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Symmetries and Invariant Solutions of the Two-dimensional Variable Coefficient KdV-Burgers Equation

T. Shanmuga Priya and B. Mayil Vaganan

Department of Applied Mathematics and Statistics,
Madurai Kamaraj University, Madurai - 625021, India
e-mail : thulas.priya@gmail.com

Abstract

We discuss the symmetries and reductions of the two-dimensional Variable Coefficient KdV-Burgers equation. The two-dimensional KdV-Burgers equation with the Variable coefficient $(u_t + u\,u_x + \lambda\,u_{xx} + \delta\,u_{xxx})_x + S(t)\,u_{yy} = 0$. We classify the one- and two-dimensional subalgebras of the symmetry algebra which is infinite-dimensional into conjugacy classes under the adjoint action of the symmetry group. Invariance under one-dimensional subalgebras provides reductions to lower-dimensional partial differential equations (PDEs). Further reductions of these PDEs to second order ordinary differential equations (ODEs) are obtained through invariance under two dimensional subalgebras.

Keywords: A (2+1)-dimensional KdV-Burgers equation with Variable Coefficient, Symmetry algebra, Conjugacy class.

1. Introduction

The Korteweg-de Vries Burgers equation

$$(u_t + u\,u_x - \nu\,u_{xx} + \mu\,u_{xxx})_x + \sigma\,u_{yy} = 0, \tag{1}$$

is a prototype example of an evolution equation in (2+1)-dimensions which is not completely integrable. Here μ, ν are real constants and $\sigma = \pm 1$. Although it has an infinite-dimensional symmetry algebra it does not have a Virasoro structure. The presence of a Virasoro algebra generally signals integrability for (2+1)-dimensional evolution equations.

Recently Senthilkumaran, Pandiaraja and Mayil Vaganan [16] reported invariant solutions of another GKdVE in the form

$$u_t + u^n u_x + \alpha(t)\, u + \beta(t) u_{xxx} = 0, \quad (2)$$

using Lie's group of infinitesimal transformations [6], [7].

In 1969, Zabolotskaya and Khokhlove derived the ZK equation

$$(u_t + u\, u_x - \beta\, u_{xx})_x + \gamma\, u_{yy} = 0\,. \quad (3)$$

Yet another model equation is derived by Kadomtsev and Petviashvili [5]

$$(u_t + u\, u_x + \epsilon^2 u_{xx})_x + \lambda\, u_{yy} = 0, \lambda = \pm 1\,. \quad (4)$$

KP equation (4) is the generalization of two spatial dimensions, x and y, of the KdV equation. But David, Levi and Winternitz [4], [5], [17] generalized KP equation to describe water waves in straits or rivers.

Gungor and Winternitz [13] transformed the generalized KP equation (GKPE)

$$(u_t + p(t)\, u\, u_x + q(t)\, u_{xxx})_x + \sigma(y,t)\, u_{yy} + a(y,t) u_y + b(y,t)\, u_{xy}$$
$$+ c(y,t)\, u_{xx} + e(y,t) u_x + f(y,t) u + h(y,t) = 0 \quad (5)$$

to its canonical form and established conditions on the coefficient functions under which (5) has an infinite-dimensional symmetry group having a Kac-Moody-Virasoro structure. In [14], they carried out the symmetry analysis of VCKP equation in the form

$$(u_t + f(x,y,t)\, u\, u_x + g(x,y,t) u_{xxx})_x + h(x,y,t)\, u_{yy} = 0 \quad (6)$$

and further classified it into equivalence classes under the local fiber preserving point transformations

$$u = U(x,y,t,\widetilde{u}(\tilde{x},\tilde{y},\tilde{t})), x = X(\tilde{x},\tilde{y},\tilde{t}), y = Y(\tilde{x},\tilde{y},\tilde{t}), t = T(\tilde{x},\tilde{y},\tilde{t}), \quad (7)$$

with non-vanishing Jacobian determinant

$$\frac{\partial U}{\partial \bar{u}} \neq 0, \frac{\partial(X,Y,T)}{\partial(\tilde{x},\tilde{y},\tilde{t})} \neq 0\,. \quad (8)$$

Thus, we can obtain the ZK equation and the GKP equation to obtain the two-dimensional generalized KdV-Burgers equation is of the form

$$(u_t + u\, u_x + \lambda\, u_{xx} + \delta\, u_{xxx})_x + S(t)\, u_{yy} = 0. \quad (9)$$

Equation (9) with S = constant is sometimes referred to as the KdV-Burgers equation. For $\lambda = 0$, (9) refers to KP equation and when $\delta = 0$, we can obtain the generalized Burgers equation [18].

In this paper, we study the symmetry properties of the generalized KdV-Burgers equation (9) by closely following the works of David, Kamran, Levi and Winternitz[5] and Gungor [10-12]. To be precise, we shall show that the generalized KdV-Burgers equation (9) admits an infinite-dimensional symmetry group and determine the corresponding Lie algebra under the adjoint action of the symmetry group in order to reduce (9) to (1+1)-dimensional PDEs and then to ODEs. The symmetry algebra is found to involve two arbitrary functions $f(t)$ and $g(t)$. Several symbolic manipulation packages are available for calculating the symmetry group of PDEs. In this work we use MathLie [19] to determine the symmetry group of KdV-Burgers equation (9).

We organize this paper as follows. In section 2, we derive the symmetry group and study the structure of the symmetry algebra of the generalized KdV-Burgers equation (9). The section 3, is devoted to the determination of physically interesting finite dimensional algebra by restricting $f(t)$ and $g(t)$ to first degree polynomials. In section 4, we give the classification of low -dimensional subalgebras of the generalized KdV-Burgers algebra, namely those of dimension $n = 1, 2$ into conjugacy classes under the adjoint action of the symmetry group of the generalized KdV-Burgers equation (9). This is done mainly to elucidate the structure of the considered infinite-dimensional Lie algebra and to establish the applicability of tools developed for classifying subalgebras of finite-dimensional Lie algebras. In section 5, we reduce the generalized equation (9) into (1+1)-dimensional PDEs using the one-dimensional subalgebras of generalized KdV-Burgers algebra. In section 6, we use two isomorphy classes of two-dimensional algebras, namely, Abelian and non-Abelian, to reduce the PDEs obtained in section 5 to ODEs and are transformed to special cases of equtaions introduced by Mayil Vaganan and Senthil Kumaran [20]. Finally, in section 8, we summarize the results of the present work.

2. The Symmetry Group and its Lie Algebra

The method for determining the symmetry group of a differential equation is straight-forward and described in several books [10, 11]. The symmetry algebra is realized by vector fields

$$V = \xi\, \partial_x + \eta\, \partial_y + \tau \partial_t + \phi\, \partial_u, \tag{10}$$

where ξ, η, τ, ϕ are the functions of *x, y, t and u*. These coefficients are to be determined from the invariance condition

$$pr^{(3)} V(E)|_{E=0} = 0. \tag{11}$$

where E = 0 is the equation under study and $pr^{(3)} V$ stands for the third prolongation of the vector field (see, for example, [6] for the general prolongation formula). Here we are faced with a group classification problem that comprises the determination of the coefficient functions in such a way that the equation admits nontrivial symmetries. Requiring the symmetry condition (11) and solving an over-determined system of linear PDEs we have,

Case I. $S(t)$ arbitrary.

For any function *S(t)* the symmetry algebra of (9) is an infinite-dimensional Lie algebra which we denote by L_P. A general element of L_P for an arbitrary $S(t) \neq$ constant is represented by

$$V = X(f) + Y(g), \tag{12}$$

where

$$X(f) = f(t)\partial x + f'(t)\partial u \quad (13)$$

$$Y(g) = g(t)\partial y - \frac{g'(t)y}{2S(t)}\partial x - \left(\frac{g'(t)}{2S(t)}\right)' y\, \partial u, \quad (14)$$

where $f(t)$ and $g(t)$ are arbitrary smooth functions and the primes denote time derivatives.

The commutation relations are

$$[X(f_1), X(f_2)] = 0,\ [X(f), Y(g)] = 0,$$

$$[Y(g_1), Y(g_2)] = X\left(\frac{1}{2S}(g_1'\, g_2 - g_1\, g_2')\right), \quad (15)$$

where [,] stands for the Lie bracket. It is readily seen that the coefficients of the vector fields $X(f)$ and $Y(g)$ multiplying ∂t are necessarily zero. This implies that the symmetry algebra does not have the structure of a Virasoro algebra. This stems from the fact that the equation under study is non-integrable. All known integrable equations in (2+1)-dimension have symmetry algebras of Virasoro type.

The vector fields $X(f)$ and $Y(g)$can be integrated to obtain the Lie group of transformations. Thus, if $u(x, y, t)$ is any solution to equation (9) then so are

$$\tilde{u} = u(x - \epsilon\, f(t), y, t) + \epsilon\, f'(t), \epsilon \in \mathbb{R} \quad (16)$$

and

$$\tilde{u} = u\left(x - \frac{1}{2S}\, g'\epsilon\left(y + \frac{g(t)\epsilon}{2}\right), y + g\epsilon, t\right) - \frac{1}{2}\left(\frac{g'}{S}\right)'\, \epsilon\, \left(y + \frac{g\,\epsilon}{2}\right), \quad (17)$$

respectively.

Now we shall show that the algebra L_p becomes larger when we specify the function $S(t)$. Here we list below such two extensions of L_p.

Case II. $S(t) = \sigma\, e^{\alpha t}, \sigma =$ Constant.
In this case, the symmetry algebra is represented by V in (12) further extended by the following additional element

$$T_\alpha = \partial_t + \frac{\alpha}{2}\, y\partial_y\, . \quad (18)$$

Case III. $S(t) = \sigma\ =$ Constant.
The symmetry algebra is L_p with the additional generator

$$T_0 = \partial_t. \quad (19)$$

The non-zero commutators amongst T_α, $X(f)$ and $Y(g)$are

$$[X(f), T_\alpha] = -X(f'),$$

$$[Y(g), T_\alpha] = Y\left(\frac{\alpha}{2} g - g'\right). \quad (20)$$

The Lie algebra L with a basis $X(f), Y(g)$ and T_α can be written as a semi-direct sum

$$L = \{X(f), Y(g)\} \oplus_S S, \tag{21}$$

where $S = T_\alpha$. For the **Case III**, we write

$$L = \{X(f), Y(g)\} \oplus_S T_0 . \tag{22}$$

3. A Finite dimensional Subalgebra by Physical Transformations

Now by restricting $f(t)$ and $g(t)$ to be a linear polynomials we obtain obvious physical symmetries spanned by

$$X \equiv X(1) = \partial_x, Y \equiv Y(1) = \partial_y,$$
$$B \equiv X(t) = t\,\partial_x + \partial_u, R \equiv Y(t) = -\frac{\sigma}{2} y\,\partial_x + t\,\partial_y, \tag{23}$$

which are space translations, Galilei transformations in the x direction and pseudorotations, respectively. The five-dimensional subalgebra $L_0 = \{T_0, X, Y, \beta, \gamma\}$ corresponding to the constant coefficient KdV-Burgers equation is solvable and has a nilpotent ideal (the Nil-radical) $NR(L_0)= \{T_0, X, Y, \beta, \gamma\}$ (see the commutataor table as shown below).

The Commutator table for the physical subalgebra L_0:

$[,]$	X	Y	T_0	β	γ
X	0	0	0	0	0
Y	0	0	0	0	$\frac{-X}{2\sigma}$
T_0	0	0	0	X	Y
β	0	0	$-X$	0	0
γ	0	$\frac{X}{2\sigma}$	-Y	0	0

4. Low-dimensional Subalgebras of the Symmetry Algebra

In order to perform the symmetry reductions in a systematic way, we need to classify subalgebras of the infinite-dimensional algebras. We use the approach followed in [5] as an adaptation of the methods developed for the classification of subalgebras of the finite dimensional algebras to infinite-dimensional ones. The difference is that we obtain differential conditions on the arbitrary functions labeling the group elements, rather than algebraic conditions on the parameters labeling the group elements of the finite-dimensional group. We present a classification of the one-dimensional subalgebras of the symmetry algebra into conjugacy classes under the adjoint action of the symmetry group. We do this individually for each algebra classified in the sections 2 and 3.

Case : A $S(t) = arbitrary \neq Constant.$

Conjugating the general element $V = X(f) + Y(g)$ by $Y(G)$ and using the commutation relations (16) we obtain

$Ad\{\exp(\epsilon\, Y(G))\}\, V = X\left(f - \frac{\epsilon}{2S}(G'g - G\, g')\right) + Y(g)$. (24)

If we choose a function $G(t)$ to be defined by

$G(t) = 2\vartheta g \int_0^t (sfg^{-2})\, u\, du + \mu\, g,$ (25)

where μ, ϑ are the arbitrary constants, as the function labeling the element $Y(G)$ of the symmetry algebra and $\epsilon = \vartheta^{-1}$ as the value of the parameter ϵ of the one parameter subgroup associated with $Y(G)$. Then it is evident that V is conjugate to $Y(g)$, if $g \neq 0$ and V is conjugate to $X(f)$if $g = 0$. Therefore it is enough to consider the two one-dimensional subalgebras namely $L_{\{p,1\}} = X(f)$ and $L_{\{p,2\}} = Y(g)$ instead of the full symmetry algebra L_p itself.

Case : B $S(t) = \sigma\, e^{\alpha t}, \alpha \neq 0.$

Similarly conjugating the general element $V = a\, T_\alpha + X(f) + Y(g),\ \alpha \neq 0$ by $X(F) + Y(G)$, we obtain

$Ad\{\exp(\epsilon\, X(F) + \delta\, Y(G))\}\, V =$

$aT_\alpha + Y\left(g - a\,\delta\left(\frac{\alpha}{2}\, G - G'\right)\right) + X\left(\left(-\frac{\delta}{2S}\right)(G'g - Gg') + f + \epsilon\, a\, F'\right).$ (26)

Therefore V is conjugate to T_α. If $a = 0$, then V is conjugate to $X(f), g = 0$.

Case : C $S(t) = \sigma = Constant.$

If we take the conjugation of $V = a\, T_0 + X(f) + Y(g),\ a \neq 0$ *by* $X(F) + Y(G)$, we obtain

$Ad\{\exp(\epsilon\, X(F) + \psi\, Y(G))\}V = aT_0 + Y(g + a\,\psi\, G') + X\left(\left(-\frac{\psi}{2\sigma}\right)(G'g - Gg') + f + \epsilon\, a\, F'\right).$ (27)

If we choose $a = 0, \psi = 1/b$ and $G(t)$ as in (27), then V is conjugate to $Y(g)$. On the other hand if we consider that $a \neq 0, \psi = 1/b, \epsilon = 1/c$ and define $F(t)$ and $G(t)$ as

$F(t) = \frac{c}{2a^2\sigma}\int(-g^2 + g' \int g(t)dt - f(t))\, dt + K,$

$G(t) = -\frac{b}{a}\int g(t)\, dt + M,$ (28)

where K and M are the arbitrary constants, then V is conjugate to T_0. If $a = g = 0$, then V is conjugate to $X(f)$.

5. Reductions to (1+1)-dimensional PDEs

The general method for performing the symmetry reduction using some specific subgroup G_0 of the full symmetry group is to first find the invariants of G_0 and rewrite (10) in terms of them. The invariants are obtained by solving the system of PDEs.

$$X_i\, I(x, y, t, u) = 0, i = 1, \dots, r, \tag{29}$$

where $\{X_1, \dots, X_r\}$ is some basis for the Lie algebra of G_0.

Below we perform reductions of (9) by one-dimensional subalgebras.

I. Subalgebra $L_{\{S,1\}} = \{X(f)\}$.

Integration of the one-dimensional vector field $X(f)$, where $f(t)$ is an arbitrary function leads to the reduction formula and the reduced PDE are,

$$u = W(y, t) + \frac{f'}{f}\, x, W_{yy} = -\frac{f''}{S\, f}. \tag{30}$$

Integrating the above eqn, we obtain an exact solution depending on three arbitrary functions of time,

$$u = -\frac{f''}{2\, S\, f}\, y^2 + \frac{f'}{f}\, x + A(t)\, y + B(t), \tag{31}$$

where $A(t)$ and $B(t)$ are the arbitrary functions.

II. Subalgebra $L_{\{S,2\}} = \{Y(g)\}$.

We use the substitution

$$u = W(\xi, \eta) - \frac{1}{4\, g(t)} \left(\frac{g'(t)}{S(t)}\right)' y^2;\ \xi = x + \frac{y^2\, g'}{4\, S(t) g(t)};\ \eta = t \tag{32}$$

and obtain the reduced PDE

$$W_t + W\, W_\xi + \lambda\, W_{\xi\xi} + \delta\, W_{\xi\xi\xi} + \frac{g'}{2\, g}\, W + N(t) - \xi \frac{S}{2\, g} \left(\frac{g'}{S}\right)' = 0, \tag{33}$$

where $N(t)$ is an arbitrary function of integration.

If $N = 0$ and $g = Constant,$ then the equation reduces to the one-dimensional KdV-Burgers equation

$$W_t + W\, W_\xi + \lambda\, W_{\xi\xi} + \delta\, W_{\xi\xi\xi} = 0. \tag{34}$$

If $\delta = 0,$ it reduces to the Burgers equation. When $\lambda = 0$, it reduces to the KdV equation. Additional reductions occur when $S(t) = \sigma\, e^{\alpha\, t}$.

III. Subalgebra $L_{\{S,3\}} = \{T_\alpha\}$.

Invariance under T implies,

$$u = W(x, \eta);\ \eta = y\, e^{\frac{-\alpha t}{2}};\ \xi = x \tag{35}$$

with W satisfying,

$$\left(W W_\xi + \lambda\, W_{\xi\xi} + \delta\, W_{\xi\xi\xi} - \frac{\alpha\, \eta}{2}\, W_\eta\right)_\xi + \sigma\, W_{\eta\eta} = 0. \tag{36}$$

6. Reductions to ODEs

One can further reduce the above-obtained PDE (36) to an ODE by imbedding T_α into two-dimensional subalgebras of the symmetry algebra. To this end we commute T_α with an element $V = X(f) + Y(g)$ and invoke that they form a two-dimensional subalgebra. This requirement implies that the functions $f(t)$ and $g(t)$ are no longer arbitrary but take some specific forms. Since there exist two isomorphic classes of two-dimensional Lie algebras, Abelian and non-Abelian, we distinguish between two algebras.

6.1 Abelian Subalgebra :

$$L_{A,1} = \left\{T_\alpha, \left(y\, e^{\frac{\alpha t}{2}}\right) + \partial_x\right\}.$$

Invariance under (37) implies that the solution has the form

$$u = R(\zeta) + \frac{\alpha^2}{16}\, y^2\, e^{-\alpha t}, \tag{37}$$

where

$$\zeta = x - y e^{\frac{\alpha t}{2}} + \frac{\alpha}{8}\, y^2\, e^{-\alpha t}, \tag{38}$$

with $R(\zeta)$ satisfying the third order ODE

$$\delta\, R''' + \lambda\, R'' + R\, R' + R' + \frac{\alpha}{4} R + \frac{\alpha^2}{8} \zeta = C_1. \tag{39}$$

For $\alpha = 0$ and integrating once then (39) reduces to the second order ODE

$$\delta\, R'' + \lambda\, R' + \frac{R^2}{2} + R = C_1 \zeta + C_2. \tag{40}$$

6.2 Non-Abelian Subalgebra :

$$L_{NA,1} = \left\{T_\alpha, \left(y\, e^{\frac{\alpha+2}{2t}}\right) + X\, e^t\right\}. \tag{41}$$

Invariance under (41) implies that the solution has the form

$$u = R(\zeta) + \frac{(\alpha^2-4)}{16}\, y^2\, e^{-\alpha t} + y\, e^{-\alpha t},$$

where

$$\zeta = x - y e^{\frac{-\alpha t}{2}} + \frac{\alpha+2}{8}\, y^2\, e^{-\alpha t}, \tag{42}$$

with $R(\zeta)$ satisfying the third order ODE

$$\delta\, R''' + \lambda\, R'' + R\, R' + R' + \frac{\alpha}{4} R + \frac{R}{2} - \frac{\zeta}{2} + \frac{\alpha^2}{8} \zeta = C_4. \tag{43}$$

For $\alpha = -2$ and integrating once then (43) reduces to the second order ODE

$$\delta\, R'' + \lambda\, R' + \frac{R^2}{2} + R = C_4 \zeta + C_5. \tag{44}$$

7. The General Form of Reductions of the Generalized

KdV-Burgers Equation$(u_t + u\, u_x + \lambda\, u_{xx} + \delta\, u_{xxx})_x + S(t)\, u_{yy} = 0$

The transformation $R(\zeta) = f^{-1}(\zeta)$ replaces the ODEs (40) and (44) respectively

$$f f'' + 2\,(f')^2 - \frac{\lambda}{\delta}\, f\, f' + \frac{f}{2\delta} + \frac{f^2}{2\delta} - (C_1\zeta + C_2)\frac{f^3}{\delta} = 0. \quad (45)$$

$$f f'' + 2\,(f')^2 - \frac{\lambda}{\delta}\, f\, f' + \frac{f}{2\delta} + \frac{f^2}{2\delta} - (C_4\zeta + C_5)\frac{f^3}{\delta} = 0. \quad (46)$$

We may write the general form of the equations (45) and (46) as

$$f f'' + a\,(f')^2 + b\, f\, f' + c\, f^2 + K\, f + g(\zeta) f^3 = 0, \quad (47)$$

which is the special case of the equation introduced by Mayil Vaganan and Senthilkumaran [20], viz.,

$$f f'' + a\,(\rho)(f')^2 + b(\rho) f\, f' + c(\rho) f^2 + d(\rho) f' + g(\rho) f^3 + kf = 0. \quad (48)$$

8. Conclusion

The results of the above sections can be summarized as follows:

- We investigated the group classification problem for the generalized (2+1)-dimensional KdV-Burgers equation.
- We found a classification of the one-dimensional subalgebras of the symmetry algebra under the adjoint (conjugate) action of the symmetry group. Next we constructed the two-dimensional subalgebras by using one-dimensional subalgebras.
- We obtained a classification of the reductions of the original equation with $S(t) = \sigma\, e^{\alpha t}$ to lower-dimensional PDEs and to second-order ODEs.

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