

Closed Form Solution of a System of Linear Difference Equations

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Abstract

In this paper, the system of linear difference equations $x_n = \frac{a_n}{c_n} y_{n-1} + \frac{b_n}{c_n}$, $y_n = \frac{d_n}{f_n} x_{n-1} + \frac{e_n}{f_n}$ is solved in closed form.

Keywords: system of linear difference equations, closed form solution.

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Introduction

Recently, there has been great interest in studying difference equation systems. One of the reasons for this is the necessity for some techniques that can be used in investigating equations arising in mathematical models describing real life situations in population biology, economics, probability theory, genetics, psychology etc. There are many papers related to the difference equations system for example,

In [1] Cinar studied the solutions of the system of difference equations

$$x_{n+1} = \frac{1}{y_n}, \quad y_{n+1} = \frac{y_n}{x_{n-1} y_{n-1}}$$

In [2] Papaschinopoulos and Schinas studied the oscillatory behavior, the boundedness of the solutions, and the global asymptotic stability of the positive equilibrium of the system of nonlinear difference equations

$$x_{n+1} = A + \frac{y_n}{x_{n-p}}, \quad y_{n+1} = A + \frac{x_n}{y_{n-p}}, \quad n = 0, 1, \dots, p, q.$$

In [3] Papaschinnopoulos and Schinas proved the boundedness, persistence, the oscillatory behavior and the asymptotic behavior of the positive solutions of the system of difference equations

$$x_{n+1} = \sum_{i=0}^k \frac{A_i}{y_{n-i}^{p_i}}, \quad y_{n+1} = \sum_{i=0}^k \frac{B_i}{x_{n-i}^{q_i}}$$

In [4,5] Özban studied the positive solutions of the system of rational difference equations

$$x_n = \frac{a}{y_{n-3}}, \quad y_{n+1} = \frac{by_{n-3}}{x_{n-q}y_{n-q}}$$

And

$$x_{n+1} = \frac{1}{y_{n-k}}, \quad y_{n+1} = \frac{y_n}{x_{n-m}y_{n-m-k}}$$

In [6,7] Clark and Kulenović investigated the global asymptotic stability

$$x_{n+1} = \frac{x_n}{a + cy_n}, \quad y_{n+1} = \frac{y_n}{b + dx_n}$$

In [8] Camouzis and Papaschinnopoulos studied the global asymptotic behavior of positive solutions of the system of rational difference equations

$$x_{n+1} = 1 + \frac{x_n}{y_{n-m}}, \quad y_{n+1} = 1 + \frac{y_n}{x_{n-m}}$$

In [9] Yang et al. considered the behavior of the positive solutions of the system of difference equations

$$x_n = \frac{a}{y_{n-p}}, \quad y_n = \frac{by_{n-p}}{x_{n-q}y_{n-q}}$$

In [10] Kulenović and Nurkanović studied the global asymptotic behavior of solutions of the system of difference Equations

$$x_{n+1} = \frac{a + x_n}{b + y_n}, \quad y_{n+1} = \frac{c + y_n}{d + z_n}, \quad z_{n+1} = \frac{e + z_n}{f + x_n}$$

In [11] Zhang et al. investigated the behavior of the positive solutions of the system of difference equations

$$x_{n+1} = A + \frac{1}{y_{n-p}}, \quad y_{n+1} = A + \frac{y_{n-1}}{x_{n-r}y_{n-s}}$$

In [12] Zhang et al. studied the boundedness, the persistence and global asymptotic stability of the positive solutions of the system of difference equations

$$x_{n+1} = A + \frac{y_{n-m}}{x_n}, \quad y_{n+1} = A + \frac{x_{n-m}}{y_n}$$

In [13] Kadry investigate the periodicity of the solutions of the stochastic system of rational difference equations

$$x_n = \frac{a}{y_{n-p}}, \quad y_n = \frac{b_n}{x_{n+p-2}}, \quad p \geq 1$$

and he developed a new analytic technique to solve it, where $a, b, x_0 = N, y_0 = M$ are independent random variables.

In [14] Kadry study the convergence and divergence of the positive solutions of the system of difference equations,

$$x_{n+1} = \frac{a}{x_n + y_n}, \quad y_{n+1} = \frac{by_n}{x_n}, \quad x_0, y_0, a, b \in [0, \infty)$$

In this paper, a closed form solution of the linear difference system of equations

$$x_n = \frac{a_n}{c_n} y_{n-1} + \frac{b_n}{c_n}, \quad y_n = \frac{d_n}{f_n} x_{n-1} + \frac{e_n}{f_n}, \quad n \in N_0 \text{ is given.}$$

The last equation yield to:

$$x_n = \frac{a_n d_{n-1}}{c_n f_{n-1}} x_{n-2} + \frac{a_n e_{n-1}}{c_n f_{n-1}} + \frac{b_n}{c_n}$$

$$y_n = \frac{a_{n-1} d_n}{c_{n-1} f_n} y_{n-2} + \frac{d_n b_{n-1}}{c_{n-1} f_n} + \frac{e_n}{f_n}, \quad n \in N$$

Last two equations can be written as:

$$x_{2n} = \frac{a_{2n} d_{2n-1}}{c_{2n} f_{2n-1}} x_{2n-2} + \frac{a_{2n} e_{2n-1}}{c_{2n} f_{2n-1}} + \frac{b_{2n}}{c_{2n}},$$

$$x_{2n-1} = \frac{a_{2n-1} d_{2n-2}}{c_{2n-1} f_{2n-2}} x_{2n-3} + \frac{a_{2n-1} e_{2n-2}}{c_{2n-1} f_{2n-2}} + \frac{b_{2n-1}}{c_{2n-1}},$$

$$y_{2n} = \frac{a_{2n-1} d_{2n}}{c_{2n-1} f_{2n}} y_{2n-2} + \frac{d_{2n} b_{2n-1}}{c_{2n-1} f_{2n}} + \frac{e_{2n}}{f_{2n}},$$

$$y_{2n-1} = \frac{a_{2n-2} d_{2n-1}}{c_{2n-2} f_{2n-1}} y_{2n-3} + \frac{d_{2n-1} b_{2n-2}}{c_{2n-2} f_{2n-1}} + \frac{e_{2n-1}}{f_{2n-1}}, \quad n \in N$$

This means that the sequences $x_{2n}, x_{2n-1}, y_n, y_{2n-1}$ are solutions of a linear first order difference equation.

By solving the last four equations, we obtain the following closed form solution of our system:

$$\begin{aligned}
x_{2n} &= x_0 \prod_{j=1}^n \frac{a_{2j} d_{2j-1}}{c_{2j} f_{2j-1}} + \sum_{i=1}^n \left(\frac{a_{2i} e_{2i-1}}{c_{2i} f_{2i-1}} + \frac{b_{2i}}{c_{2i}} \right) \prod_{s=i+1}^n \frac{a_{2s} d_{2s-1}}{c_{2s} f_{2s-1}}, \\
x_{2n-1} &= x_{-1} \prod_{j=1}^n \frac{a_{2j-1} d_{2j-2}}{c_{2j-1} f_{2j-2}} + \sum_{i=1}^n \left(\frac{a_{2i-1} e_{2i-2}}{c_{2i-1} f_{2i-2}} + \frac{b_{2i-1}}{c_{2i-1}} \right) \prod_{s=i+1}^n \frac{a_{2s-1} d_{2s-2}}{c_{2s-1} f_{2s-2}} \\
y_{2n} &= y_0 \prod_{j=1}^n \frac{a_{2j-1} d_{2j}}{c_{2j-1} f_{2j}} + \sum_{i=1}^n \left(\frac{b_{2i-1} d_{2i}}{c_{2i-1} f_{2i}} + \frac{e_{2i}}{f_{2i}} \right) \prod_{s=i+1}^n \frac{a_{2s-1} d_{2s}}{c_{2s-1} f_{2s}} \\
y_{2n-1} &= y_{-1} \prod_{j=1}^n \frac{a_{2j-2} d_{2j-1}}{c_{2j-2} f_{2j-1}} + \sum_{i=1}^n \left(\frac{b_{2i-2} d_{2i-1}}{c_{2i-2} f_{2i-1}} + \frac{e_{2i-1}}{f_{2i-1}} \right) \prod_{s=i+1}^n \frac{a_{2s-2} d_{2s-1}}{c_{2s-2} f_{2s-1}} \quad \square.
\end{aligned}$$

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