International Journal of Applied Engineering Research
ISSN 0973-4562 Volume 10, Number 12 (2015) pp. 31513-31525

http://www.ripublication.com

Some Properties of Subclass of Analytic Functions with Respect to 2k-Symmetric Conjugate Points

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Abstract

In the present paper, we introduce new subclass $\mathcal{P}_{SC}^{(k)}(\rho, \delta, \lambda, \alpha)$ of analytic function with respect to 2k-symmetric conjugate points. Such results as integral representations, convolution conditions and coefficient inequalities for this class is provided.

2000 Mathematics Subject Classification: 30C45.

Keywords and Phrases: Analytic functions, Hadamard product, 2k-symmetric conjugate points.

1. Introduction

Let $\mathcal{A}$ denote the class of functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk.

$$\Delta := \{z : z \in \mathbb{C} \text{ and } |z| < 1\}.$$

Also let $S^*(\alpha)$ and $C(\alpha)$ denote the familiar subclasses of $\mathcal{A}$ consisting of functions which are starlike and convex of order α $(0 \le \alpha < 1)$ in Δ, respectively.

Let $S_{SC}^{(k)}(\alpha)$ denote the class of functions in $\mathcal{A}$ satisfying the following inequality:

$$\mathbb{R}\left(\frac{zf'(z)}{f_{2k}(z)}\right) > \alpha. \quad (z \in \Delta), \tag{1.2}$$

where $0 \le \alpha < 1$, $k \ge 2$ is a fixed positive integer and $f_{2k}(z)$ is defined by the following equality:

$$f_{2k}(z) = \frac{1}{2k}\sum_{v=0}^{k-1}(\varepsilon^{-v} f(\varepsilon^{v} z) + \varepsilon^{v}\overline{f(\varepsilon^{v}\overline{z})}), \quad \left(\varepsilon = \exp\left(\frac{2\pi i}{k}\right); z \in \Delta\right) \tag{1.3}$$

And a function $f(z) \in \mathcal{A}$ is in the class $\mathcal{C}_{SC}^{(k)}(\alpha)$ if and only if $zf'(z) \in S_{SC}^{(k)}(\alpha)$. The class $S_{SC}^{(k)}(0)$ of functions starlike with respect to 2k-symmetric conjugate points was introduced and investigated by Al-Amiri et al. [1].

Let $\mathcal{T}$ be the subclass of $\mathcal{A}$ consisting of all functions which are of the form:

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n \quad (a_n \ge 0).$$

We denote by S^*, $\mathcal{K}$, $\mathcal{C}$ and $\mathcal{C}^*$ the familiar subclass of $\mathcal{A}$ consisting of functions which are, respectively, starlike, convex, close-to-convex and quasi-convex in Δ. Thus, by definition, we have (see, for details [4, 5, 6, 7]).

$$S^* = \left\{ f : f \in \mathcal{A} \text{ and } \mathbb{R}\left\{\frac{zf'(z)}{f(z)}\right\} > 0, \ (z \in \Delta) \right\},$$

$$\mathcal{K} = \left\{ f : f \in \mathcal{A} \text{ and } \mathbb{R}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 0, \ (z \in \Delta) \right\},$$

$$\mathcal{C} = \left\{ f : f \in \mathcal{A}, \exists g \in S^* : \text{such that } \mathbb{R}\left\{\frac{zf'(z)}{g(z)}\right\} > 0, \ (z \in \Delta) \right\} \text{ and}$$

$$\mathcal{C}^* = \left\{ f : f \in \mathcal{A}, \exists g \in \mathcal{K} : \text{such that } \mathbb{R}\left\{\frac{(zf'(z))'}{g(z)}\right\} > 0, \ (z \in \Delta) \right\}.$$

Definition 1.1.
Let $\mathcal{T}(\rho, \delta, \lambda, \alpha)$ be the subclass of $\mathcal{T}$ consisting of functions f(z) which satisfy the inequality:

$$\mathbb{R}\left(\frac{\dfrac{\lambda\delta z^3 f'''(z)+(2\lambda\delta+\lambda-\delta)z^2 f''(z)+zf'(z)}{\lambda\delta z^2 f''(z)+(\lambda-\delta)zf'(z)+(1-\lambda+\delta)f(z)}}{\rho\left(\dfrac{\lambda\delta z^3 f'''(z)+(2\lambda\delta+\lambda-\delta)z^2 f''(z)+zf'(z)}{\lambda\delta z^2 f''(z)+(\lambda-\delta)zf'(z)+(1-\lambda+\delta)f(z)}\right)+(1-\rho)}\right)>\alpha,\quad (z\in\Delta) \qquad (1.4)$$

for some α $(0 \le \alpha < 1)$, λ $(0 \le \lambda < 1)$, δ $(0 \le \delta < 1)$ and ρ $(0 \le \rho \le 1)$. If $\rho = 0$ and $\delta = 0$, a function $f(z) \in \mathcal{A}$ is in the class $\mathcal{C}(\lambda, \alpha)$. This class was first introduced and investigated by Altintas and Owa [2], then was studied by Aouf et al.[3].

We now introduce the following subclass of $\mathcal{A}$ with respect to 2k-symmetric conjugate points and obtain some interesting results.

A function $f(z) \in \mathcal{A}$ is in the class $\mathcal{P}_{SC}^{(k)}(\rho,\delta,\lambda,\alpha)$ if it satisfies the following inequality:

$$\mathbb{R}\left(\frac{\dfrac{\lambda\delta z^3 f'''(z)+(2\lambda\delta+\lambda-\delta)z^2 f''(z)+zf'(z)}{\lambda\delta z^2 f''_{2k}(z)+(\lambda-\delta)zf'_{2k}(z)+(1-\lambda+\delta)f_{2k}(z)}}{\rho\left(\dfrac{\lambda\delta z^3 f'''(z)+(2\lambda\delta+\lambda-\delta)z^2 f''(z)+zf'(z)}{\lambda\delta z^2 f''_{2k}(z)+(\lambda-\delta)zf'_{2k}(z)+(1-\lambda+\delta)f_{2k}(z)}\right)+(1-\rho)}\right)>\alpha,\quad (z\in\Delta) \qquad (1.5)$$

where $0 \le \alpha < 1$, $0 \le \lambda < 1$, $0 \le \delta < 1$, $0 \le \rho \le 1$ and $f_{2k}(z)$ is defined the equality (1.3). If $\delta = 0$, a function $f(z) \in \mathcal{A}$ is in the class $\mathcal{P}_{SC}^{(k)}(\rho,\delta,\lambda,\alpha)$ which was studied by B. Srutha Keerthi and P. Lokesh [8].

For $\delta = 0$ and $\lambda = 0$ in $\mathcal{P}_{SC}^{(k)}(\rho,\delta,\lambda,\alpha)$ we get $S_{SC}^{(k)}(\rho,\alpha)$ [10].

Lemma 1.1.
Let $\gamma \ge 0$ and $f \in \mathcal{C}$, then

$$F(z)=\frac{1+\gamma}{z^{\gamma}}\int_0^z f(t)t^{\gamma-1}dt \in \mathcal{C}.$$

This lemma is a special case of Theorem 4 in [9].

Lemma 1.2. [5]
Let $0 < \beta \le 1$ and $f \in \mathcal{C}^*$, then

$$F(z)=\frac{1}{\beta}z^{1-\frac{1}{\beta}}\int_0^z f(t)t^{\frac{1}{\beta}-2}dt \in \mathcal{C}^* \subset \mathcal{C}.$$

Lemma 1.3.
Let $0 \le \lambda \le 1$, $0 \le \delta \le 1$ and $0 \le \alpha < 1$, then we have $\mathcal{P}_{SC}^{(k)}(\rho,\delta,\lambda,\alpha) \subset \mathcal{C} \subset S$. .

Proof,

Let $F(z) = \lambda\delta z^2 f''(z) + (\lambda - \delta)zf'(z) + (1 - \lambda + \delta)f(z)$, $F_{2k}(z) = \lambda\delta z^2 f''_{2k}(z) + (\lambda - \delta)zf'_{2k}(z) + (1 - \lambda + \delta)f_{2k}(z)$ with $f(z) \in \mathcal{P}_{SC}^{(k)}(\rho, \delta, \lambda, \alpha)$, substituting z by ε^{μ} z (μ = 0, 1, 2, ..., k–1) in (1.5), we get

$$\mathbb{R}\left(\frac{\dfrac{\lambda\delta(\varepsilon^{\mu}z)^3 f'''(\varepsilon^{\mu}z)+(2\lambda\delta+\lambda-\delta)(\varepsilon^{\mu}z)^2 f''(\varepsilon^{\mu}z)+(\varepsilon^{\mu}z)f'(\varepsilon^{\mu}z)}{\lambda\delta(\varepsilon^{\mu}z)^2 f''_{2k}(\varepsilon^{\mu}z)+(\lambda-\delta)(\varepsilon^{\mu}z)f'_{2k}(\varepsilon^{\mu}z)+(1-\lambda+\delta)f_{2k}(\varepsilon^{\mu}z)}}{\rho\left(\dfrac{\lambda\delta(\varepsilon^{\mu}z)^3 f'''(\varepsilon^{\mu}z)+(2\lambda\delta+\lambda-\delta)(\varepsilon^{\mu}z)^2 f''(\varepsilon^{\mu}z)+(\varepsilon^{\mu}z)f'(\varepsilon^{\mu}z)}{\lambda\delta(\varepsilon^{\mu}z)^2 f''_{2k}(\varepsilon^{\mu}z)+(\lambda-\delta)(\varepsilon^{\mu}z)f'_{2k}(\varepsilon^{\mu}z)+(1-\lambda+\delta)f_{2k}(\varepsilon^{\mu}z)}\right)+(1-\rho)}\right)>\alpha, \qquad (1.6)$$

From inequality (1.6) we have

$$\mathbb{R}\left(\frac{\dfrac{\lambda\delta(\overline{\varepsilon^{\mu}z})^3 \overline{f'''(\varepsilon^{\mu}z)}+(2\lambda\delta+\lambda-\delta)(\overline{\varepsilon^{\mu}z})^2 \overline{f''(\varepsilon^{\mu}z)}+(\overline{\varepsilon^{\mu}z})\overline{f'(\varepsilon^{\mu}z)}}{\lambda\delta(\overline{\varepsilon^{\mu}z})^2 \overline{f''_{2k}(\varepsilon^{\mu}z)}+(\lambda-\delta)(\overline{\varepsilon^{\mu}z})\overline{f'_{2k}(\varepsilon^{\mu}z)}+(1-\lambda+\delta)\overline{f_{2k}(\varepsilon^{\mu}z)}}}{\rho\left(\dfrac{\lambda\delta(\overline{\varepsilon^{\mu}z})^3 \overline{f'''(\varepsilon^{\mu}z)}+(2\lambda\delta+\lambda-\delta)(\overline{\varepsilon^{\mu}z})^2 \overline{f''(\varepsilon^{\mu}z)}+(\overline{\varepsilon^{\mu}z})\overline{f'(\varepsilon^{\mu}z)}}{\lambda\delta(\overline{\varepsilon^{\mu}z})^2 \overline{f''_{2k}(\varepsilon^{\mu}z)}+(\lambda-\delta)(\overline{\varepsilon^{\mu}z})\overline{f'_{2k}(\varepsilon^{\mu}z)}+(1-\lambda+\delta)\overline{f_{2k}(\varepsilon^{\mu}z)}}\right)+(1-\rho)}\right)>\alpha, \qquad (1.7)$$

Note that $f_{2k}(\varepsilon^{\mu} z) = \varepsilon^{\mu} f_{2k}(z)$, $f'_{2k}(\varepsilon^{\mu}z) = f'_{2k}(z)$, $\overline{f_{2k}(\varepsilon^{\mu}\overline{z})} = \varepsilon^{-\mu}f_{2k}(z)$ and $\overline{f'_{2k}(\varepsilon^{\mu}\overline{z})} = f'_{2k}(z)$, thus, inequalities (1.6) and (1.7) can be written as

$$\mathbb{R}\left(\frac{\dfrac{\lambda\delta(\varepsilon^{\mu})^2 z^3 f'''(\varepsilon^{\mu}z)+(2\lambda\delta+\lambda-\delta)\varepsilon^{\mu} z^2 f''(\varepsilon^{\mu}z)+zf'(\varepsilon^{\mu}z)}{\lambda\delta z^2 f''_{2k}(z)+(\lambda-\delta)zf'_{2k}(z)+(1-\lambda+\delta)f_{2k}(z)}}{\rho\left(\dfrac{\lambda\delta(\varepsilon^{\mu})^2 z^3 f'''(\varepsilon^{\mu}z)+(2\lambda\delta+\lambda-\delta)\varepsilon^{\mu} z^2 f''(\varepsilon^{\mu}z)+zf'(\varepsilon^{\mu}z)}{\lambda\delta z^2 f''_{2k}(z)+(\lambda-\delta)zf'_{2k}(z)+(1-\lambda+\delta)f_{2k}(z)}\right)+(1-\rho)}\right)>\alpha, \qquad (1.8)$$

and

$$\mathbb{R}\left(\frac{\dfrac{\lambda\delta(\varepsilon^{-\mu})^2 z^3 \overline{f'''(\varepsilon^{\mu}\overline{z})}+(2\lambda\delta+\lambda-\delta)\varepsilon^{-\mu} z^2 \overline{f''(\varepsilon^{\mu}\overline{z})}+z\overline{f'(\varepsilon^{\mu}\overline{z})}}{\lambda\delta z^2 f''_{2k}(z)+(\lambda-\delta)zf'_{2k}(z)+(1-\lambda+\delta)f_{2k}(z)}}{\rho\left(\dfrac{\lambda\delta(\varepsilon^{-\mu})^2 z^3 \overline{f'''(\varepsilon^{\mu}\overline{z})}+(2\lambda\delta+\lambda-\delta)\varepsilon^{-\mu} z^2 \overline{f''(\varepsilon^{\mu}\overline{z})}+z\overline{f'(\varepsilon^{\mu}\overline{z})}}{\lambda\delta z^2 f''_{2k}(z)+(\lambda-\delta)zf'_{2k}(z)+(1-\lambda+\delta)f_{2k}(z)}\right)+(1-\rho)}\right)>\alpha, \qquad (1.9)$$

Summing inequalities (1.8) and (1.9), we can obtain

$$\mathbb{R}\left(\frac{\dfrac{\lambda\delta z^3[(\varepsilon^{\mu})^2 f'''(\varepsilon^{\mu}z)+(\varepsilon^{-\mu})^2\overline{f'''(\varepsilon^{\mu}\overline{z})}]+(2\lambda\delta+\lambda-\delta)z^2[\varepsilon^{\mu}f''(\varepsilon^{\mu}z)+\varepsilon^{-\mu}\overline{f''(\varepsilon^{\mu}\overline{z})}]+z[f'(\varepsilon^{\mu}z)+\overline{f'(\varepsilon^{\mu}\overline{z})}]}{\lambda\delta z^2 f''_{2k}(z)+(\lambda-\delta)zf'_{2k}(z)+(1-\lambda+\delta)f_{2k}(z)}}{\rho\left(\dfrac{\lambda\delta z^3[(\varepsilon^{\mu})^2 f'''(\varepsilon^{\mu}z)+(\varepsilon^{-\mu})^2\overline{f'''(\varepsilon^{\mu}\overline{z})}]+(2\lambda\delta+\lambda-\delta)z^2[\varepsilon^{\mu}f''(\varepsilon^{\mu}z)+\varepsilon^{-\mu}\overline{f''(\varepsilon^{\mu}\overline{z})}]+z[f'(\varepsilon^{\mu}z)+\overline{f'(\varepsilon^{\mu}\overline{z})}]}{\lambda\delta z^2 f''_{2k}(z)+(\lambda-\delta)zf'_{2k}(z)+(1-\lambda+\delta)f_{2k}(z)}\right)+(1-\rho)}\right)>2\alpha, \qquad (1.10)$$

Let $\mu = 0, 1, 2, ..., k-1$ in (1.10), respectively, and summing them we can get

$$\mathbb{R}\left(\frac{\dfrac{\lambda\delta z^3\frac{1}{2k}\sum_{\mu=0}^{k-1}[(\varepsilon^{\mu})^2 f'''(\varepsilon^{\mu}z)+(\varepsilon^{-\mu})^2\overline{f'''(\varepsilon^{\mu}\overline{z})}]+(2\lambda\delta+\lambda-\delta)z^2\frac{1}{2k}\sum_{\mu=0}^{k-1}[\varepsilon^{\mu}f''(\varepsilon^{\mu}z)+\varepsilon^{-\mu}\overline{f''(\varepsilon^{\mu}\overline{z})}]+z\frac{1}{2k}\sum_{\mu=0}^{k-1}[f'(\varepsilon^{\mu}z)+\overline{f'(\varepsilon^{\mu}\overline{z})}]}{\lambda\delta z^2 f''_{2k}(z)+(\lambda-\delta)zf'_{2k}(z)+(1-\lambda+\delta)f_{2k}(z)}}{\rho\left(\dfrac{\lambda\delta z^3\frac{1}{2k}\sum_{\mu=0}^{k-1}[(\varepsilon^{\mu})^2 f'''(\varepsilon^{\mu}z)+(\varepsilon^{-\mu})^2\overline{f'''(\varepsilon^{\mu}\overline{z})}]+(2\lambda\delta+\lambda-\delta)z^2\frac{1}{2k}\sum_{\mu=0}^{k-1}[\varepsilon^{\mu}f''(\varepsilon^{\mu}z)+\varepsilon^{-\mu}\overline{f''(\varepsilon^{\mu}\overline{z})}]+z\frac{1}{2k}\sum_{\mu=0}^{k-1}[f'(\varepsilon^{\mu}z)+\overline{f'(\varepsilon^{\mu}\overline{z})}]}{\lambda\delta z^2 f''_{2k}(z)+(\lambda-\delta)zf'_{2k}(z)+(1-\lambda+\delta)f_{2k}(z)}\right)+(1-\rho)}\right)>\alpha,$$

or equivalently,

$$\mathbb{R}\left(\frac{\dfrac{\lambda\delta z^3 f'''_{2k}(z)+(2\lambda\delta+\lambda-\delta)z^2 f''_{2k}(z)+zf'_{2k}(z)}{\lambda\delta z^2 f''_{2k}(z)+(\lambda-\delta)zf'_{2k}(z)+(1-\lambda+\delta)f_{2k}(z)}}{\rho\left(\dfrac{\lambda\delta z^3 f'''_{2k}(z)+(2\lambda\delta+\lambda-\delta)z^2 f''_{2k}(z)+zf'_{2k}(z)}{\lambda\delta z^2 f''_{2k}(z)+(\lambda-\delta)zf'_{2k}(z)+(1-\lambda+\delta)f_{2k}(z)}\right)+(1-\rho)}\right)$$

$$=\mathbb{R}\left(\frac{\dfrac{zF'_{2k}(z)}{F_{2k}(z)}}{\rho\left(\dfrac{zF'_{2k}(z)}{F_{2k}(z)}\right)+(1-\rho)}\right)>\alpha,$$

that is $\rho = 0$, $F_{2k}(z) \in S^*(\alpha)$, which is the class of starlike functions of order α in Δ. Note that $S^*(0) = S^*$, this implies that $F(z)=\lambda\delta z^2 f''(z)+(\lambda-\delta)zf'(z)+(1-\lambda+\delta)f(z)$.

We now split it into two cases to prove

Case (i) When $\delta = 0$, $\lambda = 0$, $f(z) = F(z) \in \mathcal{C}$.

Case (ii) When $0 < \lambda \le \delta \le 1$. From $F(z)=\lambda\delta z^2 f''(z)+(\lambda-\delta)zf'(z)+(1-\lambda+\delta)f(z)$ and $0 < \lambda \le \delta \le 1$ we have

$$f(z)=\frac{1}{\beta}z^{1-\frac{1}{\beta}}\int_0^z F(t)t^{\frac{1}{\beta}-2}dt, \quad 0<\beta\le 1.$$

Since $\gamma=\dfrac{1}{\beta}-1\ge 0$, by Lemma 1.1, we obtain that $f(z) \in \mathcal{C} \subset S$.

Hence $\mathcal{P}_{SC}^{(k)}(\rho,\delta,\lambda,\alpha)\subset\mathcal{C}\subset S$, and the proof is complete. □

2. Integral Representations

We first give some integral representations of functions in the class $\mathcal{P}_{SC}^{(k)}(\rho,\delta,\lambda,\alpha)$.

Theorem 2.1.

Let $f(z) \in \mathcal{P}_{SC}^{(k)}(\rho,\delta,\lambda,\alpha)$ with $0 < \beta \le 1$, $0 < \lambda \le 1$. Then

$$f_{2k}(z)=\frac{1}{\beta}z^{1-\frac{1}{\beta}}\int_0^z \exp\left(\frac{1}{2k}\sum_{\mu=0}^{k-1}\int_0^u \frac{2(1-\alpha)}{\zeta}\left(\frac{\omega(\varepsilon^{\mu}\zeta)}{1-\rho-(1+\rho-2\alpha\rho)\omega(\varepsilon^{\mu}\zeta)}+\frac{\overline{\omega(\varepsilon^{\mu}\overline{\zeta})}}{1-\rho-(1+\rho-2\alpha\rho)\overline{\omega(\varepsilon^{\mu}\overline{\zeta})}}\right)d\zeta\right)u^{\frac{1}{\beta}-1}du, \tag{2.1}$$

where $f_{2k}(z)$ is defined by equality (1.3), $\omega(z)$ is analytic in Δ and $\omega(0) = 0$, $|\omega(z)| < 1$.

Proof.

Suppose that $f(z) \in \mathcal{P}_{SC}^{(k)}(\rho,\delta,\lambda,\alpha)$, we know that the condition (1.5) can be written as follows:

$$\frac{\dfrac{\lambda\delta z^3 f'''(z)+(2\lambda\delta+\lambda-\delta)z^2 f''(z)+zf'(z)}{\lambda\delta z^2 f''_{2k}(z)+(\lambda-\delta)zf'_{2k}(z)+(1-\lambda+\delta)f_{2k}(z)}}{\rho\left(\dfrac{\lambda\delta z^3 f'''(z)+(2\lambda\delta+\lambda-\delta)z^2 f''(z)+zf'(z)}{\lambda\delta z^2 f''_{2k}(z)+(\lambda-\delta)zf'_{2k}(z)+(1-\lambda+\delta)f_{2k}(z)}\right)+(1-\rho)} \prec \frac{1+(1-2\alpha)z}{1-z},$$

where $\prec$ stands for the subordination. It follows that

$$\frac{\dfrac{\lambda\delta z^3 f'''(z)+(2\lambda\delta+\lambda-\delta)z^2 f''(z)+zf'(z)}{\lambda\delta z^2 f''_{2k}(z)+(\lambda-\delta)zf'_{2k}(z)+(1-\lambda+\delta)f_{2k}(z)}}{\rho\left(\dfrac{\lambda\delta z^3 f'''(z)+(2\lambda\delta+\lambda-\delta)z^2 f''(z)+zf'(z)}{\lambda\delta z^2 f''_{2k}(z)+(\lambda-\delta)zf'_{2k}(z)+(1-\lambda+\delta)f_{2k}(z)}\right)+(1-\rho)} = \frac{1+(1-2\alpha)\omega(z)}{1-\omega(z)},$$

where $\omega(z)$ is analytic in Δ and $\omega(0) = 0$, $|\omega(z)| < 1$. This yields

$$\frac{\lambda\delta z^3 f'''(z)+(2\lambda\delta+\lambda-\delta)z^2 f''(z)+zf'(z)}{\lambda\delta z^2 f''_{2k}(z)+(\lambda-\delta)zf'_{2k}(z)+(1-\lambda+\delta)f_{2k}(z)} = \frac{(1-\rho)[1+(1-2\alpha)\omega(z)]}{1-\rho-(1+\rho-2\alpha\rho)\omega(z)} \tag{2.2}$$

Substituting z by $\varepsilon^{\mu} z$ ($\mu = 0, 1, 2, ..., k-1$) in (2.2), respectively, we get

$$\frac{\lambda\delta(\varepsilon^{\mu}z)^3 f'''(\varepsilon^{\mu}z)+(2\lambda\delta+\lambda-\delta)(\varepsilon^{\mu}z)^2 f''(\varepsilon^{\mu}z)+(\varepsilon^{\mu}z)f'(\varepsilon^{\mu}z)}{\lambda\delta(\varepsilon^{\mu}z)^2 f''_{2k}(\varepsilon^{\mu}z)+(\lambda-\delta)(\varepsilon^{\mu}z)f'_{2k}(\varepsilon^{\mu}z)+(1-\lambda+\delta)f_{2k}(\varepsilon^{\mu}z)} = \frac{(1-\rho)[1+(1-2\alpha)\omega(\varepsilon^{\mu}z)]}{1-\rho-(1+\rho-2\alpha\rho)\omega(\varepsilon^{\mu}z)} \tag{2.3}$$

From (2.3), we have

$$\frac{\lambda\delta(\overline{\varepsilon^{\mu}z})^{3}\overline{f'''(\varepsilon^{\mu}z)}+(2\lambda\delta+\lambda-\delta)(\overline{\varepsilon^{\mu}z})^{2}\overline{f''(\varepsilon^{\mu}z)}+(\overline{\varepsilon^{\mu}z})\overline{f'(\varepsilon^{\mu}z)}}{\lambda\delta(\overline{\varepsilon^{\mu}z})^{2}\overline{f''_{2k}(\varepsilon^{\mu}z)}+(\lambda-\delta)(\overline{\varepsilon^{\mu}z})\overline{f'_{2k}(\varepsilon^{\mu}z)}+(1-\lambda+\delta)\overline{f_{2k}(\varepsilon^{\mu}z)}}=\frac{(1-\rho)[1+(1-2\alpha)\overline{\omega(\varepsilon^{\mu}z)}]}{1-\rho-(1+\rho-2\alpha\rho)\overline{\omega(\varepsilon^{\mu}z)}} \tag{2.4}$$

Note that $f_{2k}(\varepsilon^{\mu} z) = \varepsilon^{\mu} f_{2k}(z)$ and $\overline{f_{2k}(\varepsilon^{\mu}\overline{z})}=\varepsilon^{-\mu}f_{2k}(z)$, summing equalities (2.3) and (2.4), we can obtain

$$\begin{aligned}&\frac{\lambda\delta z^{3}[(\varepsilon^{\mu})^{3}f'''(\varepsilon^{\mu}z)+(\varepsilon^{-\mu})^{3}\overline{f'''(\varepsilon^{\mu}\overline{z})}]+(2\lambda\delta+\lambda-\delta)z^{2}[(\varepsilon^{\mu})^{2}f''(\varepsilon^{\mu}z)+(\varepsilon^{-\mu})^{2}\overline{f''(\varepsilon^{\mu}\overline{z})}]+z[(\varepsilon^{\mu})f'(\varepsilon^{\mu}z)+(\varepsilon^{-\mu})\overline{f'(\varepsilon^{\mu}\overline{z})}]}{\lambda\delta z^{2}f''_{2k}(z)+(\lambda-\delta)zf'_{2k}(z)+(1-\lambda+\delta)f_{2k}(z)}\\&=\frac{(1-\rho)[1+(1-2\alpha)\omega(\varepsilon^{\mu}z)]}{1-\rho-(1+\rho-2\alpha\rho)\omega(\varepsilon^{\mu}z)}+\frac{(1-\rho)[1+(1-2\alpha)\overline{\omega(\varepsilon^{\mu}\overline{z})}]}{1-\rho-(1+\rho-2\alpha\rho)\overline{\omega(\varepsilon^{\mu}\overline{z})}}\end{aligned} \tag{2.5}$$

Let μ = 0, 1, 2, ..., k−1 in (2.5), respectively, and summing them we can get

$$\begin{aligned}&\frac{\lambda\delta z^{3}f'''_{2k}(z)+(2\lambda\delta+\lambda-\delta)z^{2}f''_{2k}(z)+zf'_{2k}(z)}{\lambda\delta z^{2}f''_{2k}(z)+(\lambda-\delta)zf'_{2k}(z)+(1-\lambda+\delta)f_{2k}(z)}\\&=\frac{(1-\rho)[1+(1-2\alpha)\omega(\varepsilon^{\mu}z)]}{1-\rho-(1+\rho-2\alpha\rho)\omega(\varepsilon^{\mu}z)}+\frac{(1-\rho)[1+(1-2\alpha)\overline{\omega(\varepsilon^{\mu}\overline{z})}]}{1-\rho-(1+\rho-2\alpha\rho)\overline{\omega(\varepsilon^{\mu}\overline{z})}}\end{aligned} \tag{2.6}$$

From (2.6), we can get

$$\begin{aligned}&\frac{\lambda\delta z^{3}f'''_{2k}(z)+(2\lambda\delta+\lambda-\delta)z^{2}f''_{2k}(z)+zf'_{2k}(z)}{\lambda\delta z^{2}f''_{2k}(z)+(\lambda-\delta)zf'_{2k}(z)+(1-\lambda+\delta)f_{2k}(z)}-\frac{1}{z}\\&=\frac{1}{2k}\sum_{\mu=0}^{k-1}\frac{1}{z}\left(\frac{(1-\rho)[1+(1-2\alpha)\omega(\varepsilon^{\mu}z)]}{1-\rho-(1+\rho-2\alpha\rho)\omega(\varepsilon^{\mu}z)}+\frac{(1-\rho)[1+(1-2\alpha)\overline{\omega(\varepsilon^{\mu}\overline{z})}]}{1-\rho-(1+\rho-2\alpha\rho)\overline{\omega(\varepsilon^{\mu}\overline{z})}}-2\right)\end{aligned} \tag{2.7}$$

Integrating (2.7), we have

$$\log\left(\frac{\lambda\delta z^{2}f''_{2k}(z)+(\lambda-\delta)zf'_{2k}(z)+(1-\lambda+\delta)f_{2k}(z)}{z}\right)=\frac{1}{2k}\sum_{\mu=0}^{k-1}\int_{0}^{z}\frac{2(1-\alpha)}{\zeta}$$
$$\left(\frac{\omega(\varepsilon^{\mu}\zeta)}{1-\rho-(1+\rho-2\alpha\rho)\omega(\varepsilon^{\mu}\zeta)}+\frac{\overline{\omega(\varepsilon^{\mu}\overline{\zeta})}}{1-\rho-(1+\rho-2\alpha\rho)\overline{\omega(\varepsilon^{\mu}\overline{\zeta})}}\right)d\zeta.$$

That is

$$\lambda\delta z^2 f''_{2k}(z)+(\lambda-\delta)zf'_{2k}(z)+(1-\lambda+\delta)f_{2k}(z)=z\exp\frac{1}{2k}\sum_{\mu=0}^{k-1}\int_0^z\frac{2(1-\alpha)}{\zeta}$$
$$\left(\frac{\omega(\varepsilon^\mu\zeta)}{1-\rho-(1+\rho-2\alpha\rho)\omega(\varepsilon^\mu\zeta)}+\frac{\overline{\omega(\varepsilon^\mu\overline{\zeta})}}{1-\rho-(1+\rho-2\alpha\rho)\overline{\omega(\varepsilon^\mu\overline{\zeta})}}\right)d\zeta. \tag{2.8}$$

The assertion (2.1) in Theorem 2.1 can now easily be derived from (2.8). □

Theorem 2.2.
Let $f(z)\in \mathcal{P}_{SC}^{(k)}(\rho,\delta,\lambda,\alpha)$ with $k\ge 2$. Then

$$f(z)=\frac{1}{\beta}z^{1-\frac{1}{\beta}}\int_0^z\int_0^u\exp\left(\frac{1}{2k}\sum_{\mu=0}^{k-1}\int_0^\xi\frac{2(1-\alpha)}{\zeta}\right.$$
$$\left.\left(\frac{\omega(\varepsilon^\mu\zeta)}{1-\rho-(1+\rho-2\alpha\rho)\omega(\varepsilon^\mu\zeta)}+\frac{\overline{\omega(\varepsilon^\mu\overline{\zeta})}}{1-\rho-(1+\rho-2\alpha\rho)\overline{\omega(\varepsilon^\mu\overline{\zeta})}}\right)d\zeta\right) \tag{2.9}$$
$$\frac{(1-\rho)[1+(1-2\alpha)\omega(\xi)]}{1-\rho-(1+\rho-2\alpha\rho)\omega(\xi)}d\xi\, u^{\frac{1}{\beta}-2}du.$$

where $\omega(z)$ is analytic in Δ and $\omega(0)=0$, $|\omega(z)|<1$.

Proof.
Suppose that $f(z)\in \mathcal{P}_{SC}^{(k)}(\rho,\delta,\lambda,\alpha)$, from equalities (2.1) and (2.2), we can get

$$\lambda\delta z^3 f'''(z)+(2\lambda\delta+\lambda-\delta)z^2 f''(z)+zf'(z)$$
$$=(\lambda\delta z^2 f''_{2k}(z)+(\lambda-\delta)zf'_{2k}(z)+(1-\lambda+\delta)f_{2k}(z))\frac{(1-\rho)[1+(1-2\alpha)\omega(z)]}{1-\rho-(1+\rho-2\alpha\rho)\omega(z)}$$
$$=\exp\frac{1}{2k}\sum_{\mu=0}^{k-1}\int_0^\xi\frac{2(1-\alpha)}{\zeta}\left(\frac{\omega(\varepsilon^\mu\zeta)}{1-\rho-(1+\rho-2\alpha\rho)\omega(\varepsilon^\mu\zeta)}+\frac{\overline{\omega(\varepsilon^\mu\overline{\zeta})}}{1-\rho-(1+\rho-2\alpha\rho)\overline{\omega(\varepsilon^\mu\overline{\zeta})}}\right)d\zeta$$
$$\frac{(1-\rho)[1+(1-2\alpha)\omega(z)]}{1-\rho-(1+\rho-2\alpha\rho)\omega(z)}.$$

Integrating the equality, we can easily get (2.9). □

3. Convolution Conditions

In this section, we provide some convolution conditions for the class $\mathcal{P}_{SC}^{(k)}(\rho,\delta,\lambda,\alpha)$. Let f, g $\in \mathcal{A}$, where f(z) is given by (1.1) and g(z) is defined by

$$g(z) = z + \sum_{n=2}^{\infty} c_n z^n.$$

Then the Hadamard product (or convolution) f * g is defined (as usual) by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n c_n z^n = (g * f)(z).$$

Theorem 3.1.
A function $f(z) \in \mathcal{P}_{SC}^{(k)}(\rho, \delta, \lambda, \alpha)$ if and only if

$$\frac{1}{z}\Bigg\{ f * \Bigg[\lambda\delta z^2 \left(\frac{z}{(1-z)^2}[(1-e^{i\theta}) - \rho(1+(1-2\alpha)e^{i\theta})] - \frac{(1-\rho)(1+(1-2\alpha)e^{i\theta})}{2} h \right)''$$
$$+ z\left((2\lambda\delta+\lambda-\delta) \frac{z}{(1-z)^2}[(1-e^{i\theta}) - \rho(1+(1-2\alpha)e^{i\theta})] - \frac{(\lambda-\delta)(1-\rho)(1+(1-2\alpha)e^{i\theta})}{2} h \right)' \tag{3.1}$$
$$+ \left(\frac{z}{(1-z)^2}[(1-e^{i\theta}) - \rho(1+(1-2\alpha)e^{i\theta})] - \frac{(1-\lambda+\delta)(1-\rho)(1+(1-2\alpha)e^{i\theta})}{2} h \right)(z) \Bigg]$$
$$-(1-\rho)(1+(1-2\alpha)e^{i\theta}) \overline{f * \left(\frac{\lambda\delta z^2}{2} h'' + \frac{(\lambda-\rho)}{2} z h' + \frac{1-\lambda+\delta}{2} h \right)(\bar{z})} \Bigg\} \neq 0$$

for all $z \in \Delta$ and $0 \le \theta < 2\pi$, where

$$h(z) = \frac{1}{k} \sum_{v=0}^{k-1} \frac{z}{1 - \varepsilon^v z}. \tag{3.2}$$

Proof.
Suppose that $f(z) \in \mathcal{P}_{SC}^{(k)}(\rho, \delta, \lambda, \alpha)$, since the condition (1.5) is equivalent to

$$\frac{\dfrac{\lambda\delta z^3 f'''(z) + (2\lambda\delta+\lambda-\delta) z^2 f''(z) + z f'(z)}{\lambda\delta z^2 f''_{2k}(z) + (\lambda-\delta) z f'_{2k}(z) + (1-\lambda+\delta) f_{2k}(z)}}{\rho\left(\dfrac{\lambda\delta z^3 f'''(z) + (2\lambda\delta+\lambda-\delta) z^2 f''(z) + z f'(z)}{\lambda\delta z^2 f''_{2k}(z) + (\lambda-\delta) z f'_{2k}(z) + (1-\lambda+\delta) f_{2k}(z)} \right) + (1-\rho)} \neq \frac{1+(1-2\alpha)e^{i\theta}}{1-e^{i\theta}}, \tag{3.3}$$

for all $z \in \Delta$ and $0 \le \theta < 2\pi$. And the condition (3.3) can be written as follows:

$$\frac{1}{z}\{(\lambda\delta z^3 f'''(z)+(2\lambda\delta+\lambda-\delta)z^2 f''(z)+zf'(z))((1-e^{i\theta})-\rho(1+(1-2\alpha)e^{i\theta})) \\ -(1-\rho)(\lambda\delta z^2 f''_{2k}(z)+(\lambda-\delta)zf'_{2k}(z)+(1-\lambda+\delta)f_{2k}(z))(1+(1-2\alpha)e^{i\theta})\}\neq 0. \tag{3.4}$$

On the other hand, it is well known that

$$zf'(z)=f(z)*\frac{z}{(1-z)^2}. \tag{3.5}$$

And from the definition of $f_{2k}(z)$, we know

$$f_{2k}(z)=z+\sum_{n=2}^{\infty}\frac{a_n+\overline{a_n}}{2}c_n z^n=\frac{1}{2}((f*h)(z)+\overline{(f*h)(z)}), \tag{3.6}$$

where h(z) is given by (3.2). Substituting (3.5) and (3.6) into (3.4), we can easily get (3.1). This completes the proof of Theorem 3.1. □

4. Coefficient Inequalities

In this section, we provide the sufficient conditions for functions belonging to the class $\mathcal{P}_{SC}^{(k)}(\rho,\delta,\lambda,\alpha)$.

Theorem 4.1.
Let $0\le\alpha<1$, $0\le\lambda<1$, $0\le\delta<1$ and $0\le\rho<1$. If

$$\sum_{n=1}^{\infty}(\lambda\delta(nk+1)nk+(\lambda-\delta)(nk+1)+(1-\lambda+\delta)[(1-\rho)\left|(nk+1)a_{nk+1}-\mathbb{R}(a_{nk+1})\right| \\ +(1-\alpha)(\rho(nk+1)a_{nk+1}+(1-\rho)\left|\mathbb{R}(a_{nk+1})\right|] \\ +\sum_{\substack{n=2\\ n\neq lk+1}}^{\infty}(1-\lambda)n(\lambda\delta n(n-1)+(\lambda-\delta)n+(1-\lambda+\delta))\left|a_n\right|\le 1-\alpha \tag{4.1}$$

Then $f(z)\in\mathcal{P}_{SC}^{(k)}(\rho,\delta,\lambda,\alpha)$.

Proof.
It suffices to show that

$$\left| \frac{\dfrac{\lambda\delta z^3 f'''(z) + (2\lambda\delta + \lambda - \delta) z^2 f''(z) + z f'(z)}{\lambda\delta z^2 f''_{2k}(z) + (\lambda - \delta) z f'_{2k}(z) + (1 - \lambda + \delta) f_{2k}(z)}}{\rho\left(\dfrac{\lambda\delta z^3 f'''(z) + (2\lambda\delta + \lambda - \delta) z^2 f''(z) + z f'(z)}{\lambda\delta z^2 f''_{2k}(z) + (\lambda - \delta) z f'_{2k}(z) + (1 - \lambda + \delta) f_{2k}(z)}\right) + (1-\rho)} - 1 \right| < 1 - \alpha,$$

Note that for $|z| = r < 1$, we have

$$\left| \frac{\dfrac{\lambda\delta z^3 f'''(z) + (2\lambda\delta + \lambda - \delta) z^2 f''(z) + z f'(z)}{\lambda\delta z^2 f''_{2k}(z) + (\lambda - \delta) z f'_{2k}(z) + (1 - \lambda + \delta) f_{2k}(z)}}{\rho\left(\dfrac{\lambda\delta z^3 f'''(z) + (2\lambda\delta + \lambda - \delta) z^2 f''(z) + z f'(z)}{\lambda\delta z^2 f''_{2k}(z) + (\lambda - \delta) z f'_{2k}(z) + (1 - \lambda + \delta) f_{2k}(z)}\right) + (1-\rho)} - 1 \right|$$

$$= \left| \frac{\sum_{n=2}^{\infty} (1-\rho)[(\lambda\delta n(n-1) + (\lambda-\delta)n + (1-\lambda+\delta))(na_n - \mathbb{R}(a_n)b_n)]z^{n-1}}{1 + \sum_{n=2}^{\infty} [(\lambda\delta n(n-1) + (\lambda-\delta)n + (1-\lambda+\delta))(\rho n a_n + (1-\rho)\mathbb{R}(a_n)b_n)]z^{n-1}} \right|$$

$$\leq \frac{\sum_{n=2}^{\infty} (1-\rho)(\lambda\delta n(n-1) + (\lambda-\delta)n + (1-\lambda+\delta)) \left|na_n - \mathbb{R}(a_n)b_n\right| |z|^{n-1}}{1 - \sum_{n=2}^{\infty} (\lambda\delta n(n-1) + (\lambda-\delta)n + (1-\lambda+\delta))(\rho n a_n + (1-\rho)b_n \left|\mathbb{R}(a_n)\right|) |z|^{n-1}}$$

$$\leq \frac{\sum_{n=2}^{\infty} (1-\rho)(\lambda\delta n(n-1) + (\lambda-\delta)n + (1-\lambda+\delta)) \left|na_n - \mathbb{R}(a_n)b_n\right|}{1 - \sum_{n=2}^{\infty} (\lambda\delta n(n-1) + (\lambda-\delta)n + (1-\lambda+\delta))(\rho n a_n + (1-\rho)b_n \left|\mathbb{R}(a_n)\right|)}$$

where

$$b_n = \frac{1}{k}\sum_{\nu=0}^{k-1} \varepsilon^{(n-1)\nu} = \begin{cases} 1, & n = lk+1, \\ 0, & n \neq lk+1. \end{cases}$$

This last expression is bounded above by $1-\alpha$ if

$$\sum_{n=2}^{\infty} (\lambda\delta n(n-1) + (\lambda-\delta)n + (1-\lambda+\delta))[(1-\rho)\left|na_n - R(a_n)b_n\right| + (1-\alpha)(\rho n a_n + (1-\rho)b_n)\left|R(a_n)\right|)] \leq 1-\alpha, \tag{4.3}$$

Since inequality (4.3) can be written as inequality (4.1), hence f(z) satisfies the condition (1.5). This completes the proof of Theorem 4.1. □

Theorem 4.2.
Let $0 \leq \alpha < 1$, $0 \leq \lambda < 1$, $0 \leq \delta < 1$ and $0 \leq \rho \leq 1$ and $f(z) \in \mathcal{T}$. Then f(z) $\in \mathcal{TP}_{SC}^{(k)}(\rho, \delta, \lambda, \alpha)$ if and only if

$$\sum_{n=1}^{\infty}(\lambda\delta nk(nk+1)+(\lambda-\delta)(nk+1)+(1-\lambda+\delta))(nk+1-\alpha)a_{nk+1} \\ +\sum_{\substack{n=2\\ n\neq lk+1}}^{\infty}(\lambda\delta n(n-1)+(\lambda-\delta)n+(1-\lambda+\delta))na_n \leq 1-\alpha. \qquad (4.4)$$

Proof.
In view of Theorem 4.1, we need only to prove the necessity.

Suppose that $f(z) \in \mathcal{TP}_{SC}^{(k)}(\rho,\delta,\lambda,\alpha)$, then from (1.5), we can get

$$\Re\left(\frac{1-\sum_{n=2}^{\infty} n[\lambda\delta n(n-1)+(\lambda-\delta)n+(1-\lambda+\delta)]a_n z^{n-1}}{1-\sum_{n=2}^{\infty}\{\rho n(\lambda\delta n(n-1)+(\lambda-\delta)n+(1-\lambda+\delta))a_n+(1-\rho)(\lambda\delta n(n-1)+(\lambda-\delta)n+(1-\lambda+\delta))a_n b_n\}z^{n-1}}\right)>\alpha, \qquad (4.5)$$

where b_n is given by 4.2. By letting $z \to 1^-$ through real values in (4.5), we can get

$$\frac{1-\sum_{n=2}^{\infty} n[\lambda\delta n(n-1)+(\lambda-\delta)n+(1-\lambda+\delta)]a_n}{1-\sum_{n=2}^{\infty}\{\rho[n(\lambda\delta n(n-1)+(\lambda-\delta)n+(1-\lambda+\delta))a_n]+(1-\rho)(\lambda\delta n(n-1)+(\lambda-\delta)n+(1-\lambda+\delta))a_n b_n\}} \geq \alpha,$$

or equivalently,

$$\sum_{n=2}^{\infty}(\lambda\delta n(n-1)+(\lambda-\delta)n+(1-\lambda+\delta))(n-\alpha(\rho n+(1-\rho))b_n)a_n \geq 1-\alpha, \qquad (4.6)$$

substituting (4.2) into inequality (4.6), we can get inequality (4.4) easily. This completes the proof of Theorem 4.2.

Acknowledgement
The author thanks the support provided by Science and Engineering Research Board (DST), New Delhi. Project No: SR/S4/MS:716/10 with titled "On Certain Subclass of Analytic Functions with respect to 2k-Symmetric Conjugate Points".

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