpha}\widetilde{M}_T T^\alpha L(1+\Lambda_1+\Lambda_2)\Big] < 1$, Ψ is a contraction. Therefore Υ has a unique fixed point by Banach contraction principle. This completes the proof. ■

Our next existence result is based on the Krasnoselkii's fixed point theorem.

Theorem 3.4. [20] Let B be a closed convex and nonempty subset of a Banach space X. Let P and Q be two operators such that (i) $Px + Qy \in B$ whenever $x, y \in B$,(ii) P is compact and continuous, (iii) Q is a contraction mapping. Then there exists $z \in B$ such that $z = Pz + Qz$.

Now, we list the following hypothesis:

H$_4$ For each $(t, s) \in I \times I$, the functions $h_1(t, s, \cdot)$, $h_2(t, s, \cdot) : D \to X$ is continuous, and for each $x \in D$ the function $h_1(\cdot, \cdot, x), h_2(\cdot, \cdot, x) : I \times I \to X$ is strongly measurable.

H$_5$ For each $t \in J$, the function $f(t, \cdot, \cdot, \cdot) : D \times X \times X \to X$ is continuous, and for each $(x, y, z) \in D \times X$ the function $f(\cdot, x, y, z) : I \to X$ is strongly measurable.

H$_6$ There exists a continuous function $p_1 : J \to R = [0, \infty]$ such that

$$\Big\|\int_0^t h_1(t, s, x_s)ds\Big\|_X \leq p_1(t)\psi(\|x\|_D),\ \text{for every } t, s \in I \text{ and } x \in D,$$

where $\psi : [0, +\infty) \to (0, \infty)$ is a continuous non-decreasing function.

H$_7$ There exists a continuous function $p_2 : I \to R = [0, \infty]$ such that

$$\Big\|\int_0^T h_2(t, s, x_s)ds\Big\|_X \leq p_2(t)\psi(\|x\|_D),\ \text{for every } t, s \in I \text{ and } x \in D,$$

where $\psi : [0, +\infty) \to (0, \infty)$ is a continuous non-decreasing function.

H$_8$ There exists a continuous function $p_3 : I \to R = [0, \infty]$ such that

$$\Big\|f(t, x, y, z)\Big\|_X \leq p_3(t)\psi(\|x\|_D) + \|y\| + \|z\|,$$
$$\text{for every } t, s \in I \text{ and } x \in D,\ \ y, z \in X.$$

where $\psi : [0, +\infty) \to (0, \infty)$ is a continuous non-decreasing function.

H$_9$ The function $I_k : X \to X$ are continuous and there exists $\Psi > c_1$ such that

$$\Psi = \max_{1\leq k\leq m, x\in B_r}\{\|I_k(x)\|_X\}.$$

Theorem 3.5. Assume that $(H_4) - (H_9)$ are satisfied and

$$\left[\widetilde{M}_T \frac{1}{\alpha} T^\alpha L(1 + \Lambda_1 + \Lambda_2)\right] < 1.$$

Then the impulsive differential problem $(1.1) - (1.3)$ has at least one mild solution on $PC([-r, T], X)$.

Proof. Choose $r > \left[\widetilde{M}_S(r + \Psi) + \widetilde{M}_T \frac{1}{\alpha} T^\alpha \psi(r)(p_3(t) + p_1(t) + p_2(t))\right]$ and consider $B_r = \{x \in PC([-r, T], X) : \|x\|_{PC} \leq r\}$, then B_r is a bounded, closed convex subset in $PC([-r, T], X)$. Define on B_r the operators P and Q by:

$$(Px)(t) = \begin{cases} S_\alpha(t)[\phi(t) + gx(0)], & t \in [0, t_1];\\ S_\alpha(t - t_1)(x(t_1^-) + I_1(x(t_1^-))), & t \in (t_1, t_2];\\ \vdots & \\ S_\alpha(t - t_m)(x(t_m^-) + I_1(x(t_m^-))), & t \in (t_m, T]. \end{cases}$$

$$(Qx)(t) = \begin{cases} \int_0^t T_\alpha(t-s) f\left(s, x_s, \int_0^s h_1(s, \tau, x_\tau)d\tau, \int_0^T h_2(s, \tau, x_\tau)d\tau\right) ds, & t \in (0, t_1];\\ \int_0^t T_\alpha(t-s) f\left(s, x_s, \int_0^s h_1(s, \tau, x_\tau)d\tau, \int_0^T h_2(s, \tau, x_\tau)d\tau\right) ds, & t \in (t_1, t_2];\\ \vdots & \\ \int_0^t T_\alpha(t-s) f\left(s, x_s, \int_0^s h_1(s, \tau, x_\tau)d\tau, \int_0^T h_2(s, \tau, x_\tau)d\tau\right) ds, & t \in (t_m, T]. \end{cases}$$

The proof will be given in five steps:

Step 1: We show that $Px + Qy \in B_r$, whenever $x, y \in B_r$. Let $x, y \in B_r$, then

$$\|Px + Qy\|_{PC} \leq \begin{cases} \|S_\alpha(t)\|_{L(X)}[\|\phi\|_X + \|(0)\|_X] \\ \quad + \int_0^t \|T_\alpha(t-s)\|_{L(X)} \|f\left(s, x_s, \int_0^s h_1(s, \tau, x_\tau)d\tau, \int_0^T h_2(s, \tau, x_\tau)d\tau\right)\|_X ds, \ t \in (0, t_1];\\ \|S_\alpha(t - t_1)\|_{L(X)}\Big[\|x(t_1^-)\| + \|I_1(x(t_1^-))\|\Big]_X \\ \quad + \int_{t_1}^t \|T_\alpha(t-s)\|_{L(X)} \|f\left(s, x_s, \int_0^s h_1(s, \tau, x_\tau)d\tau, \int_0^T h_2(s, \tau, x_\tau)d\tau\right)\|_X ds, \ t \in (t_1, t_2];\\ \vdots \\ \|S_\alpha(t - t_m)\|_{L(X)}\Big[\|x(t_m^-)\| + \|I_1(x(t_m^-))\|\Big]_X \\ \quad + \int_{t_m}^t \|T_\alpha(t-s)\|_{L(X)} \|f\left(s, x_s, \int_0^s h_1(s, \tau, x_\tau)d\tau, \int_0^T h_2(s, \tau, x_\tau)d\tau\right)\|_X ds, \ t \in (t_m, T]. \end{cases}$$

$$\leq \begin{cases} \widetilde{M}_S(r) + \widetilde{M}_T \dfrac{T^\alpha}{\alpha} \, [\psi(r)(p_3(t) + p_1(t) + p_2(t))], & t \in (0, t_1]; \\ \widetilde{M}_S(r+\Psi) + \widetilde{M}_T \dfrac{T^\alpha}{\alpha} \, [\psi(r)(p_3(t) + p_1(t) + p_2(t))], & t \in (t_1, t_2]; \\ \vdots \\ \widetilde{M}_S(r+\Psi) + \widetilde{M}_T \dfrac{T^\alpha}{\alpha} \, [\psi(r)(p_3(t) + p_1(t) + p_2(t))], & t \in (t_m, T]. \end{cases}$$

Which implies

$$\|Px + Qy\|_{PC} \leq \Big[\widetilde{M}_S(r+\Psi) + \widetilde{M}_T \frac{T^\alpha}{\alpha}[\psi(r)(p_3(t) + p_1(t) + p_2(t))]\Big] \leq r.$$

Step 2: We show that the operator $(Px)(t)$ is continuous in B_r. For this purpose, let $\{x_n\}$ be a sequence in B_r such that $x_n \to x$ in B_r. Then for every $t \in J$, we have

$$\|(Px_n)(t) - (Px)(t)\|_X \leq \begin{cases} 0, \ t \in (0, t_1]; \\ \|S_\alpha(t-t_1)\|_{L(X)} \\ \quad (\times)\Big[\|x_n(t_1^-) - x(t_1^-)\|_X + \|I_1(x_n(t_1^-)) - x(t_1^-)\|_X\Big], t \in (t_1, t_2]; \\ \vdots \\ \|S_\alpha(t-t_m)\|_{L(X)} \\ \quad (\times)\Big[\|x_n(t_m^-) - x(t_m^-)\|_X + \|I_1(x_n(t_m^-)) - x(t_m^-)\|_X\Big], t \in (t_m, T]. \end{cases}$$

Since the functions $I_k, k = 1, 2, \ldots, m$ are continuous, $\lim_{n\to\infty} \|Px_n - Px\|_{PC} = 0$ in B_r. This implies that the mapping P is continuous on B_r.

Step 3: P maps bounded sets into bounded sets in $PC([-r, T], X)$.
Let us prove that for any $r > 0$ there exists a $\gamma > 0$ such that for $x \in B_r = \{x \in PC([-r, T], X) : \|x\|_{PC} \leq r\}$, we have $\|Px\|_{PC} \leq \gamma$. Indeed, we have for any $x \in B_r$

$$\|(Px)(t)\|_X = \begin{cases} \|S_\alpha(t)\|_{L(X)}[\|\phi\|_X + \| g(x)\|_X], & t \in [0, t_1]; \\ \|S_\alpha(t-t_1)\|_{L(X)}\Big[\|x(t_1^-)\|_X + \|I_1 x(t_1^-))\|_X\Big], & t \in (t_1, t_2]; \\ \vdots \\ \|S_\alpha(t-t_m)\|_{L(X)}\Big[\|x(t_m^-)\|_X + \|I_1(x(t_m^-))\|_X\Big], & t \in (t_m, T]. \end{cases}$$

$$\leq \begin{cases} \widetilde{M}_S(r + c_1), \ t \in (0, t_1]; \\ \widetilde{M}_S(r + \Psi), \ t \in (t_1, t_2]; \\ \vdots \\ \widetilde{M}_S(r + \Psi), \ t \in (t_m, T]. \end{cases}$$

Which implies that $\|Px\|_{PC} \leq \widetilde{M}_S(r+\Psi) = \gamma$.

Step 4: We prove that $P(B_r)$ is equicontinuous with B_r.
For $0 \leq u \leq v \leq T$, we have

$$\|(Px)(v) - (Px)(u)\|_X$$
$$\leq \begin{cases} \|S_\alpha(v) - S_\alpha(u)\|_{L(X)}[\|\phi\|_X + \| g(x)\|_X], & 0 \leq u < v \leq t_1; \\ \|S_\alpha(v-t_1) - S_\alpha(u-t_1)\|_{L(X)} & \\ \quad (\times)\Big[\|x(t_1^-)\|_X + \|I_1(x(t_1^-))\|_X\Big], \; t_1 < u < v \leq t_2; & \\ \vdots & \\ \|S_\alpha(v-t_m) - S_\alpha(u-t_m)\|_{L(X)} & \\ \quad (\times)\Big[\|x(t_m^-)\|_X + \|I_m(x(t_m^-))\|_X\Big], \; t_p < u < v \leq T. & \end{cases}$$
$$\leq \begin{cases} r\|S_\alpha(v) - S_\alpha(u)\|_{L(X)}, & 0 \leq u < v \leq t_1; \\ (r+\Psi)\|S_\alpha(v-t_1) - S_\alpha(u-t_1)\|_{L(X)}, & t_1 < u < v \leq t_2; \\ \vdots & \\ (r+\Psi)\|S_\alpha(v-t_m) - S_\alpha(u-t_m)\|_{L(X)}, & t_p < u < v \leq T. \end{cases}$$

Therefore, the continuity of the function $t \to \|S(t)\|$ allows us to conclude that

$$\lim_{u\to v} \|S_\alpha(v-t_i) - S_\alpha(u-t_i)\|_{L(X)} = 0, \; i = 1, 2, \ldots, m \text{ and}$$
$$\lim_{u\to v} \|S_\alpha(v) - S_\alpha(u)\|_{L(X)} = 0.$$

Finally, combining Step 2 to Step 4 with the Ascoli's theorem, we deduce that the operator P is compact.

Step 5: We show that Q is contraction mapping.
Let $x, y \in B_r$ and we have

$$\|(Qx)(t) - (Qy)(t)\|_X \leq \begin{cases} \displaystyle\int_0^t \|T_\alpha(t-s)\|_{L(X)} \\ \quad (\times)\|f(s, x_s, \int_0^t h_1(s,\tau,x_\tau)d\tau, \int_0^T h_2(s,\tau,x_\tau)d\tau) \\ \quad\quad -f(s, y_s, \int_0^t h_1(s,\tau,x_\tau)d\tau, \int_0^T h_2(s,\tau,x_\tau)d\tau)\|_X ds, \\ \hfill t \in (0, t_1]; \end{cases}$$

$$\leq \begin{cases} \int_0^t \|T_\alpha(t-s)\|_{L(X)} \\ \quad (\times)\|f(s, x_s, \int_0^t h_1(s,\tau,x_\tau)d\tau, \int_0^T h_2(s,\tau,x_\tau)d\tau) \\ \quad -f(s, y_s, \int_0^t h_1(s,\tau,x_\tau)d\tau, \int_0^T h_2(s,\tau,x_\tau)d\tau)\|_X ds, \\ \hfill t \in (t_1, t_2]; \\ \vdots \\ \int_0^t \|T_\alpha(t-s)\|_{L(X)} \\ \quad (\times)\|f(s, x_s, \int_0^t h_1(s,\tau,x_\tau)d\tau, \int_0^T h_2(s,\tau,x_\tau)d\tau) \\ \quad -f(s, y_s, \int_0^t h_1(s,\tau,x_\tau)d\tau, \int_0^T h_2(s,\tau,x_\tau)d\tau)\|_X ds, \\ \hfill t \in (t_m, T]. \end{cases}$$

$$\leq \begin{cases} \widetilde{M}_T \frac{T^\alpha}{\alpha} L[1+\Lambda_1+\Lambda_2]\|x-y\|_{PC}, & t \in (0, t_1]; \\ \widetilde{M}_T \frac{T^\alpha}{\alpha} L[1+\Lambda_1+\Lambda_2]\|x-y\|_{PC}, & t \in (t_1, t_2]; \\ \vdots \\ \widetilde{M}_T \frac{T^\alpha}{\alpha} L[1+\Lambda_1+\Lambda_2]\|x-y\|_{PC}, & t \in (t_m, T]. \end{cases}$$

Since $\left[\widetilde{M}_T \frac{T^\alpha}{\alpha} L[1+\Lambda_1+\Lambda_2]\right] < 1$, then Q is a contraction mapping. Hence, by the Krasnoselkii's theorem, we can conclude that $(1.1) - (1.3)$ has atleast one solution on $PC([-r, T], X)$. This completes the proof of the theorem. ∎

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