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K-Extensibility in Graph with Unique Maximum Independent Set

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Abstract

Let G be a simple graph. An independent set S of G of cardinality k is said to be extendable if S is contained in a maximum independent set of G. If S is a maximal independent set of cardinality k and S is not a maximum independent set of G, then S is not extendable. G is said to be completely extendable if every independent set of G is contained in a maximum independent set of G. In such a case G is also called a well covered graph. G is said to be trivially extendable if no independent set of cardinality less than $\beta_0(G)$ is extendable. In [9, 10] k-extendable graphs are studied. In this paper we consider graphs with unique maximum independent set and study extendability in such graphs. Also k-extendability with specific exceptions is also considered.

Keywords: Berge graph, k- extendability, well-covered graphs.

1. Introduction:

A set of vertices in a graph is independent if no two vertices in the set are joined by an edge. The maximum size of a set of independent vertices in a graph G is called the independence number of G and is denoted by $\beta_0(G)$. A set of independent vertices which attains the maximum size is referred to as a maximum independent set. A set S of independent vertices in a graph is maximal (with respect to set inclusion) if the addition to S of any other vertex in the graph destroys the independence. A maximal independent set in a graph is not necessarily maximum.

A simple graph G is called a Berge graph if every vertex of the graph is contained in a maximum independent set of the graph. Claude Berge introduced this concept in

1980. This led to the idea of k- extendability and well-covered graphs. Many papers have come out on well-covered graphs. A survey by Plummer in 1993 gives a good account of the results derived in well-covered graphs. Connected, well-covered, bipartite graphs were characterized by Ravindra.

A simple graph is said to be well covered if it has no isolated vertex and if $\beta_0(G) = i(G)$, that is every maximal independent set is maximum. This notion has been introduced by Plummer in 1970[5] and studied for bipartite graphs by Ravindra in 1977[6] and also by Berge[1]. We call a graph k-extendable if every independent set of size k is contained in a maximum independent set. This generalizes the concept of a B-graph ((i.e) 1-extendable graph) introduced by Berge and the concept of a well-covered graph. A fair amount of study has been devoted to B-graphs, for example see [1, 2, 7]. In general, there is no connection between k-extendability and j-extendability for $k \neq j$, and there are graphs which are not well-covered and which are k-extendable for given values of k. Also k-extendability does not imply (k-1)-extendability (or) vice versa.

2. k-extendable graphs

Definition :1

Let G = (V, E) be a simple graph. Let k be a positive integer, G is said to be k-extendable if every independent set of cardinality k in G is contained in a maximum independent set of G.

Example 1 In C_6 with vertex set $\{u_1, u_2, u_3, u_4, u_5, u_6\}$. $\{u_1, u_3, u_5\}$ and $\{u_2, u_4, u_6\}$ are maximum independent sets. $\beta_0(G) = 3$. C_6 is 1-extendable but not 2-extendable.

Definition 2

i. G is said to be trivially extendable if G is k-extendable only for $k = \beta_0(G)$. For example, P_5 is trivially extendable .
ii. A graph G is said to be completely extendable if G is k-extendable for every k, $1 \leq k \leq \beta_0(G)$. For example, C_4 is completely extendable.

Definition 3

A graph G is said to be well covered if every maximal independent set of G is a maximum independent set of G.

Definition 4

Let S be an independent set of G. A vertex u in V -S is called a public neighbor with respect to S, if u is adjacent to at least two vertices in S. Let S=$\{u_1, u_2, \ldots, u_k\}$ be an independent set of G. If $u \in V - S$ is a public neighbor with respect to S and u is adjacent with $u_{i_1}, u_{i_2}, \ldots, u_{i_r}$ in S, $r \geq 2$ then $(S - \{u_{i_1}, u_{i_2}, \ldots, u_{i_r}\}) \cup \{u\}$ is an independent set of cardinality $(k - r + 1) < k$.

Definition 5
Let S be an independent set of G. Let $u \in V - S$ be a public neighbor with respect to S. The public neighbor of u with respect to S is defined as $pbl(u,S) = \{x \in S:$ u and x are adjacent$\}= N(u) \cap S$. $|pbl(u,S)|$ is called public neighbor count of u with respect to S.

Theorem 1
Let G be a simple graph with a unique maximum independent set say S. Let $t = \max(|pbl(u,S)| : u \in V - S)$. Then G is not k-extendable for $k = \beta_0(G) - t + 1$.

Proof.
Let $S = \{u_1, u_2, \ldots\ldots\ldots, u_{\beta_0}\}$. Let $v \in V - S$ be such that $|pbl(v,S)| = t$.

Let $pbl(v,S) = \{u_{i_1}, u_{i_2}, \ldots\ldots\ldots, u_{i_t}\} \subset S$. Let $T = (S - pbl(v,S)) \cup \{v\}$. $|T| = \beta_0(G) - t + 1$. Since $v \in T$ and $v \in V - S$, T is not a subset of S. Therefore for the integer $k = \beta_0(G) - t + 1$, there exists an independent set namely T, which is not extendable to the unique maximum independent set S of G. Hence the proof.

Remark 1
With the hypothesis of the above theorem, G is not k-extendable, for $1 \le k \le \beta_0(G) - t + 1$.

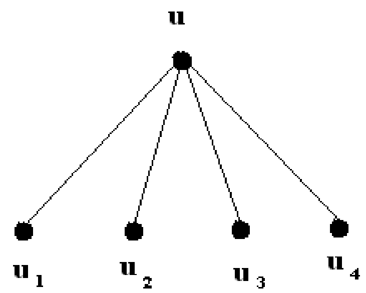
Theorem 2
Let G be a graph with a unique maximum independent set say S. Let T be a maximal independent set of G of cardinality t, $(t \ne \beta_0)$. Let $T = \{u_1, u_2, \ldots\ldots\ldots, u_t\}$. Let $t_i = |pn(u_i, T)| \ge 1, \forall i$. Then G is not k-extendable for $k = t, t + t_i - 1$ except for at most one $i, 1 \le i \le t$.

Proof.
Since G has a unique maximum independent set and T is a maximal independent set of G of cardinality t, T is not extendable. Therefore G is not k-extendable for k = t. Let $u_i \in T$. Let $T_i = pn(u_i, T)$.Then $(T - \{u_i\}) \cup T_i$ is a maximal independent set of G.

For: Suppose $T' = (T - \{u_i\}) \cup T_i$ is not maximal. Let $x \in V - T'$ be such that $T' \cup \{x\}$ is independent. If $x = u_i$, then $(T - \{x\}) \cup T_i = (T - \{u_i\}) \cup T_i$, $T' \cup \{x\} = T' \cup \{u_i\} = T \cup T_i$ is not independent, since $u_i \in T$ and $|T_i| \ge 1$, a contradiction. Then x is not adjacent with every $u_j \in T, j \ne i$ and x is not adjacent with any element of T_i .Since, $x \notin T_i$, x is not a private neighbor of u_i. Suppose x is a neighbour of u_i. Since x is not adjacent with every $u_i \in T, j \ne i$, x is a private neighbor of u_i with respect to T a contradiction. Therefore x is not adjacent to u_i. Therefore $T \cup \{x\}$ is an independent set of G, a contradiction, since T is a maximal independent set.

Suppose $(T-\{u_i\}) \cup T_i$ is extendable. Then as $(T-\{u_i\}) \cup T_i$ is maximal, $(T-\{u_i\}) \cup T_i$ coincides with the unique maximum independent set S. Therefore $T-\{u_i\}$ is a subset of S. If $T-\{u_i\}$, $T-\{u_j\}$ are subsets of S, where $i \neq j$, then $T \subset S$, a contradiction. Therefore for atmost one u_i, $(T-\{u_i\}) \cup T_i = S$. Therefore G is not k-extendable for $k = t, t+t_j - 1,\ j \neq i, 1 \leq j \leq t$.

Example 2
Let G:



$S = \{u_1, u_2, u_3, u_4\}$ is the unique maximum independent set of G. Let the maximal independent Set=$T = \{u\}$. $|T| = 1$. $pn(u,T) = \{u_1, u_2, u_3, u_4\}$ and $t_i = |pn(u_i, T)| = 4$. $(T-\{u\}) \cup pn(u,T) = \{u_1, u_2, u_3, u_4\} = S$. Therefore G is not k-extendable for k = 1. That is G is not k-extendable for $k = 1, t+t_i - 1$, for all except one i. since T contains only one element, G is not k-extendable for k = 1.

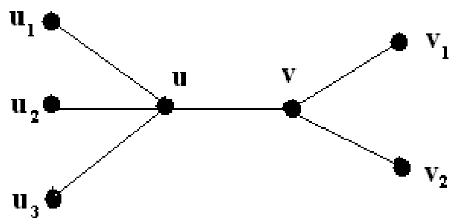
Theorem 3
Let $u, v \in V(G)$ be such that any maximum independent set of G does not contain both u and v. Suppose there exists an independent set of G of cardinality $(\beta_0(G) - 1)$ containing u (or) v. Then G is trivially k-extendable. i.e.,) G is not k-extendable, for $1 \leq k \leq (\beta_0(G) - 1)$.

Proof.
Let S be an independent set of G of cardinality$(\beta_0(G) - 1)$containing u. Let $S = \{v_1, v_2, \ldots\ldots\ldots, v_t\}$, where $t = (\beta_0(G) - 1)$.Let without loss of generality $v_1 = u$. Then, $S - \{v_2\}$, $S - \{v_2, v_3\}, \ldots\ldots\ldots\ldots\ldots$, $S - \{v_2, v_3, \ldots\ldots\ldots, v_t\}$are independent sets of G containing u. Therefore these sets are not contained in any maximum independent

set of G. Therefore G is not k-extendable for $1 \le k \le (\beta_0(G) - 1)$.

Example 3

Let G :



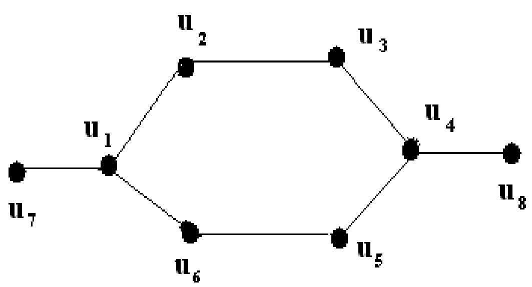
Consider the double star $D_{3,2}$. Let u and v be the centers. Let u_1, u_2, u_3 be adjacent with u and v_1, v_2 be adjacent with v in $D_{3,2}$. $D_{3,2}$ contains a unique maximum independent set namely $\{u_1, u_2, u_3, v_1, v_2\}$. u and v are not contained in this unique β_0- set. $\{u_1, u_2, u_3, v\}$ is a $(\beta_0(D_{3,2}) - 1)$-set of $D_{3,2}$ which is independent and this set contains v. $D_{3,2}$ is not k-extendable for $1 \le k \le 4$ since, $\{u\}, \{u, v_2\}, \{u, v_1, v_2\}, \{v, u_1, u_2, u_3\}$ are independent sets of $D_{3,2}$ which are not extendable.

In fact, $D_{r,2}$ where $r \ge 2$ is also an example .

Example 4

There are graphs in which $2 \le i(G) < \beta_0(G)$ such that G is not k-extandable only for $k = i(G)$.

Let G:



$i(G) = 2$ and $\beta_0(G) = 4$. $\{u_1, u_4\}$ is the unique $i(G)$-set. $\{u_2, u_4, u_6, u_7\}$, $\{u_1, u_3, u_5, u_8\}$, $\{u_3, u_6, u_7, u_8\}$, $\{u_2, u_5, u_7, u_8\}$, $\{u_2, u_6, u_7, u_8\}$, $\{u_3, u_5, u_7, u_8\}$ are maximum independent sets.

G is 1-extendable, 3-extendable but not 2-extendable.

Theorem 4

If $i(G) = \beta_0(G)$, then G is k-extendable for all k, $1 \le k \le \beta_0(G)$.

Proof.

Let G be not k-extendable for some k, say k_1, $1 \le k_1 \le \beta_0(G)$. Let S be a maximal independent set of G of cardinality k_1 such that S is not extendable. Therefore S is an independent dominating set of G with $|S| < \beta_0(G) = i(G)$, a contradiction since, $i(G)$ is independent dominating number of G. Therefore G is k-extendable for all k, $1 \le k \le \beta_0(G)$.

Theorem 5

Let G be a simple graph in which $\beta_0(G) = (i(G)+1)$. Suppose G is $(i(G)-1)$-extendable then G is k-extendable for all k, $1 \le k \le (i(G)-1)$.

Proof.

Let S be an independent set of G of cardinality k, $1 \le k \le (i(G)-1)$. S is not a maximal independent set since $i(G) > |S|$. Therefore Sis properly contained in a maximal independent set say T. Clearly, $|T| \ge i(G)$. If $|T| > i(G)$, then $|T| = \beta_0(G)$. Therefore S is extendable. Suppose $|T| = i(G)$. Consider $T_1 = T - \{u\}$, where $u \notin S$. T_1 is an independent set of cardinality $(i(G)-1)$ and $S \subset T_1$. T_1 is extendable by hypothesis and hence S is extendable Hence the theorem.

Theorem 6

Let G be a simple graph which is not well covered. Suppose G is k-extendable for all $k \le \beta_0(G) - 2$. Then $i(G) + 1 = \beta_0(G)$.

Proof.

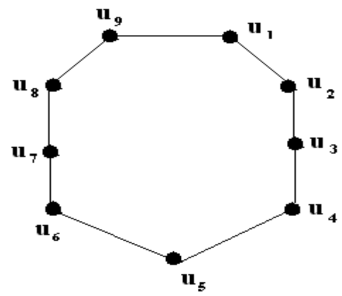
Let G be extendable for all $k \le i(G) - 2$. If $i(G) = \beta_0(G)$, then G is $(\beta_0(G)-1)$-extendable. For:

Let S be an independent set of cardinality $(i(G)-1)$. S is not a maximal independent set, since $|S| < i(G)$. Therefore S is contained in a maximal independent set of cardinality $> |S|$. Since $i(G) = \beta_0(G)$, any super set of S which is independent is of cardinality $i(G) = \beta_0(G)$. That is S is contained in a maximum independent set. Therefore G is well covered, a contradiction. Therefore $i(G) < \beta_0(G)$. That is $i(G) \le \beta_0(G) - 1$. By hypothesis, G is k-extendable for all $k \le \beta_0(G) - 2$. If

$i(G) \le \beta_0(G) - 2$, then G is $i(G)$-extendable, a contradiction, since, $i(G) < \beta_0(G)$. Therefore $i(G) > \beta_0(G) - 2$. That is $i(G) \ge \beta_0(G) - 1$. But $i(G) \le \beta_0(G) - 1$. Therefore $i(G) = \beta_0(G) - 1$. That is $i(G) + 1 = \beta_0(G)$.

Example 5

Let G = C_9



$i(G) = 3$ and $\beta_0(G) = 4$. $\{u_1, u_4, u_7\}$ is an $i(G)$-set. $\{u_1, u_3, u_5, u_7\}$, $\{u_2, u_4, u_6, u_8\}$, $\{u_1, u_3, u_5, u_8\}$, $\{u_2, u_4, u_6, u_9\}$, $\{u_1, u_4, u_6, u_8\}$, $\{u_2, u_5, u_7, u_9\}$, $\{u_1, u_3, u_6, u_8\}$, $\{u_3, u_5, u_7, u_9\}$, $\{u_2, u_4, u_7, u_9\}$ are maximum independent sets. Clearly, G is 1-extendable, 2-extendable but not 3-extendable, since $\{u_1, u_4, u_7\}$ is not contained in any maximum independent sets of C_9.

Therefore G is k-extendable for $1 \le k \le 2$. i.e.,) G is k-extendable for $1 \le k \le (\beta_0(G) - 2)$.

Here $i(G) + 1 = \beta_0(G)$.

Theorem 7

Let G be a graph. G is k-extendable for all k except $k = (\beta_0(G) - 1)$ iff $i(G) + 1 = \beta_0(G)$ and G is $(\beta_0(G) - 2)$-extendable.

Proof.

Suppose $i(G) + 1 = \beta_0(G)$ and G is $(\beta_0(G) - 2)$-extendable. Let $1 \le k \le \beta_0(G) - 3$. Let S be an independent set of cardinality k. Since $k < i(G)$, S is not maximal. Therefore S is properly contained in a maximal independent set say T. Since T is maximal $|T| \ge i(G)$. Suppose $|T| = i(G) = (\beta_0(G) - 1)$. Let $u \in T - S$. Let $T_1 = T - \{u\}$. Then T_1 is an independent set of cardinality $(\beta_0(G) - 2)$. Therefore T_1 is contained in a β_0-set of G. Since $S \subseteq T_1$, S is contained in a $\beta_0(G)$ -set of G. If $|T| > i(G)$, then $|T| = \beta_0(G)$. Therefore S is contained in a maximum independent set of G. i.e.,) G is k–extendable for all k, except $k = (\beta_0(G) - 1)$.

Conversely, Suppose G is k–extendable for all k, except $k = (\beta_0(G) - 1)$ .Then clearly G is $(\beta_0(G)-2)$-extendable $i(G) < \beta_0(G)$ and $i(G) \geq (\beta_0(G)-1)$. Therefore $i(G) = (\beta_0(G)-1)$.

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