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A General Technique for Solving 2 × 2 and 3 × 3 Systems of High Order Linear Ordinary Differential Equations with Constant Coefficients

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Abstract

In this work, a general technique for solving systems of high order linear ordinary differential equations with constant coefficients is presented. Under certain conditions, a general form solution of 2×2 and 3×3 systems is derived. In this technique, the system is efficiently reduced to a single high order ordinary differential equation. To show the efficiency of the method, illustrative examples are used.

Keywords: 2×2 and 3×3 linear systems of ODEs, High order, Constant Coefficients.

Introduction

Linear systems of ordinary differential equations (ODEs) and their applications in many discplines of engineering and science have been widely studied in the past and in the present centuries. Studying of such systems has been carried out in both analytical and numerical approaches as in [1-31]. Linear systems of ODEs can also play an important role in solving non-linear systems whenever they share the same structure of symmetry [29], which attracts the attention to develop new and efficient methods for solving those linear systems.

High order linear ODEs can be classified according to their coefficients: variables or constants. Usually, the ODEs with some variable coefficients do not have closed form solution; hence, developing methods for solving such equations enjoys a great interest, and the most popular method used in this case is the *method of reduction of order* [22]. If all coefficients are constants, then the roots of the characteristic equation are used to find the general solution when the differential equation is homogeneous; while the method of *variation of parameters* is used in the non-homogeneous case [29-31].

In this paper, a more generalizing form of the technique published in [1] is successfully derived. In [1], the authors presented a new technique for solving 2×2 and 3×3 systems of first order linear ordinary differential equations. Here, the following linear system of high order ordinary differential equations with constant coefficients is being under consideration:

$$x_i^{(n_i)}(t) = \sum_{j=1}^{k} a_{ij}\, x_j^{(m_j)}(t) + f_i(t) \; i = 1, \dots, k \,, \tag{1}$$

where $\{a_{ij}\}_{i,j=1}^{k}$ composes a constant square matrix A, and $\{f_i(t)\}_{i=1}^{k}$ are continuous functions on some interval I. (If $f_i(t) = 0$ for all $i = 1 \dots k$, then system (1) is homogeneous). It is clear that the system $X^{(n)}(t) = AX^{(m)}(t) + F(t)$ is a special case of (1) when $n_i = n$ and $m_i = m$ for all $i = 1, \dots k$. In particular, if $n_i = 1$ and $m_i = 0$ for all $i = 1, \dots k$, then system (1) is the case considered in [1]. In the next section, system (1) is reduced to a single high order linear differential equation, which obviously generalizes the findings in [1]. Illustrative examples are also presented in order to show the efficiency of the method. Conclusions and future perspectives are discussed in the last section.

Analysis of the Method and Main Results

The technique of reducing system (1) to a single high order ODE for *k=2* and *k=3* is clarified in the following two theorems. In the proof, we will assume, without loss of generality of the system, that $n_i \leq n_j$ when $i < j$ and $m_i \leq n_i$ for all $i = 1, \dots k$. Also, we will denote $x_i(t)$ by x_i throughout the paper.

Theorem 1

In system (1), for $k = 2$, if $f_2(t)$ is differentiable up to $(n_1 - m_1)$ derivatives, then the system can be reduced to

$$x_2^{(n_2+n_1-m_1)} - a_{11}x_2^{(n_2)} - a_{22}x_2^{(m_2+n_1-m_1)} + (a_{11}a_{22} - a_{12}a_{21})x_2^{(m_2)} = g(t), \tag{2}$$

where

$$g(t) = a_{11}f_2(t) - a_{21}f_1(t) - f_2^{(n_1-m_1)}(t). \tag{3}$$

Proof

For $k = 2$, system (1) is written as

$$x_1^{(n_1)} = a_{11}x_1^{(m_1)} + a_{12}x_2^{(m_2)} + f_1(t)$$
$$x_2^{(n_2)} = a_{21}x_1^{(m_1)} + a_{22}x_2^{(m_2)} + f_2(t)\,. \tag{4}$$

Let $L = n_1 - m_1$. The L^{th}-derivative of $x_2^{(n_2)}$ is

$$x_2^{(n_2+L)} = a_{21}x_1^{(m_1+L)} + a_{22}x_2^{(m_2+L)} + f_2^{(L)}(t)\,. \tag{5}$$

It is clear that if $a_{21}=0$, the solution of (4) is obvious. Hence, we assume that $a_{21}\neq 0$.
Using the first equation of (4) and replacing m_1+L by n_1 , equation (5) becomes
$$x_2^{(n_2+L)}=a_{21}\left(a_{11}x_1^{(m_1)}+a_{12}x_2^{(m_2)}+f_1(t)\right)+a_{22}x_2^{(m_2+L)}+f_2^{(L)}(\mathrm{t}). \quad (6)$$

But
$$x_1^{(m_1)}=\frac{1}{a_{21}}\left(x_2^{(n_2)}-a_{22}x_2^{(m_2)}-f_2(t)\right), \quad (7)$$

So, substituting (7) in (6) leads to
$$x_2^{(n_2+L)}=a_{11}x_2^{(n_2)}+a_{22}x_2^{(m_2+L)}-(a_{11}a_{22}-a_{21}a_{12})x_2^{(m_2)}-a_{11}f_2(t)+a_{21}f_1(t)+f_2^{(L)}(t). \quad (8)$$

Using $L=n_1-m_1$, equation (8) becomes
$$x_2^{(n_2+n_1-m_1)}-a_{11}x_2^{(n_2)}-a_{22}x_2^{(m_2+n_1-m_1)}+(a_{11}a_{22}-a_{21}a_{12})x_2^{(m_2)}=g(t)\ , \quad (9)$$

where
$$g(t)=\ a_{11}f_2(t)-a_{21}f_1(t)-f_2^{(n_1-m_1)}(t)\ . \quad (10)$$

For $n_1=n_2=1$ and $m_1=m_2=0$, system (4) is reduced to
$$x_2''-(trA)x_2'+(\det A)x_2=g(t), \quad (11)$$

which is the case considered in [1].
Now, solving (9) for x_2 can be done by the variation of parameters method when $g(t)\neq 0$, or by the characteristic roots when $g(t)=0$, and then, x_1 can be found by (7).
The following example illustrates the result of Theorem 1.

Example 1
Consider the system
$$x_1''=-4x_1'-6x_2$$
$$x_2''=3x_1'+7x_2\ ,$$

In this system, $n_1=n_2=2\ ,m_1=1\ ,m_2=0, L=n_1-m_1=1$, and $f_1(t)=f_2(t)=0$.
By (9), the system is reduce to
$$x_2'''+4x_2''-7x_2'-10x_2=0\ .$$

The characteristic polynomial is $\lambda^3+4\lambda^2-7\lambda-10$, the roots are $\lambda_1=-1\ ,\lambda_2=2\ ,\lambda_3=-5$, therefore,
$$x_2=c_1e^{-t}+c_2e^{2t}+c_3e^{-5t}\ .$$

Now, by (7),

$$x_1' = \frac{1}{3}((c_1e^{-t} + 4c_2e^{2t} + 25c_3e^{-5t}) - 7(c_1e^{-t} + c_2e^{2t} + c_3e^{-5t}))$$
$$= -2c_1e^{-t} - c_2e^{2t} + 6c_3e^{-5t} ,$$

and hence,
$x_1 = 2c_1e^{-t} - \frac{1}{2}c_2e^{2t} - \frac{6}{5}c_3e^{-5t}$ + C .

In the following, for k=3, system (1) is reduced to a single ordinary differential equation.

Theorem 2
In system (1), For $k = 3$, if $n_2 - m_2 = n_1 - m_1$ and $a_{11} + a_{22} = 0$, then the system can be reduced to the following ODE:

$$x_3^{(n_3+2n_1-2m_1)} - a_{33}x_3^{(m_3+2n_1-2m_1)} - (a_{12}a_{21} - a_{11}a_{22} + a_{13}a_{31} + a_{23}a_{32})x_3^{(m_3+n_1-m_1)} - \det(A)\, x_3^{(m_3)} = g(t) . \quad (12)$$

where

$$g(t) = (a_{11}a_{31} + a_{21}a_{32})f_1(t) + (a_{12}a_{31} + a_{22}a_{32})f_2(t) + a_{31}f_1^{(n_1-m_1)}(t) + a_{32}f_2^{(n_1-m_1)}(t) - (a_{11}^2 + a_{12}a_{21})f_3(t) + f_3^{(2n_1-2m_1)}(t) , \quad (13)$$

provided that $f_1(t)$ and $f_2(t)$ are differentiable up to $(n_1 - m_1)$ derivatives and $f_3(t)$ is differentiable up to $(2n_1 - 2m_1)$ derivatives.

Proof
For k=3, the system (1) is written as

$$x_1^{(n_1)} = a_{11}x_1^{(m_1)} + a_{12}x_2^{(m_2)} + a_{13}x_3^{(m_3)} + f_1(t)$$
$$x_2^{(n_2)} = a_{21}x_1^{(m_1)} + a_{22}x_2^{(m_2)} + a_{23}x_3^{(m_3)} + f_2(t)$$
$$x_3^{(n_3)} = a_{31}x_1^{(m_1)} + a_{32}x_2^{(m_2)} + a_{33}x_3^{(m_3)} + f_3(t). \quad (14)$$

Reducing system (14) to a high order differential equation will be done under the condition that $n_1 - m_1 = n_2 - m_2$ which keeps the generality of the method upon the case in [1]. Attempts for more generalization will be under consideration in a future work.
Let $L = 2(n_1 - m_1)$ which also equals to $2(n_2 - m_2)$.
Now, the L^{th}-derivative of $x_3^{(n_3)}$ is

$$x_3^{(n_3+L)} = a_{31}x_1^{(2n_1-m_1)} + a_{32}x_2^{(2n_2-m_2)} + a_{33}x_3^{(m_3+L)} + f_3^{(L)}(t) , \quad (15)$$

but

$$x_1^{(2n_1-m_1)} = a_{11}x_1^{(n_1)} + a_{12}x_2^{(n_2)} + a_{13}x_3^{(m_3+n_1-m_1)} + f_1^{(n_1-m_1)}(t) , \quad (16)$$

and

$$x_2^{(2n_2-m_2)} = a_{21}x_1^{(n_1)} + a_{22}x_2^{(n_2)} + a_{23}x_3^{(m_3+n_1-m_1)} + f_2^{(n_2-m_2)}(t) , \quad (17)$$

so

$$x_3^{(n_3+L)} = a_{31}(a_{11}x_1^{(n_1)} + a_{12}x_2^{(n_2)} + a_{13}x_3^{(m_3+L/2)} + f_1^{(\frac{L}{2})}(t)) + a_{32}(a_{21}x_1^{(n_1)} + a_{22}x_2^{(n_2)} + a_{23}x_3^{(m_3+L/2)} + f_2^{\left(\frac{L}{2}\right)}(t)) + a_{33}x_3^{(m_3+L)} + f_3^{(L)}(t) \; . \quad (18)$$

Rearranging terms of (18) gives

$$x_3^{(n_3+L)} = (a_{11}a_{31} + a_{21}a_{32})x_1^{(n_1)} + (a_{12}a_{31} + a_{22}a_{32})x_2^{(n_2)} + (a_{13}a_{31} + a_{23}a_{32})x_3^{(m_3+L/2)} + a_{33}x_3^{(m_3+L)} + a_{31}f_1^{(L/2)}(t) + a_{32}f_2^{(L/2)}(t) + f_3^{(L)}(t) \; . \quad (19)$$

Using the first and the second equations of (14) and rearranging terms, equation (19) becomes:

$$x_3^{(n_3+L)} = (a_{31}(a_{11}^2 + a_{12}a_{21}) + a_{21}a_{32}(a_{11} + a_{22}))x_1^{(m_1)} + \left(a_{32}(a_{22}^2 + a_{12}a_{21}) + a_{12}a_{31}(a_{11} + a_{22})\right)x_2^{(m_2)} + \left(a_{13}(a_{11}a_{31} + a_{21}a_{32}) + a_{23}(a_{12}a_{31} + a_{22}a_{32})\right)x_3^{(m_3)} + (a_{13}a_{31} + a_{23}a_{32})x_3^{(m_3+L/2)} + a_{33}x_3^{(m_3+L)} + (a_{11}a_{31} + a_{21}a_{32})f_1(t) + (a_{12}a_{31} + a_{22}a_{32})f_2(t) + a_{31}f_1^{(L/2)}(t) + a_{32}f_2^{(L/2)}(t) + f_3^{(L)}(t) \; . \quad (20)$$

Under the condition $a_{11} + a_{22} = 0$, equation (20) becomes

$$x_3^{(n_3+L)} = (a_{11}^2 + a_{12}a_{21})\left(a_{31}x_1^{(m_1)} + a_{32}x_2^{(m_2)}\right) + (a_{13}a_{31} + a_{23}a_{32})x_3^{(m_3+L/2)} + a_{33}x_3^{(m_3+L)} + (a_{13}(a_{11}a_{31} + a_{21}a_{32}) + a_{23}(a_{12}a_{31} + a_{22}a_{32}))x_3^{(m_3)} + (a_{11}a_{31} + a_{21}a_{32})f_1(t) + (a_{12}a_{31} + a_{22}a_{32})f_2(t) + a_{31}f_1^{(L/2)}(t) + a_{32}f_2^{(L/2)}(t) + f_3^{(L)}(t) \; . \quad (21)$$

But from the third equation of (14),

$$a_{31}x_1^{(m_1)} + a_{32}x_2^{(m_2)} = x_3^{(n_3)} - a_{33}x_3^{(m_3)} - f_3(t). \quad (22)$$

Substituting (22) in (21) with $L = 2(n_1 - m_1)$ and rearranging terms, equation (21) becomes:

$$x_3^{(n_3+L)} - a_{33}x_3^{(m_3+L)} - (a_{12}a_{21} - a_{11}a_{22} + a_{13}a_{31} + a_{23}a_{32})x_3^{\left(m_3+\frac{L}{2}\right)} - det(\boldsymbol{A})\, x_3^{(m_3)} = g(t) \; . \quad (23)$$

Where A is the coefficients matrix and

$$g(t) = (a_{11}a_{31} + a_{21}a_{32})f_1(t) + (a_{12}a_{31} + a_{22}a_{32})f_2(t) + a_{31}f_1^{(L/2)}(t) + a_{32}f_2^{(L/2)}(t) - (a_{11}^2 + a_{12}a_{21})f_3(t) + f_3^{(L)}(t) \; . \quad (24)$$

Which proves theorem 2. After finding x_3 by (23), Theorem 1 can be used to find the solution for x_1 and x_2.
The following example illustrates the result of theorem 2.

Example 2
Consider the system:

$$x_1'' = -x_1' + 5x_2' - \frac{64}{3}x_3 + \frac{32}{3}t\sin 2t$$
$$x_2'' = 3x_1' + x_2' - 8x_3$$
$$x_3'' = 2x_2' - 8x_3 .$$

In the above system,
$n_1 = n_2 = n_3 = 2, m_1 = m_2 = 1, m_3 = 0, f_1(t) = \frac{32}{3}t\sin 2t,\ f_2(t) = f_3(t) = 0$,
and

$$A = \begin{bmatrix} -1 & 5 & -\frac{64}{3} \\ 3 & 1 & -8 \\ 0 & 2 & -8 \end{bmatrix}.$$

By (12) and (13), the system is reduced to the following differential equation:

$$x_3^{(4)} + 8x_3'' - (15 + 1 + 0 - 16)x_3' - \det(A)x_3 = (0 + 6)\left(\frac{32}{3}\right)t\sin 2t + 0 .$$

Which is

$$x_3^{(4)} + 8x_3'' + 16x_3 = 64\,t\sin 2t.$$

The roots of the characteristic equation is used to find the complementary solution

$$(x_3)_c = c_1\cos(2t) + c_2\sin(2t) + c_3tcos(2t) + c_4tsin(2t),$$

and the method of variation of parameters is used to find the particular solution

$$(x_3)_p = -t^2\cos(2t) - \frac{2}{3}t^3\sin(2t).$$

Therefore, the general solution is

$$x_3 = (x_3)_c + (x_3)_p .$$

Conclusions and Future Perspectives

In this paper, the technique presented in [1] for solving systems of first order linear ordinary differential equations is generalized. By the general technique, high order 2×2 and 3×3 systems of homogeneous and non-homogeneous equations were efficiently reduced to a high order differential equation. For 2×2 systems, no conditions were put in order to derive the general form of the solution, while for 3×3 systems, the conditions $n_2 - m_2 = n_1 - m_1$ and $a_{11} + a_{22} = 0$ were used. As future work, the generalization of the technique for solving $n \times n$ high order systems will be under consideration. The main expected difficulty of the generalization will be the suitable choice of conditions on the coefficients matrix A .

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References

[1] Karwan. H. F. Jwamer, A. M. Rashid, 2012, New Technique for Solving System of First Order Linear Differential Equations, Applied Mathematical Sciences, 6, 64, pp. 3177 - 3183.

[2] Khanamiryan, Marianna, 2012 Quadrature Methods for Highly Oscillatory Linear and Non-Linear Systems of Ordinary Differential Equations: part II, Bit Numerical Mathematics, 52, 2, pp. 383-405.

[3] Fomin, A.I., 2012, Differential Homomorphisms of Linear Homogeneous Systems of Differential Equations, Russian Journal of Mathematical Physics, 19 ,2, pp. 159-181.

[4] Karachik, V.V., 2012, Method for Constructing Solutions of Linear Ordinary Differential Equations with Constant Coefficients, Computational Mathematics and Mathematical Physics, 52, 2, pp. 219-234.

[5] Campoamor-Stursberg, R., 2012, Systems of Second-Order Linear ODE’s with Constant Coefficients and their Symmetries. II. The Case of Non-Diagonal Coefficient Matrices, Commun. Nonlinear Sci. Numer. Simul., 17, pp. 1178-1193.

[6] Saravi, M., 2012, A Procedure for Solving Some Second-Order Linear Ordinary Differential Equations, Applied Mathematical Letters, 25, 3, pp. 191-199.

[7] Farshid Mirzaee, 2011, Differential Transform Method for Solving Linear and Nonlinear Ordinary Differential Equations, Applied Mathematical Sciences, 5, 70, pp. 3465 – 3472 .

[8] Campoamor-Stursberg, R., 2011, Systems of Second-Order Linear ODE’s with Constant Coefficients and their Symmetries, Commun. Nonlinear Sci. Numer. Simul., 16, pp. 1007-5704.

[9] Meleshko, S., 2011, Comment on “Symmetry Breaking of Systems of Linear Second-Order Ordinary Differential Equations with Constant Coefficients”, Commun. Nonlinear Sci. Numer. Simul.,16, pp. 3447–3450.

[10] Samoilenko, A.M., 2011, Some Problems of the Linear Theory of Systems of Ordinary Differential equations, Ukranian Mathematical Journal, 63, 2, pp. 278-314.

[11] Abramov, A.A., Ul'yanova, V. I., and Yukhno, L. F , 2011, A Nonlocal Problem for Singular Linear Systems of Ordinary Differential Equations, Computational Mathematics and Mathematical Physics, 51 ,7, pp. 1146-1152 .

[12] Evel'son, R.L., 2011, Matrix Method of Asymptotic Integration of a System of Linear Ordinary Differential Equations with One Elementary Divisor of Arbitrary Multiplicity, Differential Equations, 47, 5, pp. 620-626.

[13] Saravi, M., Babolian E., England, R., and Bromilow, M., 2010, System of Linear Ordinary Differential and Differential-Algebraic Equations and Pseudo-Spectral Method, Computers and Mathematics with Applications, 59, 4, pp. 1524–1531.

[14] Wafo Soh, C., 2010, Symmetry Breaking of Systems of Linear Second-Order Ordinary Differential Equations with Constant Coefficients, Commun. Nonlinear Sci. Numer. Simul., 15, pp. 139-143.

[15] Wei-Chau Xie, 2010, Differential Equations for Engineers, Cambridge University press, New York, 1st edition.

[16] Krutenko, E.V., and Levenshtam, V.B., 2009, Asymptotic Analysis of Certain Systems of Linear Differential Equations with a Large Parameter, Computational Mathematics and Mathematical Physics,49 ,12, pp 2047-2058.

[17] Lyengar, S.R.K., and Jain, R.K., 2009, Numerical Methods, New Age International (p) Limited,India.

[18] Danilin, A.R., and Kovrizhnykh, O.O., 2008, On the Asymptotics of the Solution of a System of Linear Equations with Two Small Parameters, Differential Equations, 44, 6, pp. 757-767.

[19] Raisinghania, M.A., 2008, Ordinary and Partial Differential Equations, S.Chand Company Ltd.India.

[20] Biazar, J., and Ghazvini, H., 2007, He's Variational Iteration Method for Solving Linear and Non-Linear Systems of Ordinary Differential Equations, Applied Mathematics and Computation, 191, 1, pp. 287–297 .

[21] Lara, L., 2005, A Numerical Method for Solving a System of Nonautonomous Linear Ordinary Differential Equations, Applied Mathematics and Computation,170 ,1, 86-94.

[22] Boyce, W.E, Richard, C., and Diprima, 2005, Elementary Differential Equations and Boundary Value Problems, 8th edition., John Wiley & Son.

[23] Fatma Ayaz, 2004, Solutions of the System of Differential Equations by Differential Transform Method, Applied Mathematics and Computation 147, pp. 547–567.

[24] Ashordiya, MT., and Kekeliya, NA., 2004, Well-Posedness of the Cauchy Problem for Linear Systems of Generalized Ordinary Differential Equations on an Infinite Interval, Differential Equations, 40, 4, pp. 477-490.

[25] Sarrico, C.O.R., 2002, Global Solutions of First Order Linear Systems of Ordinary Differential Equations with Distributional Coefficients, Journal of Mathematical Analysis and Applications, 276, 2, pp. 611-627.

[26] Novikov, D., 2001, Systems of Linear Ordinary Differential Equations with Bounded Coefficients May Have Very Oscillating Solutions, Proceedings of the American Mathematical Society, 129, 12, pp. 3753-3755..

[27] Carmen Chicone, 1999, Ordinary Differential Equations with Applications, Springer-Verlag, New York.

[28] Saxena, H.C, 1998, Finite Differences and Numerical Analysis, 14*th* Rev. Edn. , India.

[29] Ibragimov, NH., editor, 1993, CRC handbook of Lie group analysis of differential equations, Boca Raton.

[30] Cullen, C.G., 1991, Linear Algebra and Differential Equations, 2*nd* ed., Pws-Kent, Boston.
[31] Coddington, E.A., and Levinson, N., 1955, Theory of Ordinary Differential Equations, McGraw-Hill, New York