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Invariance of Topological Properties Using Almost Continuous Functions from Cofinite Space to Euclidean $\mathbb{R}^n$ space

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Abstract

Invariance and inverse invariance of topological properties has been investigated using continuous functions and almost continuous functions from one topological space to another topological space. This has been done in both discrete and indiscrete spaces. In this paper we show the invariance of some topological properties using almost continuous functions from a cofinite space to a Euclidean $\mathbb{R}^n$ space.

1. Introduction and Literature Review

The concept of almost continuous function was studied for real valued functions on Euclidean spaces (Blumberg, 1922). Almost continuous function had been defined differently by different authors as indicated by Prakash and Srivastava (1977). Almost continuity generalizes the notion of continuity and every continuous function is an almost continuous function even though the reciprocal may not hold. To visualize this, consider $X = Y =$ the set of all real numbers with the usual topology where the open sets are taken to be the open intervals in the real line, then the function $f: X \longrightarrow Y$ defined by: $f(x) = \begin{cases} \sin\frac{1}{x}, x \neq 0 \\ 0, x = 0 \end{cases}$ is an almost continuous function that is not continuous. Clearly this function oscillates near the point $x = 0$ hence its limit cannot exist at that point implying that the function cannot be continuous. Almost continuous mapping have been introduced in several spaces and its properties and characterization have been studied. Some of the properties and several results concerning almost continuous functions have been studied and proved (Long and McGehee, 1970). Invariance and inverse invariance of some topological properties with respect to continuous functions and almost continuous functions have been studied by Gichuki (1996). Since all continuous functions are subsets of almost continuous functions, a property that is not preserved by continuous functions cannot be preserved by almost continuous functions. It had been established that among the class of T_1- spaces, invariance of topological properties with respect to continuous bijections implies invariance of the same topological properties with respect to almost continuous bijections, and that if a property $\mathcal{P}$ is invariant of continuous functions, it must be invariant of continuous bijections and hence invariant of continuous bijections in the class of T_1- spaces (Gichuki, 1996). A remark was made that most of the interesting results with almost continuous functions are obtained with the class of T_1- spaces (Naimpally, 1966).Various notion of compactness such as countably compactness, limit point compactness, sequentially compactness and pseudocompactness have been studied by Yu (2012). The article looked at their useful properties and their relations on arbitrary topological spaces as well as on metric spaces.

2. Preliminaries

This paper considers the concept of almost continuous function as defined by Stallings (1959) in showing the invariance of topological properties from the cofinite space to the Euclidean space $\mathbb{R}^n$. The cofinite space here is a T_1 - space and the n dimensional Euclidean $\mathbb{R}^n$ space is a T_2 - space. Since every Hausdorff space is a T_1- space, the Euclidean $\mathbb{R}^n$ space satisfies the conditions of a T_1- space. Some of the topological properties shown include separability, compactness and its other notions like limit point compactness, pseudocompactness and sequential compactness.

3. Main results

A property is said to be a topological invariant (or topological property) if whenever one space possesses a given property, any space homeomorphic to it also possesses the same property. The properties of topological spaces that remain unchanged when space X is mapped onto a space Y by means of a function are said to be invariant of that function. Thus if a property $\mathcal{P}$ is known to be invariant of the function f and we want to check whether a topological space Y has the property $\mathcal{P}$, we will only need to show that $Y = f(X)$ for some function f and some topological space X having property $\mathcal{P}$. We study the behaviours of these topological properties with respect to almost continuous functions from the cofinite space to the Euclidean $\mathbb{R}^n$ spaces. These are done with respect to continuous functions and then the remark 1 noted below is used to make a conclusion.

Remark 1:

It has been shown by Gichuki (1996) that if a property $\mathcal{P}$ is invariant of continuous functions, it must be invariant of continuous bijections and hence invariant of continuous bijections in the class of T_1- spaces. Those topological properties which are invariant of continuous functions are also invariant of almost continuous functions in the class of T_1- spaces.

Theorem 1

Let f be a continuous function from a separable cofinite space $\mathcal{C}$ to the Euclidean space $\mathbb{R}^n$. If A is a countable dense subset of $\mathcal{C}$, then $f(A)$ is a countable dense subset of $f(\mathcal{C})$ in the Euclidean space $\mathbb{R}^n$.

Proof

Consider a continuous function $f: \mathcal{C} \longrightarrow \mathbb{R}^n$ where $\mathcal{C}$ is a separable cofinite space. Since A is a countable dense subset of $\mathcal{C}$, that is $\overline{A} = \mathcal{C}$, we then check that $f(\overline{A})$ is dense in $f(\mathcal{C})$ which is a subset of the Euclidean $\mathbb{R}^n$ space due to the continuity of f. Now for $f: X \longrightarrow Y$ where X and Y are topological spaces being continuous and $A \subseteq X$, letting $x \in \overline{A}$ implying that $f(x) \in f(\overline{A})$. We let $\mathcal{V}$ be a neighborhood of $f(x)$. By continuity of f, $f^{-1}(\mathcal{V})$ is an open set in X containing x. Thus we have $f^{-1}(\mathcal{V}) \cap A \neq \emptyset$ implying that $f(x) \in \overline{f(A)}$. Therefore $f(\overline{A}) \subseteq \overline{f(A)}$. From this, we can then let X to be our separable cofinite space $\mathcal{C}$ and Y to be our Euclidean space $\mathbb{R}^n$. But $A \subseteq \overline{A}$ and $\overline{A} \subseteq \mathcal{C}$. We then have $f(A) \subseteq f(\overline{A}) \subseteq \overline{f(A)} \subseteq f(\mathcal{C})$. Clearly $f(\overline{A}) \subseteq f(\mathcal{C})$ which would imply that $f(\overline{A}) = f(\mathcal{C})$. That is $f(\mathcal{C})$ has a countable dense subset $f(\overline{A})$, showing that $f(\mathcal{C})$ is separable as a subset of $\mathbb{R}^n$. Therefore separability is invariant of continuous functions from cofinite space $\mathcal{C}$ to Euclidean $\mathbb{R}^n$ space. From remark 1; separability is invariant of almost continuous functions from cofinite space $\mathcal{C}$ to Euclidean $\mathbb{R}^n$ space.

Theorem 2

Suppose $\mathcal{C}$ is a compact cofinite space and $f: \mathcal{C} \longrightarrow \mathbb{R}^n$ is a continuous function, then $f(\mathcal{C})$ is a compact subset of the Euclidean $\mathbb{R}^n$ space.

Proof

Let $\mathcal{A} = \{U_\alpha : \alpha \in \zeta\}$ be any open cover for $f(\mathcal{C})$. Then one of the members of U_α say U_β covers all but finitely many points of $f(\mathcal{C})$ since $U_\beta = f(\mathcal{C}) \setminus \{a_1, a_2, \dots, a_m\}$ for some $a_1, a_2, \dots, a_m$ in $f(\mathcal{C})$. Therefore $f(\mathcal{C}) = \bigcup_{\alpha \in \zeta} U_\alpha$

$$\mathcal{C} = f^{-1}\left(\bigcup_{\alpha \in \zeta} U_\alpha\right) = \bigcup_{\alpha \in \zeta} f^{-1}(U_\alpha)$$

Since f is continuous, $f^{-1}(U_\alpha)$ is open in $\mathcal{C}$ for some $\alpha \in \zeta$. Then $\{f^{-1}(U_\alpha) : \alpha \in \zeta\}$ is an open cover for $\mathcal{C}$. Because $\mathcal{C}$ is compact there exists $\alpha_1, \alpha_2, \dots, \alpha_m \in \zeta$ such that $\mathcal{C} = f^{-1}(U_\beta) \cup \left[\bigcup_{i=1}^m f^{-1}(U_{\alpha_i})\right] = f^{-1}(U_\beta) \cup f^{-1}\left[\bigcup_{i=1}^m U_{\alpha_i}\right] = f^{-1}\left[U_\beta \cup \left(\bigcup_{i=1}^m U_{\alpha_i}\right)\right]$ $f(\mathcal{C}) = U_\beta \cup \left(\bigcup_{i=1}^m U_{\alpha_i}\right)$. Hence $\{U_\beta \cup U_{\alpha_i} : i = 1, \dots, m\}$ is a finite subcover for $\mathcal{A}$. We therefore have compactness being invariant of continuous functions from a compact cofinite space $\mathcal{C}$ to the Euclidean $\mathbb{R}^n$ space. From remark 1; compactness is invariant of almost continuous functions from the cofinite space $\mathcal{C}$ to the Euclidean $\mathbb{R}^n$ space.

Theorem 3
Let f be a continuous function from a sequentially compact cofinite space $\mathcal{C}$ to a closed and bounded subset B of the Euclidean $\mathbb{R}^n$ space. Then $f(\mathcal{C})$ is a sequentially compact subset of the Euclidean space $\mathbb{R}^n$.

Proof
Consider a continuous function $f: \mathcal{C} \longrightarrow B$ where $\mathcal{C}$ is a sequentially compact cofinite space and B is a closed and bounded subset of the Euclidean $\mathbb{R}^n$ space. Because of continuity of f, we have $f(\mathcal{C}) \subset B$. Let $(y_1, y_2, \ldots)$ be a sequence in $f(\mathcal{C})$. Then there exists $x_1, x_2, \ldots \in \mathcal{C}$ such that $f(x_n) = y_n$ for every $n \in \mathbb{N}$. But $\mathcal{C}$ is sequentially compact, so the sequence $(x_1, x_2, \ldots)$ contains a subsequence $\left(x_{i_1}, x_{i_2}, \ldots\right)$ which converges to a point $p \in \mathcal{C}$. Now f is continuous and hence sequentially continuous, so $\left\{f\left(x_{i_1}\right), f\left(x_{i_2}\right), \ldots\right\} = \left\{y_{i_1}, y_{i_2}, \ldots\right\}$ converges to $f(p) \in f(\mathcal{C})$. Thus $f(\mathcal{C})$ is sequentially compact with a sequence $(y_1, y_2, \ldots)$ having a convergent subsequence $\left(y_{i_1}, y_{i_2}, ..\right)$. Therefore sequential compactness is invariant of continuous functions from a cofinite space $\mathcal{C}$ to Euclidean $\mathbb{R}^n$ space. From remark 1; Sequential compactness is invariant of almost continuous functions from a cofinite space $\mathcal{C}$ to the Euclidean space $\mathbb{R}^n$.

Theorem 4
Let f be a continuous function from a pseudocompact cofinite space $\mathcal{C}$ to the Euclidean space $\mathbb{R}^n$. Then $f(\mathcal{C})$ is a pseudocompact subset of the Euclidean space $\mathbb{R}^n$.

Proof
Since a compact space is pseudocompact we show that a cofinite space $\mathcal{C}$ is also pseudocompact. A cofinite space $\mathcal{C}$ is a compact space. The image of a compact space under any continuous function is compact. By Heine - Borel theorem, the compact subsets of the Euclidean space $\mathbb{R}^n$ are precisely the closed and bounded subsets. Hence a cofinite space is pseudocompact as its image under any continuous function to $\mathbb{R}^n$ is compact. To show the continuity invariance of pseudocompactness to the Euclidean $\mathbb{R}^n$ space, we consider a continuous function $f: \mathcal{C} \longrightarrow \mathbb{R}^n$. Clearly $f(\mathcal{C})$ is a compact subset of $\mathbb{R}^n$ by continuity of f. By Heine - Borel theorem $f(\mathcal{C})$ is bounded, that is f is a bounded function and $f(\mathcal{C})$ is closed since it is a compact subset of $\mathbb{R}^n$. This implies that the continuous function f attains its bounds as the supremum and infimum of $f(\mathcal{C})$ which are either in $f(\mathcal{C})$ or are the limit points. Since $f(\mathcal{C})$ is the continuous image of pseudocompact space $\mathcal{C}$, it follows that $f(\mathcal{C})$ is pseudocompact. Hence pseudocompactness is invariant of continuous functions from the pseudocompact cofinite space $\mathcal{C}$ to the Euclidean $\mathbb{R}^n$ space. From the remark 1; pseudocompactness is invariant of almost continuous functions from the cofinite space $\mathcal{C}$ to the Euclidean $\mathbb{R}^n$ space.

Theorem 5
Let $\mathcal{C}$ be a limit point compact cofinite space and f be a continuous function from $\mathcal{C}$ to the Euclidean space $\mathbb{R}^n$, then $f(\mathcal{C})$ is a limit point compact space in the Euclidean space $\mathbb{R}^n$.

Proof
Suppose $\mathcal{K}$ is a closed and bounded subset of $\mathbb{R}^n$, then it is compact since the closed and bounded subsets of the Euclidean $\mathbb{R}^n$ space are compact. If B is an infinite subset of $\mathcal{K}$, then B is also bounded and by Bolzano - Weierstrass theorem, B has a limit point p. Since $\mathcal{K}$ is closed, the limit point p of B belongs to $\mathcal{K}$, that is $\mathcal{K}$ is limit point compact. We consider a continuous function $f: \mathcal{C} \longrightarrow K$ and because of continuity of f, we have $f(\mathcal{C}) \subset K$. Since $\mathcal{C}$ contains an infinite set A whose limit point is in $\mathcal{C}$, then $f(\mathcal{C}) \subset K$ contains an infinite set $f(A)$ whose limit point is in $f(\mathcal{C})$. But $f(\mathcal{C})$ is a subset of a closed and bounded set $\mathcal{K} \subset \mathbb{R}^n$ which is also limit point compact. Clearly the cofinite space that is limit point compact is continuous invariant to the Euclidean $\mathbb{R}^n$ space. Therefore limit point compactness is continuity invariant from cofinite space $\mathcal{C}$ to the Euclidean $\mathbb{R}^n$ space. Hence from remark 1; limit point compactness is invariant with respect to almost continuous function from cofinite space $\mathcal{C}$ to the Euclidean $\mathbb{R}^n$ space.

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