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Quantum Algorithm for Knapsack Problem by Numbering Method

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Abstract

A quantum algorithm for the knapsack problem by a numbering method and its example are reported. When an optimal combination of n pieces of different weight luggage packed into the knapsack that a weight k can be put is requested, a computational complexity of a classical computation is $2^n - 1$. The computational complexity becomes about $4n$ by the quantum algorithm that uses quantum phase inversion gates, quantum inversion about mean gates and the numbering method. Therefore, a polynomial time process becomes possible.

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***Keywords*:** Quantum algorithm, knapsack problem, numbering method, computational complexity, polynomial time.

1. Introduction

The very first steps towards building a quantum computer were made by Haroche and Wineland [1]. Deutsch-Jozsa's algorithm for the rapid solution [2−4], Shor's algorithm for the factorization [3−5], Grover's algorithms for the database search [3, 6, 7] and so on are known. A quantum algorithm for the vertex coloring problem by a numbering method has recently been reported by Fujimura [8]. Its computational complexity becomes a polynomial time. The knapsack problem [3, 9] is examined by the numbering method this time. Therefore, its result is reported.

2. Knapsack Problem

As for n pieces of different weight luggage, the knapsack problem requests an optimal combination of the luggage packed into the knapsack that a weight k is assumed to be an upper bound.

3. Quantum Algorithm

It is assumed that there are n pieces of different weight x_i [$1 \leq i \leq n$. i is an integer.], and the upper bound weight k of the knapsack, and a_i [$1 \leq i \leq n$. i is the integer.] is 0 or 1. When the number of the n times repeated permutation of 0 and 1 is 2^n, $a_1 2^{n-1} + a_2 2^{n-2} + \cdots + a_n 2^0 = \sum_{i=1 \to n} a_i 2^{n-i} = U$ is the numbering datum from 0 to $2^n - 1$ [The 0-th datum is 0, 0, ⋯, 0 and 0. The $(2^n - 1)$-th datum is 1, 1, ⋯, 1 and 1.]. This method is named the numbering method for this problem. g is the minimum integer that follows $n/2 \leq g$ [$2^n/1 \leq 4^g = 2^{2g}$], because a number of combinations of an answer is 1 at least.

First of all, quantum registers $|a_1>$, $|a_2>$, ⋯, $|a_n>$, $|b_1>$, $|b_2>$, $|c>$ and $|d>$ are prepared. States of $|a_1>$, $|a_2>$, ⋯, $|a_n>$, $|b_1>$, $|b_2>$, $|c>$ and $|d>$ are a_1, a_2, ⋯, a_n, b_1, b_2, c and d, respectively.

- Step 1: Each quantum bit [=qubit] of $|a_1>$, $|a_2>$, ⋯, $|a_n>$, $|b_1>$, $|b_2>$, $|c>$ and $|d>$ is set $|0>$.

- Step 2: The Hadamard gate H [3, 4] acts on each qubit of $|a_1>$, $|a_2>$, ⋯, $|a_{n-1}>$ and $|a_n>$. It changes them for entangled states. The total states are 2^n [$=W_0$].
- Step 3: It is assumed that a quantum gate (A) changes $|b_1>$ and $|b_2>$ for $|b_1 + a_i x_i>$ and $|b_2 + a_i\ 2^{n-i}>$, respectively, at $|a_i>$. These actions are repeated sequentially from $|a_1>$ to $|a_n>$. Therefore, $|b_1>$ and $|b_2>$ become $|\sum_{i=1\to n} a_i x_i>$ and $|\sum_{i=1\to n} a_i 2^{n-i} = U>$, respectively.
- Step 4: It is assumed that a quantum gate (B) doesn't change $|c>$ in $b_1 = k$, or it changes $|c>$ for $|c + 1 + b_2>$ in the others of b_1 from $|b_1>$ and $|b_2>$.
- Step 5: It is assumed that a quantum gate (C_1) changes $|d>$ for $|1>$ in $0 \le c \le (2^n/4) - 1$, or it changes $|d>$ for $|0>$ in the others of c. As the target state for $|d>$ is 1, quantum phase inversion gates (PI) and quantum inversion about mean gates (IM) [3, 6, 7] act on $|d>$. The number of the data that is included in $0 \le c \le (2^n/4) - 1$ is $W_1 \approx 2^n/4$. When γ_1 is the minimum even integer that is $(W_0/W_1)^{1/2}$ or more, the total number that (PI) and (IM) act on $|d>$ is $\gamma_1 \approx 2$, because they are a couple. Next, an observation gate (OB) observes $|d>$, and the data of W_1 remain. Similarly, (C_i) [$2 \le i \le g - 1$. i is the integer.] changes $|d>$ for $|1>$ in $0 \le c \le (2^n/4^i) - 1$, or it changes $|d>$ for $|0>$ in the others of c. As the target state for $|d>$ is 1, (PI) and (IM) act on $|d>$. The number of the data that is included in $0 \le c \le (2^n/4^i) - 1$ is $W_i \approx 2^n/4^i$. When γ_i is the minimum even integer that is $(W_{i-1}/W_i)^{1/2}$ or more, the total number that (PI) and (IM) act on $|d>$ is $\gamma_i \approx 2$. Next, (OB) observes $|d>$, and the data of W_i remain. These actions are repeated sequentially from 2 to $g - 1$ at i.

(C_g) changes $|d>$ for $|1>$ at $c = 0$, or it changes $|d>$ for $|0>$ in the others of c. As the target state for $|d>$ is 1, (PI) and (IM) act on $|d>$. The number of the data that is included at $c = 0$ is $W_g \approx 2^n/4^g \approx 1$. When γ_g is the minimum even integer that is $(W_{g-1}/W_g)^{1/2}$ or more, the total number that (PI) and (IM) act on $|d>$ is $\gamma_g \approx 2$. Next, (OB) observes $|a_1>$, $|a_2>$, ⋯, $|a_n>$, $|b_1>$, $|b_2>$, $|c>$ and $|d>$, and one of the data of W_g remains. Therefore, one example of combinations that are $b_1 = k$ is obtained.

4. Numerical Computation

It is assumed that there are $n = 6$, $x_1 = 15$, $x_2 = 3$, $x_3 = 2$, $x_4 = 7$, $x_5 = 10$, $x_6 = 13$, $k = 19$ and $g = 3$ [$6/2=3 \leq 3 = g$].

First of all, $|a_1>$, $|a_2>$, $\cdots$, $|a_6>$, $|b_1>$, $|b_2>$, $|c>$ and $|d>$ are prepared. States of $|a_1>$, $|a_2>$, $\cdots$, $|a_6>$, $|b_1>$, $|b_2>$, $|c>$ and $|d>$ are a_1, a_2, $\cdots$, a_6, b_1, b_2, c and d, respectively.

- Step 1: Each qubit of $|a_1>$, $|a_2>$, $\cdots$, $|a_6>$, $|b_1>$, $|b_2>$, $|c>$ and $|d>$ is set $|0>$.
- Step 2: $\boxed{\text{H}}$ acts on each qubit of $|a_1>$, $|a_2>$, $\cdots$, $|a_5>$ and $|a_6>$. It changes them for entangled states. The total states are 2^6 [$=W_0$].
- Step 3: (A) changes $|b_1>$ and $|b_2>$ for $|b_1 + a_i\ x_i>$ and $|b_2 + a_i\ 2^{n-i}>$, respectively, at $|a_i>$. These actions are repeated sequentially from $|a_1>$ to $|a_6>$. Therefore, $|b_1>$ and $|b_2>$ become $|\sum_{i=1\rightarrow 6} a_i\ x_i>$ and $|\sum_{i=1\rightarrow 6} a_i\ 2^{6-i} = U>$, respectively.
- Step 4: (B) doesn't change $|c>$ at $b_1 = 19$, or it changes $|c>$ for $|c + 1 + b_2>$ in the others of b_1 from $|b_1>$ and $|b_2>$.
- Step 5: (C_1) changes $|d>$ for $|1>$ in $0 \leq c \leq (2^6/4) - 1$, or it changes $|d>$ for $|0>$ in the others of c. As the target state for $|d>$ is 1, (PI) and (IM) act on $|d>$. The number of the data that is included in $0 \leq c \leq (2^6/4) - 1$ is $W_1 \approx 2^6/4$. When γ_1 is the minimum even integer that is $(W_0/W_1)^{1/2} \approx 2 \leq 2 = \gamma_1$, the total number that (PI) and (IM) act on $|d>$ is $\gamma_1 \approx 2$. Next, (OB) observes $|d>$, and the data of W_1 remain.

(C_2) changes $|d>$ for $|1>$ in $0 \leq c \leq (2^6/4^2) - 1$, or it changes $|d>$ for $|0>$ in the others of c. As the target state for $|d>$ is 1, (PI) and (IM) act on $|d>$. The number of the data that is included in $0 \leq c \leq (2^6/4^2) - 1$ is $W_2 \approx 2^6/4^2$. When γ_2 is the minimum even integer that is $(W_1/W_2)^{1/2} \approx 2 \leq 2 = \gamma_2$, the total number that (PI) and (IM) act on $|d>$ is $\gamma_2 \approx 2$. Next, (OB) observes $|d>$, and the data of W_2 remain.

(C_3) changes $|d>$ for $|1>$ at $c = 0$, or it changes $|d>$ for $|0>$ in the others of c. As the target state for $|d>$ is 1, (PI) and (IM) act on $|d>$. The number of the data that is included at $c = 0$ is $W_3 \approx 2^6/4^3 \approx 1$. When γ_3 is the minimum even integer that is $(W_2/W_3)^{1/2} \approx 2 \leq 2 = \gamma_3$, the total number that (PI) and (IM) act on $|d>$ is $\gamma_3 \approx 2$. Next, (OB)

observes $|a_1>$, $|a_2>$, ⋯, $|a_6>$, $|b_1>$, $|b_2>$, $|c>$ and $|d>$, and one of the data of W_3 remains. For example, when a_1, a_2, a_3, a_4, a_5, a_6, b_1, b_2, c and d are 0, 0, 1, 1, 1, 0, 19, 14, 0 and 1, respectively, it is obtained that $a_1 x_1 + a_2 x_2 + a_3 x_3 + a_4 x_4 + a_5 x_5 + a_6 x_6$ is $0 \times 15 + 0 \times 3 + 1 \times 2 + 1 \times 7 + 1 \times 10 + 0 \times 13 = 19 = k$.

5. Discussion and Summary

The computational complexity of this quantum algorithm [=S] becomes the following. In the order of the actions by the gates, the number of them is n at $\boxed{H}$, n at (A), 2 at (B), g at (C_i) [$1 \le i \le g$. i is the integer.], $\sum_{i=1 \to g} \gamma_i \approx 2g$ at (PI) and (IM), and g at (OB). Therefore, S becomes $2n + 2 + 4g$. In the example of the section 4, S is 26. The computational complexity of the classical computation [=Z] is $2^n - 1 = 2^6 - 1 = 63$. After all, S/Z becomes about 1/2. When n is large enough, S becomes about $2n + 2 + 2n \approx 4n$, where g is about $n/2$. And then, S/Z is about $4n/2^n \approx n/2^n$. For example, as for $n = 100$, S/Z is about $100/2^{100} \approx 1/10^{28}$.

Therefore, the polynomial time process becomes possible.

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