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Smoothing Noisy Data Using Variational Mode Decomposition

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Abstract

This article is about the estimator of an unknown regression function which is smooth but observed with noise on a bounded interval. The method is based on applying results of the recently developed theory of variational mode decomposition. It is illustrated with simulation.

AMS subject classification:
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1. Introduction

Estimating a smooth regression function has been a very useful exploratory tool for data analysis. It is also essential in fitting a curve for high dimensional data. It helps to reduce the observational errors in the data so as the interpretation concentrates on important details of the mean dependence of the response function on the causal variable or variables. In fact, estimating the regression function is so important in the present information age as it becomes most challenging to synthesize information, achieve understanding, and to derive insight from increasingly massive, time-varying, noisy and possibly conflicting data sets [11].

This can be achieved either by the parametric estimation when we assume that the mean response has some pre specified functional form or by the nonparametric estimation when there is no reference to a specific form. Nonparametric modeling of regression does not project the observed data into a Procrustean bed of fixed parametrization due to the doubt concerning the appropriate parametric form or about the assumptions on the parameters. Nonparametric simple regression is often called "scatter plot smoothing" because an important application is to tracing a smooth curve through a scatter plot of y against x. Thus the main objective of these studies is to draw conclusion on the underlying trend of the data observed in various real time situations [9]. Though the major research on nonparametric regression dates back to 1960's, several methods of nonparametric curve fitting have been proposed in the recent years and is still an expanding area of ongoing research. During the 1990's the nonparametric regression literature was dominated by linear and nonlinear wavelet estimators. The purpose of this paper is to derive an alternative method to estimate a nonparametric regression function based on variational mode decomposition (VMD).

The rest of the paper is organized as follows. We brief about the existing regression function estimation procedure in Section 2. A review of the VMD model is given in Section 3. We present our VMD based approach to estimate the regression function in Section 4. Section 5 discusses the results based on the simulation. The paper concludes with Section 6.

2. Regression Function Estimation

Let us consider the nonparametric fixed design regression model

$$Y_i = f(x_i) + \varepsilon_i \text{ for } i = 1, \ldots, n \tag{1}$$

where Y_i's are noisy observations of the regression function f and ε_i's are i.i.d normal random variables with zero mean and unit variance. We suppose, without loss of generality, that x_i's are within the unit interval [0, 1] with $x_1 = 0$, $x_n = 1$.

The common methods of nonparametric regression are kernel estimation, local-polynomial regression which is a generalization of kernel estimation, smoothing splines and orthogonal series methods.

The kernel approach is conceptually simple and is to obtain and use appropriate weights w_{ij} to yield fitted values $\hat{y}_i = \sum_{j=1}^{n} w_{ij} y_j$. Out of many methods of determining weights the one that was proposed by Nadarya (1964, 1965) and Watson (1964) is more common. Due to its weight structure it is also known as local constant estimator. It was understood that this local constant estimation nearer to the boundary has very high bias. Due to this limitation of the kernel estimation, the local linear estimation method was proposed to address this boundary problem. But, it is found by simulation that the local linear method does not perform always better than the local constant method. In fact, when the regression function is quite flat, the local constant estimate does better. Meanwhile, the better performance in few cases by local linear estimates over local constant estimators triggered to think of local polynomial estimators too. Though all these local linear and local polynomial methods reduce bias, the variance increases with the order of the polynomial. Thus the curve fitted with such methods has too much rapid local variation and hence are too wiggly which may not be acceptable. But, the solution for such penalized regression problems were given by the smoothing spline method which minimizes $S(h) = \sum [y_i - f(x_i]^2 + h \int [f''(x)]^2 dx$ where h is the roughness penalty.

Orthogonal series estimating procedure is yet another estimator which is widely used as it is simple in structure and easy to compute and analyze mathematically. It also has well developed mathematical theory. But, the quality of this estimation procedure depends very much on the orthogonal system selected [3]. The linear and nonlinear wavelet estimators are also a subset of orthogonal series estimators which are implemented through fast algorithm and hence computationally better than the other classical orthogonal series estimators [3]. Despite these facts wavelet estimation can be made only when the sample values are taken at dyadic points in general and wavelet basis functions are not data-adaptive in nature.

Meanwhile, the empirical mode decomposition (EMD) was introduced by Huang et al. [7] to decompose the given function recursively into different modes or sub signals of separate frequency bands. It has been shown that EMD shares important similarities of wavelets [8] and has been widely used by signal and image processing community. Despite these facts, EMD algorithm could not be modeled mathematically and has few apparent limitations. These shortcomings are overcome by non-recursive VMD model proposed by Konstantin Dragomiretskiy and Dominique Zosso where the modes are extracted concurrently. VMD is interpreted as signal decomposition with a wavelet-packet transform filter bank structure, which is very different from the EMD which is wavelet-like in structure. Therefore, VMD is an adaptive decomposition method with more refinement time-frequency divisions than EMD [10]. These developments in signal and image processing problems lead to this article by making an effort to estimate the regression function using VMD algorithm and is found to be justified by the results obtained for our simulation study. We present in the next section a brief review of VMD model before we describe our estimation procedure.

3. Review of Variational Mode Decomposition

The VMD model looks for a number of modes f_k and their respective center frequencies ω_k, such that $\sum f_k = Y$. It has important relations to Wiener filter denoising. The goal of VMD is to decompose an input signal into a discrete number of modes, that have sparsity in its bandwidth while reproducing the input. In other words, we require each mode k to be mostly compact around a center pulsation.

The modes and the respective centre frequencies are obtained by minimizing the constrained variational problem

$$min_{f_k,\omega_k} \sum_k \left\| \partial_x \left[\left(\delta(x) + \frac{j}{\pi x} \right) * f_k(x) \right] e^{-j\omega_k x} \right\|_2^2$$

such that $\sum_k f_k = Y$, where δ represents the dirac delta function and $j = \sqrt{-1}$.

The above minimization procedure is resulted while the effort is made to estimate the bandwidth of a mode by obtaining the analytic function corresponding to the real mode using the Hilbert transform $\mathcal{H}$ as $f_k(x) + j\mathcal{H} f_k(x)$. The analytic function becomes essential in order to have the positive sided frequency spectrum alone of the mode which is assumed to be centered at ω_k. The resulting analytic function is demodulated by multiplying by $e^{-j\omega_k x}$ so that to shift the centre of the frequency of the mode to the origin [2] [1]. In fact, the modes are derived as optimal functions using calculus of variations [2].

The solution of the constrained problem is shown to be the saddle point of the Lagrangian unconstrained problem

$$\mathcal{L}(f_k, \omega_k, \lambda) = \alpha \sum_k \left\| \partial_x \left[\left(\delta(x) + \frac{j}{\pi x} \right) * f_k(x) \right] e^{-j\omega_k x} \right\|_2^2 + \left\| f - \sum f_k \right\|^2 + \langle \lambda, f - \sum f_k \rangle$$

in which α denotes the balancing parameter of the data-fidelity constraint and λ is the Lagrangian multiplier. The solution will be found in a sequence of iterative sub-optimizations called alternate direction method of multipliers.

The solution of the above unconstrained optimization problem at $n+1$ th iteration is given in frequency domain by,

$$\hat{f}_k^{(n+1)} = \left(\hat{f} - \sum_{i \neq k} \hat{f}_i + \frac{\hat{\lambda}}{2} \right) \frac{1}{(1 + 2(\omega - \omega_k))^2} \tag{2}$$

.

$$\omega_k^{(n+1)} = \frac{\int_0^\infty \omega |f_k(\omega)|^2 d\omega}{\int_0^\infty |f_k(\omega)|^2 d\omega} \tag{3}$$

where $\hat{g}$ for any g will denote the Fourier transform of g. It is important for our proposed problem of estimating the regression function to note the presence of Wiener filter structure in the iterative algorithm of the Fourier transform of the mode $\hat{f}_k$. However, the required mode can be obtained from the real part of inverse Fourier transform.

This discussion will suffice to present the procedure of the proposed VMD based regression function estimation.

4. The Proposed VMD based estimator

In this section we present our approach of obtaining the estimate of the regression function $f(x)$ of the model given in (2.1) from the noise added observations Y_i for $i = 1, 2, \ldots, n$. We understand that two modes will be enough for our problem of estimating the regression function by eliminating the noise from the observations. The Hilbert transform of the assumed modes are obtained as convolution with the impulse response since the Hilbert transform is linear time invariant and hence the analytic function of the modes are constructed. The resulting analytic function is demodulated by suitable multiplication of complex exponential function involving the central frequencies of the modes.

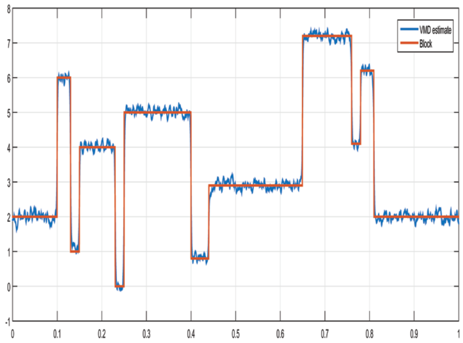
The modes and the corresponding centre frequencies are obtained by solving the unconstrained Lagrangian optimization problem using alternate direction method of multipliers. Though the original formulation of the optimization problem is continuous, it is carried out over the vector whose components are the discrete sampled values and the mode vectors are obtained which might be assumed to be the sampled values of the required mode function. As per the discussion made by Upadhyay and Pachori [1], a large value of α is used so as to eliminate the noise. This is iterated until some stopping criteria is met. The resulting first mode gives us the vector consisting of required estimate of the regression function $f(x)$ whereas the eliminated noise constitute the second mode. Thus obtained the estimate of the regression function from the noisy observation Y_i. Thus we could accomplish smoothing the noisy data using VMD.

There is much similarity between the proposed VMD based estimation and the wavelet estimation. The decomposition of the given function is the key idea based on which the estimation is done in both of these methods. Despite this fact, the decomposition in wavelet method is recursive and is depending on the basis function we intend to use whereas it is nonrecursive and it depends very much on the function which we wish to estimate. In addition to this key difference, the samples we use in wavelets should come from the dyadic points otherwise a suitable wavelet must be chosen or constructed whereas no such restriction is imposed in the case of estimation using VMD.

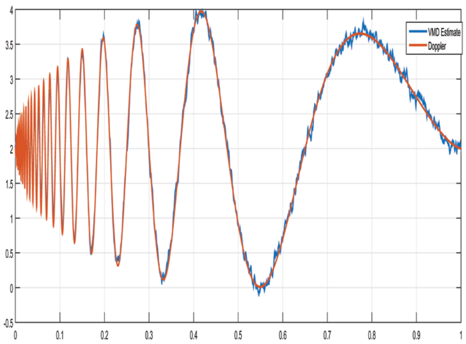
5. Simulation Results and Discussion

In this section we will investigate the performance of the proposed estimator numerically. Of course, the proposed estimation procedure introduced in Section 4 is easily implementable in MATLAB. We implemented our numerical works in MATLAB R 2014 b. For this purpose, we consider the noise added observations Y_i's given in equation (2.1) where $n = 40000$. In fact, the signal to noise ratio of our problem is 10. We have considered only two modes as it is discussed earlier in the previous section. The first mode f_1 results in the estimate of the regression function. In fact, modes are obtained as vectors of the same size as Y_i. However, we obtain the smooth plot of the first mode to compare with $f(x)$. The stopping criteria is fixed by taking the ratio of the l_2 norm of the difference between the two successive iterations to the l_2 norm of the previous iteration not to exceed e^{-7}. The resulting estimates are very much comparable with the estimates of any other existing estimating procedures.

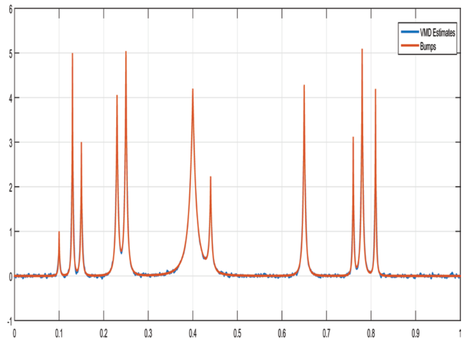
In the numerical study, we use $\alpha = 70000$, so as to work well in eliminating noise. Though the error is not so different for the choice of $\alpha = 70000$ and $\alpha = 5000$ there is a significant visual difference in the estimate. For a given value of α the error decreases when n increases.



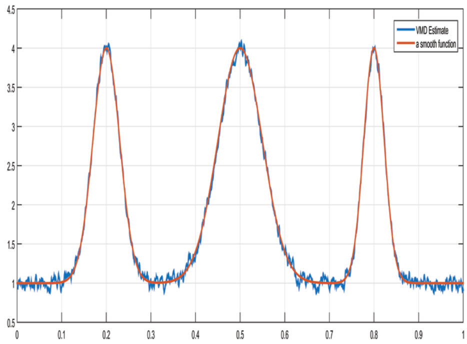
(a) Estimate of Blocks



(b) Estimate of Doppler



(c) Estimate of Bumps



(d) Estimate of $\mathfrak{m}(x)$

Figure 1: VMD estimates

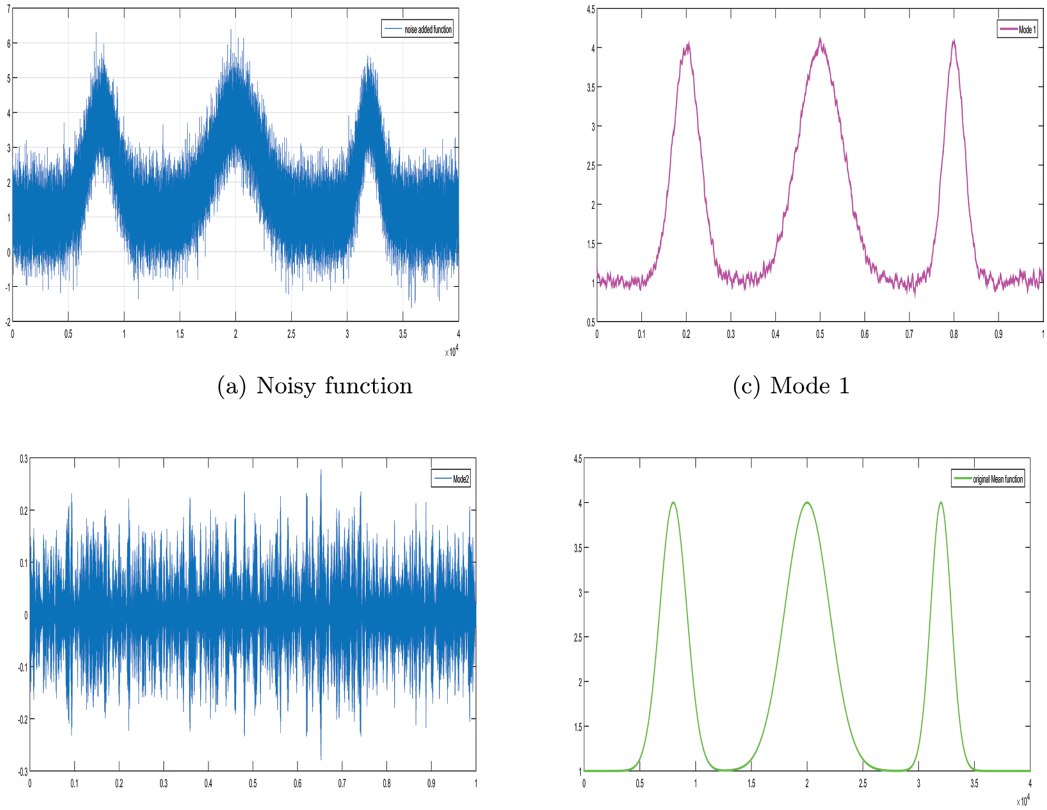
The functions Bumps, Blocks, Doppler [6] and

$$\mathfrak{m}(x) = 1 + 4(e^{-550(x-0.2)^2} + e^{-200(x-0.5)^2} + e^{-950(x-0.8)^2}) \tag{4}$$

are estimated using VMD estimation procedure. These estimates are depicted in Figure 1. In fact, we demonstrate in Figure 2 the way VMD works in extracting the noise and catches the smooth regression function. Visual comparison between the estimates of the regression function using wavelet and VMD is also done. The Haar wavelet is used for this purpose of regression function estimation and the first level approximation is considered. In fact, we considered the estimates of Bumps for this purpose.

6. Conclusion

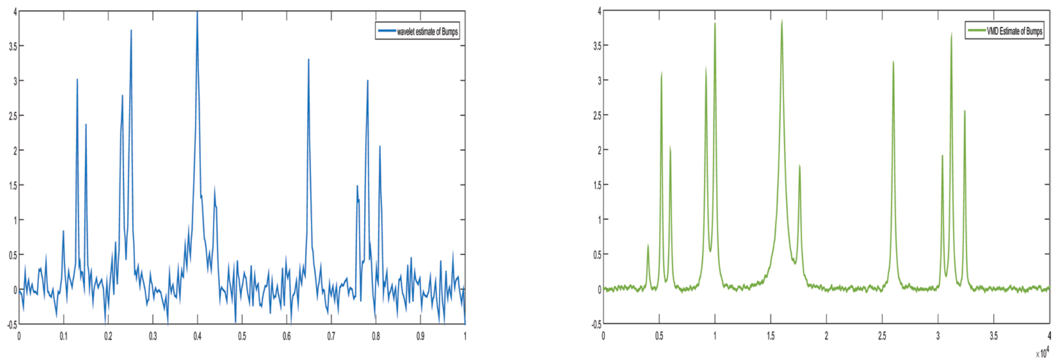
In this paper we tried a VMD based approach to estimate the regression function in the nonparametric regression model. We succeeded in obtaining an adaptive estimation procedure for the regression function. We have explained our method with the numerical examples of obtaining estimates without any prior knowledge of the function. It is noticed that the difference between our proposed VMD based estimate and the linear



(a) Noisy function (c) Mode 1

(b) Mode 2 (d) True Mean function

Figure 2: VMD estimating Procedure



(a) Wavelet Estimate of Bumps (b)VMD Estimate of Bumps

Figure 3: Comparison of Wavelet and VMD estimates

wavelet estimate is not much significant. However, we have many advantages of VMD based estimate over wavelet methods. There will be no restriction on the number of samples. In fact, it will be a major problem to choose the wavelet system to obtain better results whereas it will be done inherently by VMD method. Therefore, this has led to the conclusion that the VMD based approach is better in inhomogeneous function estimation and comparable to wavelet estimates in the case of smooth function.

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