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Symmetries and the Differential Form for Nonlinear Boltzmann Equation

Varun Jerath and V. G. Gupta

Department of Mathematics, University of Rajasthan, Jaipur, India.
Email: varun.jerath@yahoo.in

Abstract:

In this paper Differential form method is used to obtain the determined equations of Non linear boltzmann equation. Later on the Lie symmetries have been discussed for the problem. Finally group invariant solutions have been obtained.

Introduction :

In a pioneer work, Harrison and Estabrook [2], introduced the method of writing differential equations or system of differential equations in terms of differential forms and finnding their symmetries. Later on, Papachristou and Harrison [3-5], generalized the method to vector valued or Lie algebra-valued differential forms and used in the two-dimensional Dirac equation and the Yang-Mills free field equations in Minkowski space-time. Waller [6] used 1-form and contraction in nonlinear diffusion equations arising in plasma physics.

Recently, Davison and Kara [1] treated Burgers equation to obtain potential and approximate symmetries using differential form method. In the present study, to obtain the determined equations of nonlinear diffusion equation with convection term, it is assumed that when the differential forms are zero then their Lie derivatives are also zero. Later on, the symmetries of four special cases of the problem have been considered. Finally, the group invariant solutions are obtained for all the cases of the problem. This method save considerable work in complicated cases, specially in cases where not all forms in the Ideal are of the same rank.

Non linear boltzmann equation:

Consider non linear Boltzmann equation $u_{xt}+ u_x +u^2 = 0$, which has an associated auxillary system which we write here for convenience :

$u_x = v$

$v_t + v + u^2 = 0$ (1)

we introduce the 2-forms

$\alpha = dudt - vdxdt$

$= u_x dxdt - vdxdt$

$\beta = dvdx + (v+u^2)dtdx$

$= v_t dtdx + vdtdx + u^2 dtdx$

Which return the system (1) when annulled. To calculate a symmetry

$$X = \xi\,\partial_x + \tau\partial_t + \phi\partial_u + \eta\partial_v$$

of (1), we calculate the lie derivatives of these forms. First,

$L_X\alpha = X \square d\alpha + d(X \square \alpha)$

$= X \square (-dvdxdt) + d(-v\xi dt - \tau du + v\tau dx + \varphi dt)$

$= (-v\xi_x - v\tau_t + \varphi_x - \eta)dxdt + (-\xi - v\xi_v + \varphi_v + \xi)dvdt + (-v\xi_u + \tau_t + \varphi_u)dudt$

$+ (-\tau_x - v\tau_u)dxdu + (-\tau_v)dvdu + (v\tau_v)dvdx$

When $\alpha = \beta = 0$, we have $dudt = vdxdt$ and $dvdx = (v + u^2)dxdt$

$L_X\,\alpha\big|_{(\alpha=\beta=0)} = (-v\xi_x + \varphi_x - \eta - v^2\xi_u + v\varphi_u + v^2\tau_v + u^2 v\tau_v)dxdt + (-v\xi_v + \varphi_v)dvdt$

$+ (-\tau_x - v\tau_u)dxdu + (-\tau_v)dvdu$

And we may now split the coefficients of dtdx, dvdt, dxdu, dvdu to obtain

dtdx : $-v\xi_x + \varphi_x - \eta - v^2\xi_u + v\varphi_u + v^2\tau_v + u^2 v\tau_v = 0$ (2)

dvdt : $-v\xi_v + \varphi_v = 0$ (3)

dxdu : $-\tau_x - v\tau_u = 0$ (4)

dvdu : $\tau_v = 0$ (5)

Next,

$L_X\beta = X \square d\beta + d(X \square \beta)$

$= X \square (dvdxdt + 2ududtdx) + d(-\xi dv - v\xi dt - u^2\xi dt + \eta dx + \tau vdx + \tau u^2 dx)$

$= (-v\xi_x - u^2\xi_x - \eta_t - v\tau_t - u^2\tau_t - \eta - 2u\varphi)dxdt + (-\xi_x - \eta_v - v\tau_v - u^2\tau_v)dxdv$

$+ (-\xi_u)dudv + (-\xi_t + v\xi_v + u^2\xi_v)dtdv + (-v\xi_u)dudt + (\eta_u + v\tau_u + u^2\tau_u)dudx$

When $\alpha = \beta = 0$, we have $dudt = vdxdt$ and $dvdx = (v + u^2)dxdt$

$L_X\,\beta\big|_{(\alpha=\beta=0)} = (-\eta_t - v\tau_t - u^2\tau_t - \eta - 2u\varphi + v\eta_v + v^2\tau_v + vu^2\tau_v + u^2\eta_v + u^2 v\tau_v + u^4\tau_v$

$-$

$v^2\xi_u)dxdt + (-\xi_u)dudv + (\eta_u + v\tau_u + u^2\tau_u)dudx +$

$(-\xi_t + v\xi_v + u^2\xi_v)dtdv$

And we may now split the coefficients of dxdt, dudv, dudx, dtdv to obtain

dxdt : $-\eta_t - v\tau_t - u^2\tau_t - \eta - 2u\varphi + v\eta_v + v^2\tau_v + vu^2\tau_v + u^2\eta_v$
$+u^2v\tau_v + u^4\tau_v - v^2\xi_u = 0$ (6)

dudv : $-\xi_u = 0$ (7)

dudx : $\eta_u + v\tau_u + u^2\tau_u = 0$ (8)

dtdv : $-\xi_t + v\xi_v + u^2\xi_v = 0$ (9)

Solving equations (2) –(9) we get the following results

$\xi = c_1x + c_2$ (10)

$\tau = -c_3e^t + c_4$ (11)

$\varphi = -c_1u + c_3ue^t$ (12)

$\eta = vc_3e^t - 2vc_1$ (13)

thus we have the symmetry generators

$X_1 = x\,\partial_x - u\,\partial_u - 2v\,\partial_v$

$X_2 = \partial_x$

$X_3 = -e^t\,\partial_t + ue^t\,\partial_u + ve^t$

$X_4 = \partial_t$

The symmetry X_1 and X_3 are the only genuine potential symmetry of the nonlinear Boltzmann equation. In the absence of the auxiliary variable v, i.e., for the case v = 0 the symmetry generators called the Lie point symmetry generators. The commutation relation between the Lie point symmetry generators or vector fields is given by the following table:

	X_1	X_2	X_3	X_4
X_1	0	$-X_2$	0	0
X_2	X_2	0	0	0
X_3	0	0	0	$-X_3$
X_4	0	0	X_3	0

The one-parameter groups G_i (i = 1, 2, 3, 4) generated by the X_i are given by using $\exp(\varepsilon X_i)(x, t, u)$ as follows

$G_1 : (xe^{\varepsilon}, t, ue^{-\varepsilon})$, $G_2 : (x+\varepsilon, t, u)$,
$G_3 : (x, \exp(-e^t)+t-1, \exp(e^t)u)$, $G_4 : (x, t+\varepsilon, u)$

Since each group G_i is a symmetry group.

The solution of Boltzmann equation corresponding to its different symmetry groups G_i (i =1,2,3,4) are obtained by using $\tilde{u} = g \cdot u = g \cdot f(x, t)$ as follows

$u^{(1)} = e^{-\varepsilon} f(xe^{-\varepsilon}, t)$, $u^{(2)} = f(x - \varepsilon, t)$, $u^{(3)} = \exp(e^t) f(x, T)$, with $T = \exp(-e^t)+t-1$ and $u^{(4)} = f(x, t - \varepsilon)$

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