Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 9, Number 6 (2013), pp. 559-566

http://www.ripublication.com/gjpam.htm

# On Intuitionistic Fuzzy β-sub algebras of β-algebras

**K.Sujatha[1], M.Chandramouleeswaran[2] and P.Muralikrishna[3]**

[1]*Assistant Professor, Department of Mathematics,*
*SBK College, Aruppukottai - 626 101 Tamilnadu.*
*ksujatha203@gmail.com*
[2]*Associate Professor and Head, Department of Mathematics,*
*SBK College, Aruppukottai - 626 101 Tamilnadu.*
*moulee59@gmail.com*
[3]*Assistant Professor, School of Advanced Sciences,*
*VIT University, Vellore - 632 014. Tamilnadu*
*prakasam.muralikrishna@yahoo.com*

**Abstract**

In [1] author's introduced the notion of fuzzy β-algebras. In this paper, we introduce the notion of the Intuitionistic Fuzzy- structures of β-algebras, and prove some interesting results.

**AMS Classification:** 08A72, 03E72

**Key words :** *β- algebra, Intuitionistic Fuzzy β- subalgebra*

## 1.Introduction

In 1996, Y.Imai and Iseki [3] introduced two classes of algebras originated from the classical and non-classical propositional logic. These algebras are known as BCK and BCI algebras. It is known that the notion of BCI-algebra is a generalization of BCK-algebras in the sense that the class of BCK algebras is a proper subclass of the class of BCI –algebras [4]. In 2002, J.Neggers and H.S.Kim introduced the notion of B-algebras, which is another generalization of BCK-algebras [7]. Also they introduced the notion of β-algebras [ 8] where two operations are coupled in such a way as to reflect the natural coupling, which exists, between the usual group operation and its associated B-algebra, which is naturally defined by the group. In 2012, Young Hee Kim and Keun Sook So, gave a new approach of β-algebras and proved some related results [12].

In 1965, L.A.Zadeh [13 ] introduced a new notion of fuzzy set, to evaluate the modern concept of uncertainty in real life. The notion of fuzzy sets is a generalization of the notion of crisp sets in which the boundaries are not crisp or sharp. The study of fuzzy algebraic structures was initiated by A.Rosenfeld [10] .Hu and Li introduced the notion of fuzzy subring. K.T.Atanasov introduced the notion of Intuitionistic Fuzzy set in 1986, in which not only the membership value is considered but also we consider non-membership values [2]. In 1991, X.Ougen [9] defined fuzzy subset in BCK-algebras and investigated some properties. In 1963 Y.B.Jun [6] applied it to BCI-algebras. In 2012,P.K.Sharma [11] introduced the t-Intuitionistic Fuzzy Subrings and investigated some properties. Recently in 2013 the authors of [1], introduced the concept of Fuzzy β-sub algebras of β-algebras.

Motivated by this, we are interested in connecting the two notions β-algebras and Intuitionistic Fuzzy Sets. In this paper we discuss Intuitionistic Fuzzy concept of β-sub algebras of β-algebra and investigate some of their properties.

## 2.PRELIMINARIES

In this section we recall some basic definitions and results that are needed in the sequel.

**Definition 2.1. [3]**

A BCK-algebra (X,*,0) is a non-empty set X with a constant 0 and a binary operation * satisfying the following axioms:

1. $((x * y) * (x * z)) * (x * z) = 0$
2. $(x * (x * y)) * y = 0$
3. $x * x = 0$
4. $x * y = 0$ and $y * x = 0 \Rightarrow x = y$
5. $0 * x = 0$ for all $x, y \epsilon X$.

**Definition 2.2. [4]**

A BCI-algebra (X,*,0) is a non-empty set X with a constant 0 and a binary operation * satisfying the following axioms:

1. $((x * y) * (x * z)) * (x * z) = 0$
2. $(x * (x * y)) * y = 0$.
3. $x * x = 0$
4. $x * y = 0$ and $y * x = 0 \Rightarrow x = y$ for all $x, y \epsilon X$.

**Definition 2.3.[8]**

A β-algebra is a non empty set X with a constant 0 and a binary operation + and - satisfying the following axioms:

1. $x + x = 0$ and $x - x = 0$
2. $(0 - x) + x = 0$
3. $(x - y) - z = x - (z + y)$ for all $x, y \epsilon X$.

**Example 2.4.**
Let X = {0,1,2,3} be a set with constant 0 and binary operations + and - are defined on X by the following Cayley's tables.

| + | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 3 | 2 | 1 |
| 1 | 1 | 0 | 3 | 2 |
| 2 | 2 | 1 | 0 | 3 |
| 3 | 3 | 2 | 1 | 0 |

| - | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 3 | 2 | 1 |
| 1 | 1 | 0 | 3 | 2 |
| 2 | 2 | 1 | 0 | 3 |
| 3 | 3 | 2 | 1 | 0 |

Then $(X, +, -, 0)$ is called a β-algebra.

**Definition 2.5.**
A non empty subset A of a β-algebra $(X, +, -, 0)$ is called a β- sub algebra of X, if $x+y \in A$ and $x-y \in A$ for all $x, y \in A$.

**Example 2.6.**
In the above example of the β-algebra, the subset A = {0, 2} is a β- sub algebra of X.

**Definition 2.7.[2].**
An Intuitionistic fuzzy set (IFS) in a nonempty set X is defined by

$$A = \{< x, \mu_A(x), \vartheta_A(x) > / x \in X\}$$

where $\mu_A : X \to [0,1]$ is a membership function of A and $\vartheta_A : X \to [0,1]$ is a non membership function of A satisfying $0 \le \mu_A(x) + \vartheta_A(x) \le 1$ for all $x \in X$.

## 3. Intuitionistic fuzzy β- sub algebras

**Definition 3.1.**
Let $(X, +, -, 0)$ be a β-algebra. Then $A = \{< x, \mu_A(x), \vartheta_A(x) > / x \in X\}$ is called an Intuitionistic fuzzy (IF) β-algebra of X, if following conditions are satisfied.

1.$\mu_A(x+y) \ge \min\{\mu_A(x), \mu_A(y)\}$ and $\vartheta_A(x+y) \le \max\{\vartheta_A(x), \vartheta_A(y)\}$

2.$\mu_A(x-y) \ge \min\{\mu_A(x), \mu_A(y)\}$ and $\vartheta_A(x-y) \le \max\{\vartheta_A(x), \vartheta_A(y)\}$ where $0 \le \mu_A(x) + \vartheta_A(x) \le 1$ for all $x \in X$.

**Example 3.2.**
For the β-algebra defined in the example 2.4. is the IF subset of X.

$$\mu_A(x) = \begin{cases} .8, & x = 0 \\ .5, & x = 1,2 \\ .3, & x = 3 \end{cases} \quad \text{and} \quad \vartheta_A(x) = \begin{cases} .1, & x = 0 \\ .5, & x = 1,2 \\ .4, & x = 3 \end{cases}$$

is an IF β-sub algebra of X.

**Definition 3.3.**
Let $(X,+,-,0)$and $(Y,+,-,0)$be two β-algebras. Let $A = \{< x, \mu_A(x), \vartheta_A(x) > / x \in X\}$ and $B = \{< y, \mu_B(y), \vartheta_B(y) > / y \in Y\}$ be IF subsets in X and Y. The Cartesian product of A and B, denoted by, A x B is defined to be the set $A \times B = \{< (\mu_A \times \mu_B)(x,y), (\vartheta_A \times \vartheta_B)(x,y) > / x \in X \times Y\}$ where $(\mu_A \times \mu_B) : X \times Y \to [0,1]$ is given by $(\mu_A \times \mu_B)(x,y) = \min\{\mu_A(x), \mu_B(y)\}$ and $(\vartheta_A \times \vartheta_B): X \times Y \to [0,1]$ is given by $(\vartheta_A \times \vartheta_B)(x,y) = \max\{\vartheta_A(x), \vartheta_B(y)\}$ for all $x \in X$ and $y \in Y$.

**Definition 3.4.**
Let $(X,+,-,0)$and $(Y,+,-,0)$ be two β-algebras .A mapping $f : X \to Y$ .is said to be a β- homomorphism, if $f(x_1 + x_2) = f(x_1) + f(x_2)$ and $f(x_1 - x_2) = f(x_1) - f(x_2)$ for all $x_1, x_2 \in X$.

**Theorem 3.5.**
If A and B are two IF β- sub algebras of X, then A ∩ B is also an IF β-sub algebra of X.

**Proof:**
Let x, y ∈ X,

$$
\begin{aligned}
(\mu_A \cap \mu_B)(x+y) &= \min\{\mu_A(x+y), \mu_B(x+y)\} \\
&\geq \min(\min\{\mu_A(x), \mu_A(y)\}, \min\{\mu_B(x), \mu_B(y)\}) \\
&= \min(\min\{\mu_A(x), \mu_B(x)\}, \min\{\mu_A(y), \mu_B(y)\}) \\
&= \min\{(\mu_A \cap \mu_B)(x), (\mu_A \cap \mu_B)(y)\}
\end{aligned}
$$

$$
\begin{aligned}
(\vartheta_A \cap \vartheta_B)(x+y) &= \max\{\vartheta_A(x+y), \vartheta_B(x+y)\} \\
&\leq \max(\max\{\vartheta_A(x), \vartheta_A(y)\}, \max\{\vartheta_B(x), \vartheta_B(y)\}) \\
&= \max(\max\{\vartheta_A(x), \vartheta_B(x)\}, \max\{\vartheta_A(y), \vartheta_B(y)\}) \\
&= \max\{(\vartheta_A \cap \vartheta_B)(x), (\vartheta_A \cap \vartheta_B)(y)\}
\end{aligned}
$$

Similarly, one can prove that
$(\mu_A \cap \mu_B)(x-y) \geq \min\{(\mu_A \cap \mu_B)(x), (\mu_A \cap \mu_B)(y)\}$ and
$(\vartheta_A \cap \vartheta_B)(x-y) \leq \max\{(\vartheta_A \cap \vartheta_B)(x), (\vartheta_A \cap \vartheta_B)(y)\}$

The above theorem can be extended for any family of IF β-sub algebra of X as follows.

**Corollary 3.6.**
Let {Ai / i ∈ I} be an arbitrary family of IF β-sub algebra of X. Then $\cap A_i$ is also a IF β- sub algebra of X.

**Theorem 3.7.**
Let X be a β-algebra. Let $A = \{< x, \mu(x), \vartheta(x) > / x \in X\}$ be an IF β- subalgebra of X. Then

1. $\mu$ (x) $\leq \mu$ (0) and $\vartheta$ (x) $\geq \vartheta$ (0) .
2. $\mu$ (x) $\leq \mu(x^*) \leq \mu$ (0) and $\vartheta$ (x) $\geq \vartheta$ $(x^*) \geq \vartheta$ (0) where $x^* = 0 - X$.

**Proof :**
For any $x \in X$,
$$\mu(0) = \mu(x - x) \geq \min\{\mu(x), \mu(x)\} = \mu(x).$$

similarly, $\vartheta(0) = \vartheta(x - x) \leq \max\{\vartheta(x), \vartheta(x)\} = \vartheta(x)$.
$$\mu(x^*) = \mu(0 - x) \geq \min\{\mu(0), \mu(x)\} = \mu(x).$$

That is $\mu(x^*) \geq \mu(x)$.
Hence , $\mu(x) \leq \mu(x^*) \leq \mu(0)$.
$$\vartheta(x^*) = \vartheta(0 - x) \leq \max\{\vartheta(0), \vartheta(x)\} = \vartheta(x).$$

That is $\vartheta(x^*) \leq \vartheta(x)$.
Hence $\vartheta(x) \geq \vartheta(x^*) \geq \vartheta(0)$.

**Theorem 3.8.**
Let A be an IF-β sub algebra of a β- algebra of X. Let $X_A = \{x \in X \backslash \mu(x) = \mu(0) \text{ and } \vartheta(x) = \vartheta(0)\}$. Then $X_A$ is also a β-sub algebra of X.

**Proof:**
Let A be an IF-β sub algebra of a β-algebra of X.
Let $x, y \in X_A$, then $\mu(x) = \mu(0)$ and $\vartheta(x) = \vartheta(0)$
$\mu(y) = \mu(0)$ and $\vartheta(y) = \vartheta(0)$

For, $\mu_A(x + y) \geq \min\{\mu_A(x), \mu_A(y)\} = \min\{\mu_A(0), \mu_A(0)\} = \mu_A(0)$
$\mu_A(x + y) \geq \mu_A(0)$

Now, $\mu_A(0) = \mu_A(0 + 0) \geq \min\{\mu_A(0), \mu_A(0)\} = \min\{\mu_A(x), \mu_A(y)\} = \mu_A(x + y)$
$\Rightarrow \mu_A(x + y) = \mu_A(0)$.

For, $\vartheta_A(x + y) \leq \max\{\vartheta_A(x), \vartheta_A(y)\} = \max\{\vartheta_A(0), \vartheta_A(0)\} = \vartheta_A(0)$
$\vartheta_A(x + y) \leq \vartheta_A(0)$

Now, $\vartheta_A(0) = \vartheta_A(0+0) \leq \max\{\vartheta_A(0), \vartheta_A(0)\} = \max\{\vartheta_A(x), \vartheta_A(y)\} = \vartheta_A(x+y)$
$\Rightarrow \vartheta_A(x + y) = \vartheta_A(0)$.

Therefore $x + y \in X_A$. $x, y \in X$.
Similarly we can prove that, $x - y \in X_A$.
Thus $X_A$ is a β-sub algebra of X.

**Theorem 3.9.**
Let A be an IF β- algebra of X. Let $X_A = \{x \in X / \mu(x) = \mu(0) \& \vartheta(x) = 1 - \vartheta(0)\}$. Then $X_A$ is a β-sub algebra of X.

**Proof :**

Let x, y $\in X_A$, then $\mu(x) = \mu(0)$ and $\vartheta(x) = 1 - \vartheta(0)$

$\mu(y) = \mu(0)$ and $\vartheta(y) = 1 - \vartheta(0)$

$$\mu_A(x + y) \geq \min\{\mu_A(x), \mu_A(y)\}$$
$$= \min\{\mu_A(0), \mu_A(0)\} = \mu_A(0)$$
$$\mu_A(x + y) \geq \mu_A(0)$$

Now, $\mu_A(0) = \mu_A(0 + 0) \geq \min\{\mu_A(0), \mu_A(0)\} = \min\{\mu_A(x), \mu_A(y)\} = \mu_A(x + y) \Rightarrow \mu_A(x + y) = \mu_A(0)$.

$$\vartheta_A(x + y) \leq \max\{\vartheta_A(x), \vartheta_A(y)\}$$
$$= \max\{1 - \vartheta_A(0), 1 - \vartheta_A(0)\}$$
$$= 1 - \vartheta_A(0)$$
$$\vartheta_A(x + y) \leq 1 - \vartheta_A(0)$$

Now, $1 - \vartheta_A(0) = 1 - \vartheta_A(0-0)$

$$\leq 1 - \max\{\vartheta_A(0), \vartheta_A(0)\}$$
$$= \min\{1 - \vartheta_A(0), 1 - \vartheta_A(0)\}$$
$$\leq \max\{1 - \vartheta_A(0), 1 - \vartheta_A(0)\}$$
$$= \max\{\vartheta_A(x), \vartheta_A(y)\}$$
$$= \vartheta_A(x+y)$$
$$\Rightarrow \vartheta_A(x + y) = 1 - \vartheta_A(0).$$

Therefore $x + y \in X_A$.

Similarly we can prove that, $x - y \in X_A$.

**Theorem 3.10.**

Let A and B be two IF β-sub algebras of $X$ *and* $Y$ respectively. Then $A \times B$ is also an IF β- sub algebra of $X \times Y$.

**Proof:**

$A = \{< x, \mu_A(x), \vartheta_A(x) > / x \in X\}$ and

$B = \{< y, \mu_B(y), \vartheta_B(y) > / y \in Y\}$ be IF- β-sub algebras in X and Y.

Take $x = (x_1, x_2)$, $y = (y_1, y_2) \in X \times Y$ and $\mu = \mu_A \times \mu_B$.

$$\mu(x + y) = \mu((x_1, x_2) + (y_1, y_2))$$
$$= (\mu_A \times \mu_B)((x_1 + y_1), (x_2 + y_2))$$
$$= \min\{\mu_A(x_1 + y_1), \mu_B(x_2 + y_2)\}$$
$$\geq \min\{\min\{\mu_A(x_1), \mu_A(y_1)\}, \min\{\mu_B(x_2), \mu_B(y_2)\}\}$$
$$= \min\{\min\{\mu_A(x_1), \mu_B(x_2)\}, \min\{\mu_A(y_1), \mu_B(y_2)\}\}$$
$$= \min\{(\mu_A \times \mu_B)(x_1, x_2), (\mu_A \times \mu_B)(y_1, y_2)\}$$
$$= \min\{\mu(x), \mu(y)\}$$

Similarly, $\mu(x - y) \geq \min\{\mu(x), \mu(y)\}$.

And also, $\vartheta(x+y) = \vartheta((x_1+y_1), (x_2+y_2))$

$$= (\vartheta_A \times \vartheta_B)((x_1 + y_1), (x_2 + y_2))$$
$$= \max\{\vartheta_A(x_1 + y_1), \vartheta_B(x_2 + y_2)\}$$
$$\leq \max\{\max\{(\vartheta_A x_1), \vartheta_A(y_1)\}, \max\{\vartheta_B(x_2), \vartheta_B(y_2)\}\}$$

$= \max\{\max\{\vartheta_A(x_1), \vartheta_B(x_2)\}, \max\{\vartheta_A(y_1), \vartheta_B(y_2)\}\}$
$= \max\{(\vartheta_A \times \vartheta_B)(x_1, x_2), (\vartheta_A \times \vartheta_B)(y_1, y_2)\}$
$= \max\{\vartheta(x), \vartheta(y)\}$
$\vartheta(x+y) \leq \max\{\vartheta(x), \vartheta(y)\}$

Similarly, we prove that $\vartheta(x-y) \leq \max\{\vartheta(x), \vartheta(y)\}$.

Hence the Cartesian product of $A \times B$ is also an IF β- sub algebra of $X \times Y$.

**Theorem 3.11.**
If $A = \{< x, \mu_A(x), \vartheta_A(x) >/x \in X\}$ is an IF –β sub algebra of X, then so is $\odot A$, where $\odot A = \{< x, \mu_A(x), \overline{\mu_A}(x) >/x \in X\}$

**Proof:**
Let A be an IF –β sub algebra of X.

Let x, y $\in X$.

$\overline{\mu_A}(x+y) = 1 - \mu_A(x+y)$
$\geq 1 - \min\{\mu_A(x), \mu_A(y)\}$
$\geq \max\{1 - \mu_A(x), 1 - \mu_A(y)\}$
$= \min\{1 - \mu_A(x), 1 - \mu_A(y)\}$
$= \min\{\overline{\mu_A}(x), \overline{\mu_A}(y)\}$

Similarly, $\overline{\mu_A}(x-y) \geq \min\{\overline{\mu_A}(x), \overline{\mu_A}(y)\}$

Hence $\odot A$ is an IF –β sub algebra of X.

**Definition 3.12.**
Let f: X → Y be a function. Let A and B be two IF-β sub algebra of X and Y. Then the inverse image of B under f is defined as $f^{-1}(B) = \{< x, f^{-1}(\mu_B(x)), f^{-1}(\vartheta_B(x)) > / x \in X\}$

Such that $f^{-1}(\mu_B(x)) = \mu_B(f(x))$ and $f^{-1}(\vartheta_B(x)) = \vartheta_B(f(x))$.

**Theorem 3.13.**
Let $(X, +, -, 0)$ and $(Y, +, -, 0)$ be two β- algebras and Let $f: X \to Y$ be a homomorphism. If A is an IF-β sub algebra of Y, then $f^{-1}$ (A) is an IF-β sub algebra of X.

**Proof:**
Let A be an IF-β sub algebra of Y.
For x, y $\in X$.

$f^{-1}(\mu_A(x+y)) = \mu_A(f(x+y))$
$= \mu_A(f(x) + f(y))$
$\geq \min\{\mu_A(f(x), \mu_A(f(y))\}$
$= \min\{f^{-1}(\mu_A(x)), f^{-1}(\mu_A(y))\}$

$f^{-1}(\vartheta_A(x+y)) = \vartheta_A(f(x+y))$
$= \vartheta_A(f(x) + f(y))$
$\leq \max\{\vartheta_A(f(x), \vartheta_A(f(y))\}$
$= \max\{f^{-1}(\vartheta_A(x)), f^{-1}(\vartheta_A(y))\}$

Analogously, one can prove that $\mu_A$(x-y) $\geq \min\{f^{-1}(\mu_A(x)), f^{-1}(\mu_A(y))\}$ and $\vartheta_A$(x-y) $\leq \max\{f^{-1}(\vartheta_A(x)), f^{-1}(\vartheta_A(y))\}$

Hence $f^{-1}$is a IF $\beta$ – sub algebra of X.

**Theorem 3.14.**
Let $(X,+,-,0)$ and $(Y,+,-,0)$ be two β- algebras .Let f: X → Y be an endomorphism.
If A is IF $\beta$ – sub algebra of X, define $f(A) = \{< x, \mu_f(x)) = \mu(f(x)), \vartheta_f(x) = \vartheta(f(x)) > / x \in X\}$. Then $f(A)$ is IF $\beta$ – sub algebras of Y.

**Proof :**
Let x,y $\in X$.

$$\mu_f(x+y)) = \mu(f(x+y)) = \mu(f(x)+f(y)) \geq \min\{\mu(f(x), \mu(f(y))\}$$
$$= \min\{\mu_f(x), \mu_f(y))$$

Similarly, we prove that $\mu_f(x-y) \geq \min\{\mu_f(x), \mu_f(y))\}$

Further,

$\vartheta_f(x+y) = \vartheta(f(x+y)) = \vartheta(f(x)+f(y)) \leq \max\{\vartheta(f(x), \vartheta(f(y))\} = \max\{\vartheta_f(x), \vartheta_f(y))\}$ Similarly, we prove that $\vartheta_f(x-y)) \leq \max\{\vartheta_f(x), \vartheta_f(y))\}$.

Hence $f(A)$ is an IF $\beta$ – sub algebra of Y.

## References :


[1] M.AbuAyub Anasri and M.Chandramouleeswaran, IJMSE, Vol.7, No.V,Sep-2013.
[2] K.T.Atanasov, Intuitionistic Fuzzy sets and systems.J.Math.Appl. 18(1987).
[3] Y.imai and K.Iseki, On axiom systems of propositional calculi, Proc.Japan Academy.(1973),19-22.
[4] Iseki K. An introduction to theory of BCK-algebra,Math Japan(1973),1-26.
[5] Iseki, ON BCI-algebras,Math semin.Notes.Kobe Univ. 11(1983),313-320.
[6] Y.B.Jun, Closed ideals in BCI-algebras, 38 (1)(1993),pp 199-202.
[7] Neggers and Kim Hee Sik,On B-algebras,Math.Solvaca.no 5,517-530.
[8] Neggers and Kim Hee Sik,On β-algebras,Math.Solvaca.no 5,517-530.
[9] X.Ougen, Fuzzy BCK-algebras, Math.Japaninca, 36(5)(1991),pp 935-942.
[10] A.Rosenfeld,Fuzzy Groups, J.Math.Anal.Appl. 35(1971),512-517.
[11] P. K. Sharma, t- Intuitionistic Fuzzy Subrings. IJMS, Vol. 11, No. 3-4, pp. 265-275.
[12] Young Hee Kim and Keun Sook So, β-algebras and realted topics, commun. Korean Math. Soc., 27(2012).
[13] L.A.Zadeh, fuzzy sets,inform.control (1965),338-353.