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Characterizations of Distributions in a Open Subset of R^n. Positive distributions

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Abstract

We adapt the definitions of ordered vector space and locally convex lattice to the setting of the complex scalar field. As a consequence of the generalization of the result by Garrett Birkhoff that the order dual of a Banach lattice coincides with its topological dual, to Fréchet end LF lattices that we obtain, we show that O being a open set in the complex plane, the space H(O) being a Fréchet lattice through a partial order, the analytic functionals in O are the order bounded linear functionals. Each analytic functional in O is the difference of two positive analytic functionals. Next, we consider a LF lattice E(Ω) related to the space D(Ω) of test functions and we show that each positive linear functional on E(Ω) corresponds to an unique distribution in Ω and conversely. This way we obtain the positive distributions, each distribution in Ω being the difference of two positive distributions. We obtain an application to bounded linear opeators on the Sobolev space to a Banach space.

Introduction

We consider the partial order z≤z′ meaning that Rez≤Rez′ and Imz≤Imz′ for complex numbers z and z′ and the concepts of ordered vector space and locally convez lattice follow with the corresponding condition for the scalar product. In paragraph 2, The setting for complex locally convex lattices, we obtain the generalization of the result by Garrett Birkhoff to Frréchet and LF lattices in the Abstract. The characterization of Radon measures in Ω as the order bounded linear functionals on $C\_\{c\}^{0}(\Omega)$ follows in paragraph 3, Applications to distributions and analytic functionals. We consider a partial order < in H(O), O an open subset of the complex plane, for which H(O) is a complex LF lattice. The order dual and the topological dual of the space coincide, hence the analytic functionals in Ω are the order bounded linear functionals. Also we consider a LF lattice $(E(\Omega),<)$ such that each continuous linear functional on $E(\Omega)$ corresponds to an unique distribution in Ω and conversely. In this way, we obtain the positive distributions in Ω, which are such that they correspond to positive linear functionals on $E(\Omega)$. Each distribution in Ω is the difference of two positive distributions.

The setting for complex locally convex lattices

Recall that X being a real vector space equipped with a partial order $\leq$, we say that X is a vector lattice ([1], [4]) if $x+z\leq y+z$ whenever $x\leq y$, $\alpha x\geq 0$ if $x\geq 0$ and α is a non negative scalar and, further, there exist in X the elements $x\vee y=\sup\{x,y\}$ and $x\wedge y=\inf\{x,y\}$ for each $x,y\in X$. We may adapt the definition to complex vector spaces putting $\lambda x\geq 0$ for $x\geq 0$ and complex $\lambda\geq 0$ in the understanding of the Introduction. We then put $x^{+}=x\vee 0, x^{-}=(-x)\vee 0$ and $|x|=x\vee(-x)$. Notice that $x=x^{+}-x^{-}$ and $|x|=x^{+}+x^{-}$ ([1]) hence $|x|\geq 0$. This way we obtain complex vector lattices and we may as well consider, following [4], the <definition/>X being a complex Hausdorff locally convex space that also is a vector lattice, we say that X is a locally convex lattice if it has a base $P=\{p_{\alpha}:\alpha\in A\}$ of continuos seminorms such that each $p_{\alpha}(x)\leq p_{\alpha}(y)$ whenever $|x|\leq|y|$.

X being a vector lattice, we denote $X^{+}=\{x^{+}:x\in X\}$. We consider a complex vector lattice X in the folowing.

<definition/>(Following [1]) We say that the linear functional T on X is positive if $Tx\geq 0$ for each $x\in X^{+}$.

<notation/>For $x,y\in X$ such that $x\leq y$, we write $[x,y]$ for the interval $[x,y]=\{z\in X:x\leq z\leq y\}$.

<definition/>(Following [1]) For X a vector lattice and T a linear functional on X, we say that T is order bounded if the image $T([x,y])=\{Tz:z\in[x,y]\}$ is a bounded set of scalars for each interval $[x,y]$.

We see easily that each positive linear functional on X is order bounded, due of $Tx\leq Tz\leq Ty$ for each $z\in[x,y]$.

<lemma/>Each order bounded linear functional T on X is the difference $Tx=T^{+}x^{+}-T^{-}x^{-}$. Here, $T^{+}x^{+}=\sup\{Ty:0\leq y\leq x\}$ and $T^{-}x^{-}=\sup\{-Ty:0\leq -y\leq x\}$ are positive operators on X^{+}.

<proof/>See [1], p. 15.

<definition/>We say that a Fréchet space which is a locally convex lattice is a Fréchet lattice.

<definition/>If the LF space $X=\lim_{M\in N}X_{M}$ is equipped with a partial order $<$ such that each space $(X_{M},<)$ is a Fréchet lattice, we say that X is a LF lattice.

<theorem/>For X a Fréchet lattice or a LF lattice and T a positive linear functional on X, it holds that T is continuous.

<proof/>Let $P=\{p_{n}:n=0,1,2,...\}$ be a base of continuous seminorms for the Fréchet lattice X such that $p_1\leq p_2\leq ...$ Suppose that T is linear positive, not continuous. We then have that, for each n=1,2,..., there is some $x_{n}\in X$ such that $p_{n}(x_{n})=1$ and $|Tx_{n}|\geq n^3$. Here, we may $|x_{n}|>0$ for some given x_{n}. Consider the series $\sum_{n=1}^{\infty}x_{n}/n^2$. We have that $p_{m}(\sum_{j=n}^{M}x_{j}/j^2)\leq\sum_{j=n}^{M}p_{m}(x_{j}/j^2)\leq\sum_{j=n}^{M}p_{j}(x_{j}/j^2)\leq\sum_{j=n}^{M}1/j^2\rightarrow_{n,M\rightarrow\infty}0$ (we consider $m\leq n$). Hence the series is a Cauchy sequence, it converges to $x=\sum_{n=1}^{\infty}x_{n}/n^2$ and we find that each $x_{n}/n^2\leq\sum_{j=1}^{n}x_{j}/j^2\leq x$ (we see that for $0\leq u_{n+1}\leq u_{n}$, n=1,2,... and $u_{n}\rightarrow u$ it follows $u\geq 0$, for $u<0$ implies $u_{n}-u\geq -u>0$, $p_{m}(u_{n}-u)\geq p_{m}(u)>0$ (n=1,2,...) for $p_{m}\in P$, contradicting that $\lim_{n\rightarrow\infty}p_{m}(u_{n}-u)=0$). Thus $n\leq|Tx_{n}/n^2|\leq Tx$ for each n, we obtain a contradiction. As for $X=\lim_{n\in N}X_{n}$ a LF lattice, each X_{n} a Fréchet space, if T is not continuous then it is not continuous on some X_{n} and the theorem follows from the above.

Recall ([1]) that the vector space of all order bounded functionals on a vector lattice X is the order dual of X. Clearly that the definition adapts to complex vector lattices.

<corollary/>The order dual of a Fréchet or a LF lattice coincides with its topological dual. Each continuous linear functional is the difference of two positive linear functionals.

<proof/>Clearly that for T a continuos linear functional on the space, each z in the interval [x,y] satisfying that $|z|\leq|x|\vee|y|$, we have that $p_{\alpha}(z)\leq p_{\alpha}(|x|\vee|y|)$ for each seminorm p_{α}; [x,y] being bounded, its image through T is a bounded set of scalars, T is order bounded. The theorem follows from Theorem 2.7. and Lemma 2.4. as wished.

Applications to Radon measures and analytic functionals

Recall ([5]) the spaces $C^{m}(\Omega)$ where Ω is a open subset of R^{N}. We consider the seminorms $\|\phi\|_{j,K}=\max_{|\alpha|\leq j}\{\max\{|D^{\alpha}\phi(x)|:x\in K\}\}$ on $C^{m}(\Omega)$ ($j\leq m<\infty$) and $\{\|.\|_{j,K}:j=0,1,2,...\}$ on $C^{\infty}(\Omega)$ where K ranges over the class K of the compact subsets of Ω, $D^{\alpha}=((\partial/(\partial x_1)))^{\alpha_1}...((\partial/(\partial x_{N})))^{\alpha_{N}}$, $\alpha=(\alpha_1,...,\alpha_{N})\in N_0^{N}$. Also recall the spaces $C_{K}^{m}(\Omega)=\{\phi\in C^{m}(\Omega):supp(\phi)\subset K\}$

$C_{c}^{m}(\Omega)=\{\phi\in C^{m}(\Omega):supp(\phi)$ is compact$\}$ $(0\leq m\leq\infty)$ where, $supp(\phi)=\{x\in\Omega:\phi(x)\neq 0\}$. Leting $C_{c}^{0}(\Omega)$ equipped with the LF space topolgy that is the strict indutive limit of the Fréchet spaces $C_{K}^{0}=(\{\phi\in C^{0}(\Omega):supp(\phi)\subset K\},\|.\|_{0,K}\})$, K ranging over K, its dual is the space of Radon measures in Ω.

Consider the ordering for scalar functions $\phi\leq\psi$ meaning that $\phi(x)\leq\psi(x)$ for each x, the ordering in the compex plane as above. We have that $\sup\{\phi,\psi\},\inf\{\phi,\psi\}\in C_{K}^{0}(\Omega)$ if $\phi,\psi\in C_{K}^{0}(\Omega)$, these spaces are vector lattices. Also for each seminorm $\|.\|_{0,K}$ on $C_{K}^{0}(\Omega)$, clearly that $|\phi|\leq|\psi|$ implies $\|\phi\|_{0,K}\leq\|\psi\|_{0,K}$. Here, $|\phi|(x)=|\phi(x)|,|\psi|(x)=|\psi(x)|$. We thus have that $C_{K}^{0}(\Omega)$ is a Fréchet lattice, $C_{c}^{0}(\Omega)$ is a LF lattice when equipped with the LF space topology $\lim_{K\in K}(C_{K}^{0}(\Omega),\|.\|_{0,K})$.

<theorem/>The linear functional T on $C_{c}^{0}(\Omega)$ is a Radon measure if and only if it is order bounded, in which case there exist positive Radon measures T_1,T_2 in Ω such that $<T,\phi>=<T_1,\phi^{+}><-T_2,\phi^{-}>$ for each $\phi\in C_{c}^{0}(\Omega)$.

<proof/>This follows by Theorem 2.7. and Corollary 1.

Recall that O being a open subset of the complex plane, the space H(O) is determined by the analytic complex functions in O, equipped with the Fréchet space topology of $(C^{0}(\Omega),\{\|.\|_{0,K}:K\in K\})$ where we cosider $\Omega\subset R^2$ through the identification $x+iy\leftrightarrow(x,y)$. The dual H′(O) is the space of analytic functionals in O.

We consider the partial order $\phi<\psi$ in H(O) meaning that $D^{j}\phi(z)\leq D^{j}\psi(z)$ $(z\in\Omega,j=0,1,2,...)$

<theorem/>Letting O be as above, the space H(O) is a Fréchet lattice.

<proof/>In fact, H(O) is a Fréchet space. Also, there exist $\phi\vee\psi=\sum_{n=0}^{\infty}(a_{n}\vee b_{n})(z-a)^n$ for each $\phi(z)=\sum_{n=0}^{\infty}a_{n}(z-a)^n$ and $\psi(z)=\sum_{n=0}^{\infty}b_{n}(z-a)^n$ in $\{z\in O:|z-a|<r$, r the least of the radius of convergence of the power series at each point $a\in O$. Analogously for $\phi\wedge\psi$, we see easily that H(O) is a vector lattice. We have that for $|\phi|<|\psi|$, $|\phi(z)|\leq|\psi(z)|$ holds for $|z-a|<r$ and it follows $\|\phi\|_{0,K}\leq\|\psi\|_{0,K}$ for each seminorm $\|.\|_{0,K},K\in K$ defined analogously as above. The theorem follows.

Following Theorem 2.7., we have that each positive linear functional on H(O) is continuous. We put

<definition/>We say that the analytic functional T in O is positive if it is a positive linear functional on H(O).

<theorem/>For O a open subset of C, the analytic functionals in O are the order bounded linear functionals on (H(O),<). Each analytic functional in O is the difference

of two positive analytic functionals.

<proof/>This follows from Theorem 2.8. and Corollary 1.

Positive distributions

In the following, we consider a non empty subset Ω of R^{N} and the test functions ϕ in $C_{c}^{\infty}(\Omega)$.

Recall that $D(K,\Omega)=(C_{K}^{\infty}(\Omega),\{\|.\|_{j,K}:j\in N_0\})$ is a Fréchet space, the space of distributions in Ω is the LF space $D(\Omega)=\lim D(K(M),\Omega)$ where $(K(M))_{M=1}^{\infty}$ is an increasin sequence of compact subsets of Ω such that $\cup_{M=1}^{\infty}K(M)=\Omega$.

Also recall that S being a subset of R^{N} we say that S is of the first category if it is the countable union of sets C such that $\mathrm{int}(S)=\varphi$.

<remark/>If S is a subset of R^{N} of the first category, then $\lambda(S)=0$, λ for the Lebesgue measure.

<proof/>In fact we have $\lambda(S)\leq\lambda(S)=\inf\{\lambda(O):O \text{ open},S\subset O\}=0$ ([2], Theorem 15.5., Definition 15.4., pp. 113, 112).

<remark/>For ϕ a test function in Ω, the set Z of the points x in supp(ϕ) such that $\phi(x)=0$ is of the first category.

<proof/>In fact, notice that $\phi(x)=0$ $(x\in Z)$. If int(Z) contains a non void open set C we conclude the contradiction that $\mathrm{supp}(\phi)\subset\mathrm{supp}(\phi)\backslash C$ (see above).

<definition/>For two functions ϕ,ψ in $C_{c}^{\infty}(\Omega)$ we put $\phi<\psi$ meaning that $D^{\alpha}\phi\leq D^{\alpha}\psi$ in Ω for each $\alpha\in N_0^{N}$.

<lemma/>Given a test functions ϕ the fuction $|\phi|$ is differentiable a.e.

<proof/>In fact, we have that $\phi(x)\neq 0$ a.e. in supp(ϕ) by Remark 4.2 and Remark 4.1. Hence $|\phi|(x)=|\phi(x)|>0$ a.e. in supp(ϕ) and it follows that for $x\in\mathrm{supp}(\phi)\backslash C$, either $\phi(x)>0$ or $\phi(x)<0$ where $\lambda(C)=0$. Hence, assuming $\phi(x)>0$, we have that $|\phi|(y)=\phi(y)$ $(y\in(x-\delta,x+\delta)$ and there exists $D^{\alpha}|\phi|(x)=D^{\alpha}\phi(x)$. For $\phi(x)<0$ we have $D^{\alpha}|\phi|(x)=-D^{\alpha}\phi(x)$, the lemma follows.

<proposition/>For ϕ,ψ test functions in Ω, the functions $\phi\vee\psi$ and $\phi\wedge\psi$ are differentiable a.e.

<proof/>This follows by the above lemma using that $\phi\vee\psi=(\phi+\psi+|\phi-\psi|)/2$ and $\phi\wedge\psi=(\phi+\psi-|\phi-\psi|)/2$ ([1], Theorem 1.7. (2), p. 5).

<proposition/>For $\psi=\phi$ a.e, where ϕ is a test function, it holds that there exists $D^{\alpha}\psi(x)$ a.e. for each $\alpha\in N_0^{N}$.

<proof/>In fact, for each $x\in\Omega\backslash Z(\alpha)$ related to $D^{\alpha}\phi, D^{\alpha}\psi$ (see Remark 4.2.) we have $\psi(y)=\phi(y)$ in a neighborhood of x, hence there exists $D^{\alpha}\psi(x)$ where $\lambda(Z(\alpha))=0$ if $|\alpha|=1$. Next, if $|\beta|=2$, there exists $D^{\beta}\psi(x)$ for all $x\in\Omega\backslash(Z(\alpha)\cup Z(\beta))$ where $\lambda(Z(\alpha)\cup Z(\beta))\le\lambda(Z(\alpha))+\lambda(Z(\beta))=0$. The remark follows.

We consider the space $F(\Omega)$ that is the span of the set $S(\Omega)$ of all functions $\phi_1\vee...\vee\phi_{n}$ where $\phi_1,...,\phi_{n}\in C_{c}^{\infty}(\Omega)$. It follows from above that $F(\Omega)\subset I(\Omega)=\{\psi\in C^{\Omega}:\exists\phi\in C_{c}^{\infty}(\Omega),\psi(x)=\phi(x)$ a.e.$\}$. Clearly that $(F(\Omega),<)$ is a Riesz space due of $\phi\wedge\psi=-((-\phi)\vee(-\psi))$.

<notation/>We denote by $E(\Omega)$ the vector space that is determined by the equivalence classes $[\phi]=\{\psi:\psi(x)=\phi(x)$ a.e.$\}$ where $\phi\in D(\Omega)$.

<definition/>(Folowing [7]) We say that the ubspace W of the Riesa space $(X;<)$ is an ideal if $a\in W$ whenever $|a|\le|b|$ where $b\in W$

<lemma/>The subspace [0] is an ideal of $F(\Omega)$.

<proof/>In fact, if $|\psi|<|\phi|$ and $\|\phi\|_{j,K}=0$ clearly that $\|\psi\|_{j,k}=0$.

<definition/>For $\phi,\psi\in E\Omega)$ we put $[\phi]<[\psi]$ if and only if there exist elements $\phi_1\in[\phi],\psi_1\in[\psi]$ such that $\phi_1<\psi_1$.

Notice that given $\psi_2\in[\psi]$ in the above definition, also $\phi_1<\psi_2$. This follows from $\phi<\psi$ if and only if $D^{\alpha}\phi(x)\le D^{\alpha}\psi(x)$ a.e., $\phi,\psi\in E(\Omega)$, $\psi_1(x)=\psi_2(x)$ a.e.

<remark/>The space $E(\Omega)$ is a Riesz space such that $[\phi]\vee[\psi]=[\phi\vee\psi]$ and $[\phi]\wedge[\psi]=[\phi\wedge\psi]$.

<proof/>This follow by [7], Theorem 19.5. p. 127 and p. 128.

<notation/>For $[\phi]\in E(\Omega)$, we let supp([ϕ])= supp(ϕ).

For $[\phi]\in E(\Omega),\psi\in[\phi]$ it holds that $\text{essup}\{|D^{\alpha}\psi(x)|:x\in K\}=\sup\{|D^{\alpha}\phi(x):x\in K\}$ where K is any compact subset of Ω. Putting $p_{m,K}([\phi])=\|\phi\|_{m,K}$ (m=0,1,2,...) we see easily that $p_{m,K}$ is a seminorm in $E_{K}(\Omega)=\{[\phi]\in E(\Omega):\text{supp}([\phi])\subset K\}$. Letting $([\phi_{n}])$ be a Cauchy sequence in $(E(\Omega),\{p_{m,K}:m=0,1,2,...\})$ we have that $\phi_{n}\to\phi$ in $D(\Omega)$, some $\phi\in C_{c}^{\infty}(\Omega)$ hence $[\phi_{n}]$ converges to $[\phi]$ in $(E(\Omega),\{p_{m,K}:m=0,1,2,...\})$, hence $(E(\Omega),\{p_{m,K}:m=0,1,2,...\})$ is a Fréchet space.

<theorem/>The spaces $(E_{K}(\Omega),\{p_{m,K}:m=0,1,2,...\})$ are Fréchet lattices.

<proof/>In fact we have that $p_{m,K}([\phi])\leq p_{m,K}([\psi])$ for $[\phi]<[\psi]$ (see above). The theorem follows by definition.

<definition/>Following the analogue to the LF space $D(\Omega)$ in [5], we may consider the LF space $E(\Omega)=\lim(E_{K(M)}(\Omega),\{p_{m,K}:m=0,1,2,...\})$ where $E_{K(M)}(\Omega)=\{[\phi]\in E(\Omega):supp(\phi)\subset K(M)\}$, $(K(M))$ a sequence of compact subsets of Ω such that $\cup[K(M):M=1,2,...]=\Omega$.

<theorem/>For Ω a open subset of R^{N}, $E(\Omega)$ is a LF lattice.

<proof/>This follows from above.

<definition/>We say that a continuous linear functional on $E(\Omega)$ is a global distribution in Ω.

<definition/>We call global positive distributions in Ω the positive linear functions on the LF lattice $E(\Omega)$.

Notice that for $\phi_1,\phi_2\in[\phi]$ we have $\phi_1(x)=\phi_2(x)$ a.e. hence $<T,\phi_1-\phi_2>=0,<T,\phi_1>=<T,\phi_2>$ for T a global distribution on Ω.

For each distribution T in Ω, if we put $<T,[\phi]>=<T,\phi>$ we have that it holds that to every compact subset K of Ω there is a constant C such that, for all $[\phi]\in E(\Omega)$ such that $supp(\phi)\subset K$, we have $|<T,[\phi]>|=|<T,\phi>\leq C\|\phi\|_{m,K}$. We have that the associated T to T is a global distribution in Ω if and only if T is a distribution in Ω. This follows from Proposition 21.1. in [5], p. 222.

<theorem/>The global distributions T in Ω are the order bounded linear functionals on $E(\Omega)$. There exist positive global distributions T_1,T_2 in Ω such that $<T,[\phi]>=<T_1,[\phi]^+>-<T_2,[\phi]^->$ for each $[\phi]\in E(\Omega)$.

<proof/>This follows by Theorem 2.7. and Corollary 1. as wished.

<definition/>Letting Ω be a open subset of R^{N} we say that the distribution T in Ω is a positive distribution in Ω if the associated global distribution T in Ω is positive.

<theorem/>Each distribution T in Ω satisfies that there exist two positive distributions in Ω such that $<T,\phi>=<T_1\phi^+>-<T_2,\phi^->$ for each test function ϕ.

<proof/>This follows by Corollary 1.

As an application of the foregoing, consider the related even Sobolev space $W_{even}^{k,p}(\Omega)$ where Ω is a open subset of R^{N}, $1\leq p<\infty,k=1,2,...$ of the complex functions f on Ω such that the derivatives $D^{\alpha}f$ are in $L^{p}(\Omega)$ $(|\alpha|\leq k)$ ([6]). Here, we identify f with the distribution $<T_{f},\phi>=\int_{\Omega}f\phi$ $(\phi\in D(\Omega))$.

Following Definition 4.7. and Definition 4.3. we have that, putting $T_{f}\geqslant T_{g}\Leftrightarrow D^{\alpha}T_{f}\succ D^{\alpha}T_{g}$ for $|\alpha|$ even, $W^{k,p}(\Omega)$ is a Banach lattice (we equip the space with the norm $\|f\|_{k,p}=(\max\{\int_{\Omega}|D^{\alpha}f|^{p}:|\alpha|\leq k,\alpha \text{ even}\})^{1/p})$. We have that $\langle D^{\alpha}T_{f},\phi\rangle\geq 0$ ($|\alpha|$ even, we then have that T_{f} is positive if and only if $\phi\geq 0$, see just before Theorem 3.1.) and we may apply Theorem 2.7. to the Banach space $W_{even}^{k,p}(\Omega)$ obtaining

<theorem/>Each positive operator $T:W_{even}^{k,p}(\Omega)\rightarrow X$ where X is a Banach lattice, is continuous.

<proof/>This follows from the above as wished.

Notice that, using that each Banach space actually being a Banach lattice through a given order (see ([3]), the above theorem has possible applications in general settings.

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