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Rings with (*x*, *y*, *x*) in the Center

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ABSTRACT

In [1] right alternative rings satisfying the weak Novikov identity$(w, x, yz) = y(w, x, z)$ are studied. Kleinfeld and Smith [2] proved that a semiprime flexible ring with weak Novikov identity is associative. In this paper, we replace the weak Novikov identity $(w, x, yz) = y(w, x, z)$ with $(w, x, yz) = (w, x, y)z$. We prove that in a nonassociative ring R with $[(x, y, x), R] = 0$ and the weak Novikov identity $(w, x, yz) = (w, x, y)z$, then it is flexible i.e., $(x, y, x) = 0$. Next we prove that the associator ideal I is anticommutative and alternative. Using these results we show that R is associative.

Keywords : Nonassociative ring, semiprime, center, Associator ideal, Char. $\neq n$.

AMS Classifications :17D05

Throughout this paper, R represents a nonassociative ring of char. $\neq 2$ satisfying the weak Novikov identity

$$(w, x, yz) = (w, x, y)z, \tag{1}$$

for all w, x, y, z in R.

A ring R is semiprime if for any ideal A of R, $A^2 = 0$ implies $A = 0$ and C defined as $C = \{c \in N(R) / [c, R] = 0\}$ is called the center of R. A ring R is of char. $\neq n$ if $nx = 0$ implies $x = 0$ for all x in R and n a natural number. The associator ideal I consists of all finite sums of associators and right multiples of associators. As a consequence of (1), we observe that the associator ideal I of R consists of all finite sums of associators.

First we prove the following Lemmas:

***Lemma 1*:** If R is a nonassociative ring satisfying $(x, y, x) \in C$ and $(w, x, yz) = (w, x, y)z$ for all w, x, y, z in R, then it is flexible, that is, $(x, y, x)=0$.

***Proof* :** By hypothesis $[(x, y, x), R] = 0$ (2)

By taking $w = y = x$, $x = y$ and $z = (x, y, x)$ in (1) and using $(x, y, x) \in C$, we get
$(x, y, x(x, y, x)) = (x, y, (x, y, x)x) = (x, y, (x, y, x))x = 0$.

Now by using (1), we get $(x, y, x)\,(x, y, x) = 0$. That is, $(x, y, x)^2 = 0$.
Since (x, y, x) is in C, this implies that
$(x, y, x) = 0$. (3)

This completes the proof of the lemma.
We use the Teichmuller identity
$(wx, y, z) – (w, xy, z) + (w, x, yz) = w(x, y, z) + (w, x, y)z$, (4)

which is valid in every ring.
By using (1) in (4), we get
$(wx, y, z) – (w, xy, z) = w(x, y, z)$. (5)

By replacing w with z in (5) and using (3), we get
$(zx, y, z) = z(x, y, z)$.

Now by using (3) on both sides of the above equation and applying (1), (3), we get
$(z, y, zx) = z(z, y, x)$,

That is
$(z, y, z)x = 0 = z(z, y, x)$,

Therefore
$z(z, y, x) = 0$. (6)

Linearization of (6) yields
$z(w, y, x) = – w(z, y, x)$. (7)

For arbitrary element a, b, x, y, z in R, using 1 several times, we observe that
$p = (a, b, (x, y, z)) = (a, b, (xy)z – x(yz)) = (a, b, xy)z – (a, b, c)yz$
$= ((a, b, x)y)z – (a, b, c)yz = ((a, b, x), y, z)$.

Therefore $p = (a, b, (x, y, z)) = ((a, b, x), y, z)$.

Now by using (3), we get
$(a, b, (x, y, z)) = – (z, y, (a, b, x))$. (8)

By replacing x with z and using (3), we get
$(z, y, (a, b, z)) = 0.$ (9)

Now by linearizing (9), we get
$(z, y, (a, b, w)) = -(w, y, (a, b, z)).$ (10)

Now we combine (3) with (10). Then
$(w^{\alpha}, y, (a^{\alpha}, b, z^{\alpha})) = (\mathrm{sgn}\alpha)\,(w, y, (a, b, z)),$ (11)

where α is any permutation on the set $\{w, a, z\}$.

From(11), we have
$(a, b, (x, y, z)) = -(x, b, (a, y, z)).$ (12)

By using (12) and (8), we get
$(a, b, (x, y, z)) = (a, y, (z, b, x)).$

Now using (3), we obtain
$(a, b, (x, y, z)) = -(a, y, (x, b, z)).$ (13)

Thus we have
$(w, y^{\beta}, (a, b^{\beta}, z)) = (\mathrm{sgn}\,\beta)\,(w, y, (a, b, z)),$ (14)

Where β is any permutation on the set $\{y, b\}$. Now by using (7), we get
$(a, b, c)\,(x, y, z) = -x\,((a, b, c), y, z).$

Using (3) and by taking $c = z$ in above equation, we get
$(a, b, z)\,(x, y, z) = x(z, y, (a, b, z)),$

now by using (9), we have
$(a, b, z)\,(x, y, z) = 0.$

By linearizing this equation, we get
$(a, b, z)\,(x, y, c) = -(a, b, c)\,(x, y, z)$ (15)

At this point (15) and (3) implies
$(a^{\gamma}, b, c^{\gamma})\,(x^{\gamma}, y, z^{\gamma}) = (\mathrm{sgn}\,\gamma)\,(\mathrm{a, b, c})\,(x, y, z),$ (16)

Where γ stands for permutation on the set $\{a, c, x, z\}$. Also using (7) and (3), we get
$q = (a, b, c)\,(x, b, z) = -x((a, b, c), b, z) = x(z, b, (a, b, c)).$

Now by using (14), we obtain
$q = -x(z, b, (a, b, c)).$
Therefore
$q = -q$ implies $2q = 0.$

Since R is of char. $\neq 2$, we get $q = 0$.
That is
$(a, b, c)\,(x, b, z) = 0$.

Now by linearizing this equation, we get
$$(a, b, c)\,(x, y, z) = -\,(a, y, c)\,(x, b, z). \tag{17}$$

At this point combining (17) and (14), we get
$$(a, b, c)\,(x, y, z) = -\,(x, y, z)\,(a, b, c). \tag{18}$$

This implies the following result.

Lemma 2 : The associator ideal I is anticommutative.
Next we have

Lemma 3 : The associator ideal I is alternative.

Proof : Let q be an alternative element in I and w, x, y, z are arbitrary elements in R. Then by using (7), we get $wq(z, x, y) = -\,z(wq, x, y)$.

Now by using (3) and (1) twice in the same order, we get
$wq(z, x, y) = -\,z(wq, x, y) = z(y, x, wq) = z(y, x, w)q = -\,z(w, x, y)q$
$= -\,z(w, x, yq) = w((z, x, y)q)$, by using (7) and (1)
$= -\,w(q(z, x, y))$, by Lemma 2.

Therefore $wq(z, x, y) = -\,w(q(z, x, y))$. Then
$$(wq)p = -\,w(qp) \text{ or } w(qp) = -\,(wq)p, \tag{19}$$

Where p, q are in I and w in R.
Now we assume that r an element of I. Then by using (19), we get
$(p, q, r) + (p, r, q) = (pq)r - p(qr) + (pr)q - p(rq)$
$= -\,p(qr) - p(qr) - p(rq) - p(rq) = -\,2p(qr) - 2p(rq)$
$= -\,2p(qr + rq)$

From Lemma 2 it implies $qr + rq = 0$. So that $(p, q, r) + (p, r, q) = 0$.
At this point I is both flexible and right alternative, hence alternative.

Lemma 4 : If S is an anticommutative alternative ring of char. $\neq 2$, then $(S^2)(S^2) = 0$.

Proof : For arbitrary elements w, x, y, z in S, we have $(xy)(zx) = x(yz)x = -\,x^2(yz) = 0$, using alternative identities and anticommutativity. Linearizing this identity results in $(wy)(zx) = -(xy)(zw)$. Applying this in conjunction with anticommutativity leads to $(wy)(zx) = (zx)(wy)$. However (wy) also anticommutes with (zx) so that $2(wy)(zx) = 0$. Since S is of char. $\neq 2$, we have $(wy)(zx) = 0$. So that $(S^2)(S^2) = 0$.

Main Result:

Theorem 1: If R is a semiprime ring of char. $\neq 2$ with $(x, y, x) \in C$ satisfying the Novikov identity $(w, x, yz) = (w, x, y)z$ is associative.

***Proof* :** Let p, q be arbitrary elements of I, I the associator ideal of R and z an arbitrary element of R. Then $z(qp) = -(zp)q$ by (19). Thus I^2 is a left ideal of R. Also $(p, q, z) = -(z, q, p) = -(zq)p + z(qp)$ is an element of I^2. Hence $(pq)z = (p, q, z) + p(qz) \in I^2$. Thus I^2 is an ideal of R. Then Lemmas 1, 2 and 3 imply that the ideal I^2 of R squares to 0.

Since R is semiprime, $I = 0$.

Hence R is associative.

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