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Some Universals in Strong Semilattices of Modules

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Abstract

Two universal arrows in the category $PMod-K$ of partial modules (strong semilattices of modules) over a partial ring K (strong semilattice of rings) are introduced . The category Set_{E_K} of strong semilattices of sets is developed and free partial modules over K on objects in Set_{E_K} are constructed . A theory of tensor products for $PMod-K$ is also developed.

Keywords: Free partial modules,Tensor products

1.Introduction

This paper is a successor of $[1]$ in the direction of developing universal constructions for partial modules over partial rings. A partial ring is introduced in $[3]$ and proved to be precisely a strong semilattice of rings . Given a partial ring K, we have the notion of a partial module over K on one hand , and the notion of a strong semilattice of modules over K on the other , which are proved in $[1]$ to be equivalent . All products and coproducts exist in the category of partial modules over $K[1]$. In the present work , we introduce free partial modules and tensor product for partial modules.These constructions, that comprising Sections 2 and 3 respectively, are based on the notion of a strong semilattice of sets, which is discussed in a categorical setting in Section 1 . We refer to $[1]$ for notations and terminology that are not sufficiently explained here . Other references in semigroups , strong semilattices of groups and rings , categories , and rings and modules are listed at the end.

2.The category of semilattices of sets

A strong semilattice of sets consists of a (disjoint) union A of sets,

$A = \cup A_x, x \in E_A$, indexed by a semilattice E_A , and for any $x, y \in E_A$ with $x \geq y$, a function of sets $\varphi_{x,y} : A_x \to A_y$ such that $\varphi_{x,x}$ is the identity function of A_x , for all $x \in E_A$ and for every $x, y, z \in E_A$ with $x \geq y \geq z, \varphi_{y,z} \circ \varphi_{x,y} = \varphi_{x,z}$.

If A is a strong semilattice E_A of sets A_x , we may write

$A = \rho\lfloor E_A; A_x, \varphi_{x,y} \rfloor$

If B is another strong semilattice E_B of sets B_x ,we may write , when no confusion exists

$B = \rho\lfloor E_B; B_x, \varphi_{x,y} \rfloor$

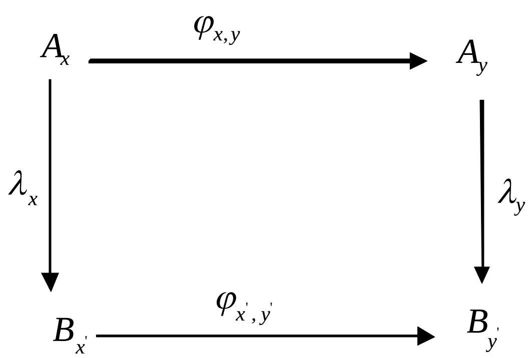
where $\varphi_{x,y}$ is the connecting mapping $\varphi_{x,y} : B_x \to B_y$ $(x \geq y)$. Let E be an arbitrary but fixed semilattice.

Let Set_E consists of objects and morphisms as follows ; An object A in Set_E is a strong semilattice of sets

$A = \rho\lfloor E_A; A_x, \varphi_{x,y} \rfloor$

together with on isomorphism of semilattices , $\sigma_A : E_A \to E$. If A and B are objects in Set_E , we have an isomorphism of semilattices $\sigma_B^{-1} \sigma_A : E_A \to E_B$. We may denote this isomorphism by σ_{BA} , and for each $x \in E_A$, denote the element $\sigma_B^{-1} \sigma_A (x)$ by x' . In other words , we write $\sigma_{BA}(x) = x'$ for every $x \in E_A$. It follows immediately that $x \geq y$ in E_A if and only of $x' \geq y'$ in E_B.

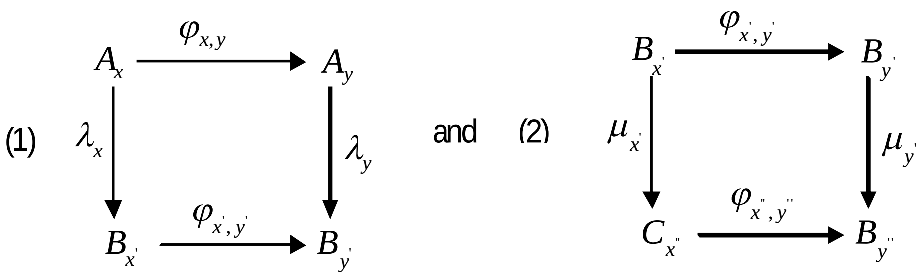
A morphism λ in Set_E with domain object A in set_E and codomain object B in set_E is a collection $\lambda = (\lambda_x)_{x \in E_A}$ of functions $\lambda_x : A_x \to B_{\sigma_B^{-1}\sigma_A(x)}$; i.e. $\lambda_x : A_x \to B_{x'}$ for all $x \in E_A$, such that the diagram



commutes, for every x and y in E_A with $x \geq y$.

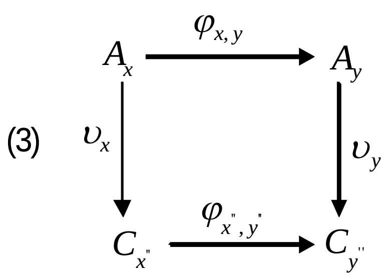
Lemma 2.1 *Set_E is a category.*

Proof. Let A, B and C be objects in Set_E. Let $\lambda : A \to B$ and $\mu : B \to C$ be morphisms in Set_E. For each $x \in E_A$, let ν_x be the composite function $\nu_x = \mu_{x'} \lambda_x : A_x \to C_{x''}$. Define $\nu = (\nu_x)_{x \in E_A}$. Let $x \geq y$ in E_A. We have two commutative diagrams



a

where $x' = \sigma_{BA}\ (x) = \sigma_B^{-1}\ \ \sigma_A\ (x)$ and $x'' = \sigma_{CB}\ (x') = \sigma_C^{-1}\ \ \sigma_B\ (x')$, for every $x \in E_A$. Consider the diagramd



We have

$$\begin{aligned}
\phi_{x'',y''}\, \nu_x &= \phi_{x'',y''}\left(\mu_{x'} \lambda_x\right) = \left(\phi_{x'',y''} \mu_{x'}\ \lambda_x\right) \\
&= \left(\mu_{y'}\ \phi_{x',y'}\right)\lambda_x = \mu_{y'}\left(\phi_{x',y'}\ \lambda_x\right) \\
&= \mu_{y'}\left(\lambda_y\, \phi_{x,y}\right) = \left(\mu_{y'}\ \lambda_y\right)\phi_{x,y} \\
&= \nu_y\, \phi_{x,y}.
\end{aligned}$$

Hence the diagram (3) commutes. Thus $\nu = \mu\lambda$ is a morphism in Set_E. Clearly, the operation on morphisms defined by $\mu\lambda = \ \nu$ is associative, and $C_A = \left(C_{A_x}\right)_{x \in E_A}$

(where $i_{A_x} : A_x \to A_x$ is the identity mapping for all $x \in E_A$), satisfies $\lambda\, i_A = \lambda$ and $i_A\, \mu = \mu$, and the proof is complete.

Let K be an arbitrary but fixed partial ring, namely, K is a strong semilattice E_K of rings K_u. In symbols

$$K = \rho\lfloor E_K; K_u, \psi_{u,v} \rfloor$$

There is a category $PMod - K$ of left $K-$ partial modules and $K-$ partial module homomorphisms $[[1], \text{Lemma}\, 3.3]$. We have

Lemma 2.2 *There exists a faithful (forgetful) functor*

$$U : PMod - K \to \text{Set}_{E_K}$$

which sends every left $K-$ partial module $M = \rho\lfloor E_M; M_e, \varphi_{e,f} \rfloor$ to the strong semilattice of sets $UM = \rho\lfloor E_M; M_e, \varphi_{e,f} \rfloor$ in Set_{E_K} (forgetting the structure of M_e), together with the isomorphism $\sigma_M : E_M \to E_K$, and sends every left $K-$ partial module homomorphism $\alpha = (\alpha_e)_{e \in E_M} : M \to N$ to the morphism $U\alpha = (\alpha_e)_{e \in E_M}$ in Set_{E_K}, where $\alpha_e : M_e \to N_{e'}$ $\left(e' = \sigma_{MN}(e)\right)$ is viewing as mapping of sets . In other words, the functor U is completely determined by the usual forgetful functors,

$$U_u : Mod - K_u \to Set \quad , u \in E_K ,$$

where $Mod - K_u$ is the category of left $K_u -$modules and Set is the usual category of sets and functions of sets .

Proof. Follows from $[1]$, Lemma 3.2 (i), (ii).

3.Free Partial Modules

In this section, $K = \rho\lfloor E_K; K_u, \psi_{u,v} \rfloor$ is a fixed partial ring and $U : PMod - K \to Set_{E_K}$ is the functor of Lemma 2.2.

Theorem 3.1 *Let* $A = \rho\lfloor E_A; A_x, \varphi_{x,y} \rfloor$ *be an object in* Set_{E_K} *with the isomorphism* $\sigma_A : E_A \to E_K$. *Then there exists a pair* (F, η) *where* F *is an object of* $PMod - K$ *and* η *is a morphism* $\eta : A \to UF$ *in* Set_{E_K} *subject to the universal property that, whenever* (B, ζ) *is a another such a pair , there exists a unique* $PMod - K$ *morphism* $\zeta^* : F \to B$ *such that* $U\zeta^*\eta = \zeta$.

Proof. For each $x \in E_A$, let $F_{\bar{x}}$ be the left free $K_{\sigma_A(\bar{x})}-$ module with basis A_x and zero $\bar{x}$. Let $u, v \in E_K$ with $u \geq v$. Then $F_{\bar{y}}$ is a left K_u- module, where $y = \sigma_A^{-1}(v)$, with action given by, for $r \in K_u$ and $a \in F_{\bar{y}}$,

$$r \cdot a = \psi_{u,v}\, r \cdot a,$$

(cf.[1], Lemma 2.3). By the universal property of free modules, the mapping $A_x \to F_{\bar{y}}$, given by $a \mapsto \varphi_{x,y}\, a$, where $x = \sigma_A^{-1}(u)$, extends uniquely to a left K_u- module homomorphism

$$\bar{\varphi}_{\bar{x},\bar{y}} : F_{\bar{x}} \to F_{\bar{y}}$$

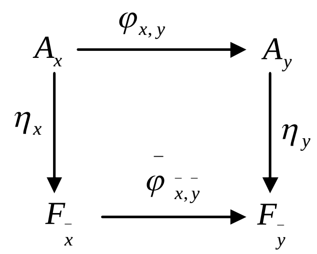
such that for any free generator $a \in F_{\bar{x}}, \bar{\varphi}_{\bar{x},\bar{y}}(a) = \varphi_{x,y}(a)$ $,\varphi_{x,y}(a)$ is in A_y which is the set of free generators in $F_{\bar{y}}$). Let $E_F = \left\{\bar{x} : x \in E_A\right\}$ with $\bar{x} \geq \bar{y}$ if and only if $x \geq y$ if and only if $\sigma_A(x) \geq \sigma_A(y)$. Clearly, for every $\bar{x} \in E_F$, $\bar{\varphi}_{\bar{x},\bar{x}}$ is the identity module homomorphism on $F_{\bar{x}}$. For all $\bar{x}, \bar{y}, \bar{z} \in E_F$ with $\bar{x} \geq \bar{y} \geq \bar{z}$, and any element $\sum r_i\, a_i$, finite sum in $F_{\bar{x}}$ ($r_i \in K_u, a_i \in A_x, \sigma_A(x) = u$), we have

$$\begin{aligned}
\left(\bar{\phi}_{\bar{y},\bar{z}}\, \bar{\phi}_{\bar{x},\bar{y}}\right)\left(\sum r_i\, a_i\right) &= \bar{\phi}_{\bar{y},\bar{z}}\left(\bar{\phi}_{\bar{x},\bar{y}}\left(\sum r_i\, a_i\right)\right) \\
&= \bar{\phi}_{\bar{y},\bar{z}}\left(\sum r_i\, \phi_{x,y}(a_i)\right) \\
&= \sum r_i\, \bar{\phi}_{\bar{y},\bar{z}}\left(\phi_{x,y}(a_i)\right) \\
&= \sum r_i\left(\phi_{y,z}\, \phi_{x,y}(a_i)\right) \\
&= \sum r_i\, \phi_{x,z}(a_i) = \bar{\phi}_{\bar{x},\bar{z}}\left(\sum r_i\, a_i\right).
\end{aligned}$$

Hence $\bar{\phi}_{\bar{y},\bar{z}}\, \bar{\phi}_{\bar{x},\bar{y}} = \bar{\phi}_{\bar{x},\bar{z}}$. It follows that $F = \bigcup_{\bar{x} \in E_F} F_{\bar{x}}$ is a strong semilattice of left K_u- modules ($u = \sigma_A(x)$)

$$F = \rho\left[E_F; F_{\bar{x}}, \bar{\phi}_{\bar{x},\bar{y}}\right],$$

that is, F is an object of $PMod-K$, with the isomorphism $\sigma_F : E_F \to E_K, \bar{x} \mapsto \sigma_A(x)$. For each $x \in E_A$, let η_x be the natural embedding of the set A_x in the left $K_{\sigma(x)=u}$ – module $F_{\bar{x}}$, that is the mapping of sets $\eta_x : A_x \to F_{\bar{x}}, a \mapsto \left(1+\bar{x}\right)a = a$, which satisfies the universal property of the free K_u- module $F_{\bar{x}}$. Let $x \geq y$ in E_A, and consider the diagram



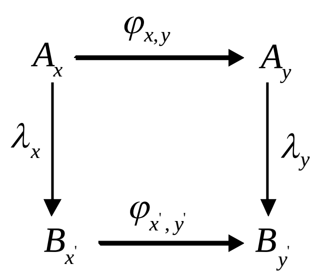
Let $a \in A_x$, then

$$\bar{\phi}_{\bar{x},\bar{y}}\,\eta_x\left(a\right) = \bar{\phi}_{\bar{x},\bar{y}}\left(a\right) = \phi_{x,y}\left(a\right) = \eta_y\,\phi_{x,y}\left(a\right)$$

and so the diagram commutes. It follows that $\eta = (\eta_x)_{x \in E_A}$ is a morphism $\eta : A \to UF$ in Set$_{E_K}$. Let (B, λ) be such that, B is an object of $PMod-K$, with isomorphism $\sigma_B : E_B \to E_K$, and λ is a morphism $\lambda : A \to UB$ in Set$_{E_K}$. Thus λ is a collection $\lambda = (\lambda_x)_{x \in E_A}$ of functions of sets

$$\lambda_x : A_x \to B_{\sigma_{BA}}(x) = B_{x'}$$

such that the diagram



commutes for all $x, y \in E_A$ with $x \geq y$. For each $x \in E_A$, the universal property of $\left(F_{\bar{x}}, \eta_x \right)$ implies that there exists a unique left $K_u -$module homomorphism $\lambda_x^* : F_{\bar{x}} \to B_x$, such that $\lambda_x^* \, \eta_x = \lambda$. Actually, λ_x^* is defined by

$$\lambda_x^*\left(\sum r_i \, a_i\right) = \sum r_i \;\; \lambda_x(a_i), a_i \in A_x, r_i \in K_u.$$

Define $\lambda^* : F \to B$, as follows: For any $a \in F$, say $a = \sum r_i \, a_i \in F_{\bar{x}}$, for some $x \in E_A$ with $\sigma_A(x) = u, r_i \in K_u, a_i \in A_x$, let

$$\lambda^*(a) = \lambda_x^*\left(\sum r_i \, a_i\right) = \sum r_i \, \lambda_x^*(a_i) = \sum r_i \, \lambda_x(a_i).$$

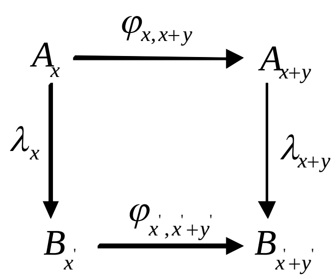
If $a = \sum r_i \, a_i$ is an element in $F_{\bar{x}}$ as above and $b = \sum s_j \, b_j \in F_{\bar{y}}, b_j \in A_y, s_j \in K_v = K_{\sigma_A(y)}$, for some $y \in E_A$, then $a + b \in F_{\bar{x}+\bar{y}}$, and we have

$$\begin{aligned}
\lambda^*\left(a+b\right) &= \lambda_{x+y}^*\left(a+b\right) = \lambda_{x+y}^*\left(\sum r_i \, a_i + \sum s_j \, b_j\right) \\
&= \lambda_{x+y}^*\left(\bar{\phi}_{\bar{x},\bar{x}+\bar{y}}\left(\sum r_i \, a_i\right) + \bar{\phi}_{\bar{y},\bar{x}+\bar{y}}\left(\sum s_j \, b_j\right)\right) \\
&= \lambda_{x+y}^*\left(\sum r_i \, \phi_{x,\,x+y} \, a_i + \sum s_j \, \phi_{y,\,x+y} \, b_j\right) \\
&= \sum \psi_{u,u+v} \, r_i \, \lambda_{x+y}^* \, \eta_{x+y} \, \phi_{x,\,x+y} \, a_i \\
&\quad + \sum \psi_{v,u+v} \, s_j \, \lambda_{x+y}^* \, \eta_{x+y} \, \phi_{y,\,x+y} \, b_j \\
&= \sum \psi_{u,u+v} \, r_i \, \lambda_{x+y} \, \phi_{x,\,x+y} \, a_i + \sum \psi_{v,u+v} \, s_j \, \lambda_{x+y} \, \phi_{y,\,x+y} \, b_j \\
&= \sum \psi_{u,u+v} \, r_i \;\; \phi_{x,\,x+y} \, \lambda_x \, a_i + \sum \psi_{v,u+v} \;\; s_j \, \phi_{y,\,x+y} \, \lambda_y \, b_j \\
&= \phi_{x,\,x+y}\left(\sum r_i \, \lambda_x^* \, \eta_x \, a_i\right) + \phi_{y,\,x+y}\left(\sum s_j \, \lambda_y^* \, \eta_y \, b_j\right) \\
&= \phi_{x,\,x+y} \, \lambda_x^*\left(\sum r_i \, a_i\right) + \phi_{y,\,x+y} \, \lambda_y^*\left(\sum s_j \, b_j\right) \\
&= \lambda_x^* \, a + \lambda_y^* \, b = \lambda^* \, a + \lambda^* \, b.
\end{aligned}$$

Let $r \in K_u$, for some $u \in E_K$ and let $a = \sum_{\text{finite sum}} r_i \, a_i \in F_{\bar{x}}$, for some $x \in E_A$, ($r_i \in K_{\sigma_A(x)}, a_i \in A_x$). Let $y = \sigma_A^{-1}(u)$. Thus $\sigma_A(y) = u$, and $\bar{y} = \sigma_F^{-1}(u)$. We have,

$$
\begin{aligned}
\lambda^*(ra) &= \lambda^*\left(\psi_{u,u+\sigma_{A(x)}}\, r\cdot\bar{\phi}_{\bar{x},\sigma_F^{-1}(u)+\bar{x}}\, a\right)\\
&= \lambda^*\left(\psi_{u,u+\sigma_F(\bar{x})}\, r\cdot\bar{\phi}_{\bar{x},\bar{x}+\bar{y}}\left(\sum r_i\, a_i\right)\right)\\
&= \lambda^*_{x+y}\left(\psi_{u,u+\sigma_F(\bar{x})}\, r\cdot\sum r_i\, \phi_{x,x+y}\, a_i\right)\\
&= \psi_{u,u+\sigma_F(\bar{x})}\, r\left(\sum r_i\, \lambda_{x+y}\, \phi_{x,x+y}\, a_i\right)\\
&= r\cdot\left(\sum r_i\, \phi_{x',x'+y'}\, \lambda_x\, a_i\right)
\end{aligned}
$$

by commutativity of the diagram



$$
\begin{aligned}
&= r\cdot\phi_{x',x'+y'}\left(\sum r_i\, \lambda_x\, a_i\right)\\
&= r\cdot\phi_{x',x'+y'}\, \lambda^*_x\left(\sum r_i\, a_i\right)\\
&= r\cdot\lambda^*_x\, a = r\cdot\lambda^*(a).
\end{aligned}
$$

Thus λ^* satisfies PMH1 and PMH2 (cf.[1], Sec.3). If $\bar{x}\in E_F$, and so $\sigma_F(\bar{x}) = \sigma_A(x)$ (for a unique $x\in E_A$) then, since $\bar{x}$ is the zero of $F_{\bar{x}}$, we have $\lambda^*(\bar{x}) = x'$ the zero of $B_{x'}$, where $x' = \sigma_B^{-1}\,\sigma_A(x)$. Thus, $\sigma_F(\bar{x}) = \sigma_A(x) = \sigma_B(x') = \sigma_B\,\lambda^*(\bar{x})$, and so PMH3 holds. It follows that λ^* is a left K partial module homomorphism. Let $a\in A$, say $a\in A_x$, for some $x\in E_A$. Then $\eta\, a = \eta_x\, a\in F_{\bar{x}}$. Thus we have

$$U\,\lambda^*\,\eta(a) = U\,\lambda^*(\eta\, a) = U\,\lambda^*(\eta_x a) = \lambda^*_x\,\eta_x\, a = \lambda_x\, a = \lambda\, a.$$

Hence, $U\,\lambda^*\,\eta = \lambda$. Finally, the uniqueness of λ^* follows from that of each λ^*_x.

4.Tensor Products

In this section we introduce some theory of tensor products for partial modules parallel to corresponding results in classical tensor products of modules over rings . Our technique allows to extend other results (e.g. those for commutative rings or rings with identities). Throughout the section, unless stated otherwise, K is an arbitrary partial ring , i.e., strong semilattice of rings, $K = \rho\lfloor E_K; K_u, \psi_{u,v} \rfloor$

Let A_K and ${}_K B$ be right and left partial modules over K respectively, with isomorphisms of semilattices $\sigma_A : E_A \to E_K$ and $\sigma_B : E_B \to E_K$ respectively. As usual, for each $e \in E_A$, the element $\sigma_B^{-1}\sigma_A(e)$ in E_B is denoted $e^{'}$, and the isomorphism

$$\sigma_B^{-1}\sigma_A : E_A \to E_B, e \mapsto e^{'}$$

is denoted σ_{BA}. Let C be any $\square_{E_K}$ − module (cf[1], Example 3.1) and so, C is a strong semilattice E_C of abelian groups with isomorphism $\sigma_C : E_C \to E_K$.

Let $A \times_K B$ be the product of A and B. Viewing as $\square_{E_K}$ − partial modules, let $A \times B$ be the product of A and B in the category $PMod - \square_{E_K}$, [cf.[1], See 3]. Equivalently, $A \times B$ is a strong semilattice of abelian groups

$$A \times B = \rho\left[E_{A\times B}; A_e \times B_{e^{'}}, \phi_{(e,e^{'}),(f,f^{'})}\right]$$

where $E_{A\times B}$ is the semilattice $E_{A\times B} = \left\{\left(e, e^{'}\right) : e \in E_A\right\}$with $\left(e, e^{'}\right) \le \left(f, f^{'}\right)$ if and only if $e \le f$ and $e^{'} \le f^{'}$ and $\sigma_{A\times B}$ is the isomorphism $E_{A\times B} \to E_K, (e,e^{'}) \mapsto \sigma_A(e) = \sigma_B(e^{'})$. $A \times B$ has the additional structure:

For $r \in K$ and $(a,b) \in A \times B$, there are well defined elements (ar, b) and (a, rb) in $A \times B$. We then have the following definition, we call a mapping $\lambda : A \times B \to C$ in Set${}_{E_K}$ (where C is a $\square_{E_K}$ − partial module which is also , forgetting structure, an object in Set${}_{E_K}$), middle linear if for all $a, a_i \in A, b, b_i \in B, r \in K \ (i = 1,2)$ with $e_a = e_{a_i}, e_b = e_{b_i} = \sigma_{BA}(e_a)$, $i = 1,2$ and $e_r = \sigma_A(e_a)$, we have

(i) $\lambda(a_1 + a_2, b) = \lambda(a_1, b) + \lambda(a_2, b)$

(ii) $\lambda(a, b_1 + b_2) = \lambda(a, b_1) + \lambda(a, b_2)$

(iii) $\lambda(ar, b) = \lambda(a, rb)$

The proof of the following lemma is easy and omitted.

Lemma 4.1 *Let* A, B, C *and* λ *be as above. For any* $(a,b) \in A \times B$, *we have*

(i) $e_{\lambda(a,b)} = \lambda(e_a, b) = \lambda(a, e_b)$

(ii) $-\lambda(a,b)=\lambda(-a,b)=\lambda(a,-b).$

We now construct a tensor product for partial modules that extends the usual one for modules over rings. Our approach follows that given for modules in [6].

We assume that we are given a right $K-$ partial module A_K and a left $K-$ partial module ${}_K B.$ When no confusion may exist, we omit the subscribt K and simply write A and $B.$ For each $e\in E_B,$ we write $\sigma_B^{-1}\sigma_A(e)=\sigma_{BA}(e)=e',$ and $\sigma_A(e)=\sigma_B(e')=u_e.$ Viewing as right and left $K_{u_e}-$ modules, A_e and $B_{e'}$ have a tensor product $A_e\otimes_{K_{u_e}} B_{e'}$. We set

$$A\otimes_K B=\bigcup_{\substack{e\in E_A\\ (u\in E_K)}} A_e\otimes_{K_{u_e}} B_{e'}.$$

Theorem 4.1 $A\otimes_K B$ *is an abelian partial group (equivalently a* $\mathbb{Z}_{E_K}-$ *partial module).*

Proof. For each $e\in E_A,$ the identity of the abelian group $A_e\otimes B_{e'}$ is $e\otimes e'$ and we have $e\otimes e'=a\otimes e'=e\otimes b$ for all $a\in A_e$ and $b\in B_{e'}.$ Thus $A\otimes_K B$ may be viewed as union of (disjoint) abelian groups indexed by the semilattice

$$E_{A\times B}=\{(e,e'):e\in E_A,\sigma_{BA}(e)=e'\}$$

where, $(e,e')\geq(f,f')$ in $E_{A\times B}$ if and only if $e\geq f$ in E_A and $e'\geq f'$ in $E_B.$ We have the isomorphism of semilattices

$\sigma_{A\otimes B}:E_{A\times B}\to E_K,(e,e')\mapsto u_e,$ where $u_e\in E_K$ is defined as above by $\sigma_A(e)=\sigma_B(e')=u_e.$ We identify the element $(e,e')\in A\times B$ with $e\otimes e'$ which is the identity of the abelian group $A_e\otimes B_{e'},$ and we have a union of abelian groups

$$A\otimes_K B=\bigcup_{e\otimes e'\in E_{A\otimes_K B}} A_e\otimes B_{e'}$$

indexed by the semilattice $E_{A\otimes_K B}=\{e\otimes e':e\in E_A\}$ Every element in $E_{A\otimes_K B}$ will be written in the unique standard form $e\otimes e',$ where $e\in E_A$ and $e'=\sigma_{BA}(e).$ Thus for all $a\in A_e, b\in B_{e'},$ we write $a\otimes e'=e\otimes e'$ and $e\otimes b=e\otimes e'.$ We define structure maps on $A\otimes_K B$ as follows: For $(e,e')\geq(f,f'),$ that is $e\otimes e'\geq f\otimes f'$ in $E_{A\otimes B},$ equivalently, $e\geq f$ in E_A and $e'\geq f'$ in $E_B,$ define

$$\varphi \quad : \quad A_e \times B_{e'} \to A_f \otimes_{K_{u_f}} B_{f'}$$

$$(a,b) \quad \mapsto \quad \phi_{e,f}(a) \otimes \phi_{e',f'}(b).$$

If $a_1, a_2 \in A_e, b \in B_{e'}$, then

$$\varphi(a_1 + a_2, b) \quad = \quad \phi_{e,f}(a_1 + a_2) \otimes \phi_{e',f'}(b)$$

$$= \quad (a_1 + a_2 + f) \otimes (b + f').$$

$$\phi(a_1, b) + \phi(a_2, b) \quad = \quad (a_1 + f) \otimes (b + f') + (a_2 + f) \otimes (b + f')$$

$$= \quad ((a_1 + f) + (a_2 + f)) \otimes (b + f')$$

$$= \quad (a_1 + a_2 + f) \otimes (b + f').$$

Thus
$\phi(a_1 + a_2, b) = \phi(a_1, b) + \phi(a_2, b)$.

Similarly, for any $a \in A_e, b \in B_{e'}$ and $q \in \square$, we have

$$\varphi(qa, b) \quad = \quad (qa + f) \otimes (b + f')$$

$$= \quad (qa + qf) \otimes (b + f')$$

$$= \quad q(a + f) \otimes (b + f')$$

$$= \quad (a + f) \otimes q(b + f')$$

$$= \quad (a + f) \otimes (qb + f')$$

$$= \quad \varphi(a, qb).$$

It follows that ϕ is middle and hence extends uniquely to a homomorphism of abelian groups ($\square_{E_K}$ − modules)

$$\phi_{e\otimes e', f\otimes f'} : A_e \otimes_{K_{u_e}} B_{e'} \to A_f \otimes_{K_{u_f}} B_{f'}$$

which is defined on generators by,

$$a \otimes b \mapsto (a + f) \otimes (b + f').$$

Clearly, $\phi_{e\otimes e', e\otimes e'}$ is the identity map on $A_e \otimes B_{e'}$ for all $e \in E_A$. Let $e \otimes e' \geq f \otimes f' \geq g \otimes g'$ in $E_{A\otimes B}$, that is $e \geq f \geq g$ and $e' \geq f' \geq g'$ in E_A and E_B respectively. Then, for $a \otimes b \in A_e \otimes_{K_{u_e}} B_{e'}$, we have

$$\left(\phi_{f\otimes f',g\otimes g'}\cdot\phi_{e\otimes e',f\otimes f'}\right)(a\otimes b) = \phi_{f\otimes f',g\otimes g'}\left((a+f)\otimes\left(b+f'\right)\right)$$
$$= (a+f+g)\otimes\left(b+f'+g'\right)$$
$$= (a+g)\otimes\left(b+g'\right).$$

Also,
$$\phi_{e\otimes e',g\otimes g'}(a\otimes b)=(a+g)\otimes\left(b+g'\right).$$

Thus, $\phi_{f\otimes f',g\otimes g'}\cdot\phi_{e,e',f\otimes f'}=\phi_{e\otimes e',g\otimes g'}$. It follows that $A\otimes_K B$ is an abelian partial group, which is precisely a strong semilattice of abelian groups
$$A\otimes_K B=\rho\left[E_{A\otimes_K B}, A_e\times B_{e'}, \phi_{e,e',f\otimes f'}\right].$$

Corollary 4.1 *For any* $a\otimes b\in A\otimes_K B$, *say* $a\in A_e, b\in B_{e'}$, *and any* $g\otimes g'\in E_{A\otimes_K B}$, *we have*
$$(a\otimes b)+\left(g\otimes g'\right)=(a+g)\otimes\left(b+g'\right).$$

Proof. Since $A\otimes_K B$ is a strong semilattice of groups, the addition in $A\otimes_K B$ is given by the structure maps. Moreover, it is easy to see that addition in $E_{A\otimes_K B}$ is given by $\left(e\otimes e'\right)+\left(g\otimes g'\right)=(e+g)\otimes\left(e'+g'\right)$, for any $e\otimes e'$ and $g\otimes g'$ in $E_{A\otimes_K B}$, (notice that $E_{A\times B}$ is isomorphic to $E_{A\otimes_K B}$). Thus we have

$$\left(a\otimes b\right)+\left(g\otimes g'\right) = \phi_{e\otimes e',e+g\otimes e'+g'}\; a\otimes b+\phi_{g\otimes g',e+g\otimes e'+g'}\; g\otimes g'$$
$$= \left(a+e+g\otimes b+e'+g'\right)+\left(e+g\otimes e'+g'\right)$$
$$= \left(a+g\otimes b+g'\right)+\left(e+g\otimes e'+g'\right)$$
$$= (a+g)\otimes\left(b+g'\right)$$

(since $a+g\otimes b+g'\in A_{e+g\otimes e'+g'}$).

Lemma 4.2 *For each* $e\in E_A$, *let* $\xi_{\left(e,e'\right)}$ *be the usual canonical middle linear map*
$$\xi_{\left(e,e'\right)}: A_e\times B_{e'}\to A_e\otimes_{K_{u_e}} B_{e'}, (a,b)\mapsto a\otimes b.$$

Let ξ be the map
$$\xi: A\times B\to A\otimes_K B$$

defined as follows. For $(a,b)\in A\times B,$ say $(a,b)\in A_e\times B_{e'},$ for some $e\in E_A,$ let $\xi(a,b)=\xi_{(e,e')}(a,b).$ Then ξ is a morphism in the category Set$_{E_K}$. Moreover, ξ is middle linear .

Proof. For $e\otimes e'\geq f\otimes f'$ in $E_{A\otimes_K B}$, we have a diagram

$$\begin{array}{ccc} A_e\times B_{e'} & \xrightarrow{\ \varphi_{(e,e'),(f,f')}\ } & A_f\times B_{f'} \\ \Big\downarrow{\scriptstyle \xi_{(e,e')}} & & \Big\downarrow{\scriptstyle \xi_{(f,f')}} \\ A_e\otimes_{K_{u_e}} B_{e'} & \xrightarrow{\ \varphi_{e\otimes e',f\otimes f'}\ } & A_f\otimes_{K_{u_f}} B_{f'} \end{array} \qquad (1)$$

Let $(a,b)\in A_e\times B_{e'}$. Then
$\xi_{(f,f')}\cdot\phi_{(e,e'),(f,f')}(a,b)=\xi_{(f,f')}\left(a+f,b+f'\right)=(a+f)\otimes\left(b+f'\right).$

On the other hand
$\phi_{e\otimes e',f\otimes f'}\cdot\xi_{(e,e')}(a,b) \;=\; \phi_{e\otimes e',f\otimes f'}(a\otimes b)$
$= \;(a+f)\otimes\left(b+f'\right).$

Thus the diagram (1) commutes, and so ξ is a morphism in Set$_{E_K}$. Clearly, $\xi=\left(\xi_{(e,e')}\right)$ is middle linear, since each $\xi_{(e,e')}$ is so and the proof is complete.

Given a right $K-$partial module A and a left $K-$partial module $B,$ then in view of Theorem 4.1 and Lemma 4.2, we call the □$_{E_K}$ $-$ partial module $A\otimes_K B$ together with the middle linear map $\xi,$ the *tensor product* of A and B over K. This is justified by showing the universality of $\xi,$ and this is the aim of our next theorem.

Theorem 4.2 *Let* A, B *and* ξ *be as above. For any* $\square_{E_K}-$ *partial module* C *with isomorphism* $\sigma_C : E_C \to E_K$ *(forgetting structures,* $A \times B$ *and* C *are objects in Set*$_{E_K}$*) and any middle linear morphism* $\lambda : A \times B \to C$ *in Set*$_{E_K}$*, there is a unique* $\square_{E_K}-$ *partial module homomorphism*

$$\bar{\lambda} : A \otimes_K B \to C$$

which is (forgetting structure) a morphism in Set$_{E_K}$, such that

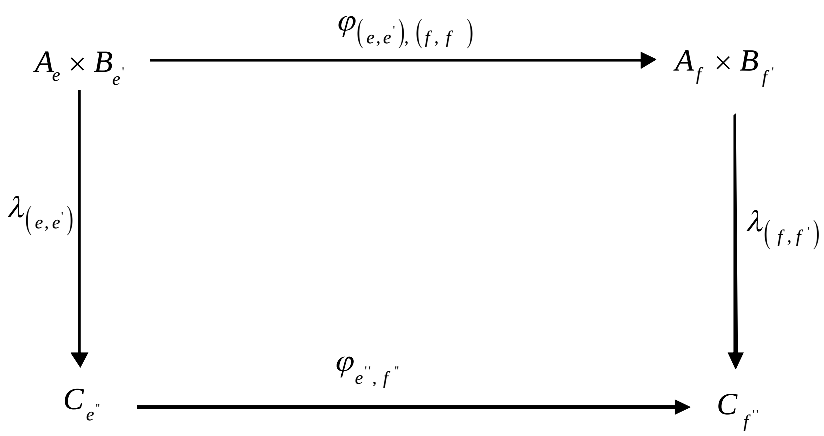
$$\bar{\lambda}\,\xi = \lambda.$$

Proof. As usual, for any $e \in E_A$, we write

$\sigma_A(e) = u_e = \sigma_B(e') = \sigma_C(e''), e' \in E_B, e'' \in E_C, u_e \in E_K$, and

$\sigma_{BA}(e) = e' = \sigma_{BC}(e'')$, etc.

Let $e, f \in E_A$ with $e \geq f$. Since λ is a morphism in Set$_{E_K}$, we have a commutative diagram



(2)

the universal property of tensor products of R_u-modules implies that there is a unique homomorphism of abelian groups

$$\bar{\lambda}_{(e,e')} : A_e \otimes B_{e'} \to C_{e''}$$

such that

$$\bar{\lambda}_{(e,e')}\,\xi_{(e,e')} = \lambda_{(e,e')} \text{ (for every } e \in E_A). \qquad (3)$$

Moreover, for any generator $a \otimes b \in A_e \otimes B_{e'}$, we have

$$\bar{\lambda}_{(f,f')} \cdot \phi_{e\otimes e', f\otimes f'} \left(a \otimes b\right) \;=\; \bar{\lambda}_{(f,f')} \cdot \phi_{e\otimes e', f\otimes f'} \left(\xi_{(e,e')} \left(a, b\right) \right)$$

$$= \bar{\lambda}_{(f,f')} \cdot \xi_{(f,f')} \; \phi_{(e,e'),(f,f')} \left(a, b\right) \quad \text{(by(1))}$$

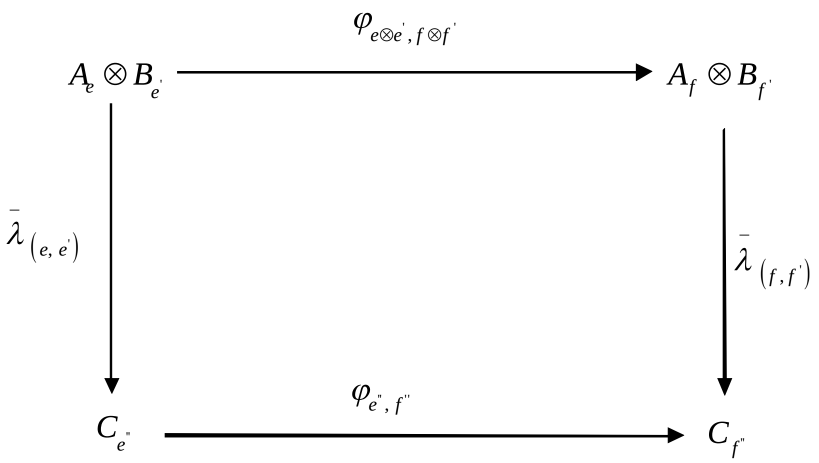
$$= \bar{\lambda}_{(f,f')} \, \phi_{(e,e'),(f,f')} \left(a, b\right) \quad \text{(by(3))}$$

$$= \phi_{e^{\cdot}, f^{\cdot}} \cdot \bar{\lambda}_{(e,e')} \left(a, b\right) \quad \text{(by(2))}$$

$$= \phi_{e'', f''} \; \bar{\lambda}_{(e,e')} \, \xi_{(e,e')} \left(a, b\right) \quad \text{(by(3))}$$

$$= \phi_{e'', f''} \; \bar{\lambda}_{(e,e')} \left(a \otimes b\right).$$

Hence the diagram



(4)

commutes. Define $\bar{\lambda} : A \otimes_K B \to C$ on generators as follows, for any $(a \otimes b) \in A \otimes_K B$, say $a \otimes b \in A_e \otimes_{K_{u_e}} B_{e'}$, for some (and unique) $e \in E_A$, $\bar{\lambda}\left(a \otimes b\right) = \bar{\lambda}_{(e,e')}\left(a \otimes b\right)$. Let $a \otimes b,\ c \otimes d \in A \otimes_K B$, say $a \otimes b \in A_e \otimes_{K_{u_e}} B_{e'}$ and $c \otimes d \in A_f \otimes_{K_{u_f}} B_{f'}$ $(e, f \in E_A)$. Then

$$a \otimes b + c \otimes d = \left(a + f \otimes b + f'\right) + \left(c + e \otimes d + e'\right).$$

Thus

$$\bar{\lambda}(a\otimes b+c\otimes d) = \bar{\lambda}_{(e+f,e'+f')}\left((a+f\otimes b+f')+(c+e\otimes d+e')\right)$$

$$= \bar{\lambda}_{(e+f,e'+f')}(a+f\otimes b+f')+\bar{\lambda}_{(e+f,e'+f')}(c+e\otimes d+e')$$

$$= \bar{\lambda}_{(e+f,e'+f')}\xi_{(e+f,e'+f')}(a+f,b+f')$$

$$+\bar{\lambda}_{(e+f,e'+f')}\xi_{(e+f,e'+f')}(c+e,d+e')$$

$$= \lambda_{(e+f,e'+f')}(a+f,b+f')+\lambda_{(e+f,e'+f')}(c+e,d+e')$$

$$= \lambda_{(e+f,e'+f')}\phi_{(e,e'),(e+f,e'+f')}(a,b)$$

$$+\lambda_{(e+f,e'+f')}\phi_{(f,f'),(e+f,e'+f')}(c,d)$$

$$= \phi_{e'',e''+f''}\lambda_{(e,e')}(a,b)+\phi_{f'',e''+f''}\lambda_{(f,f')}(c,d)$$

$$= \lambda_{(e,e')}(a,b)+\lambda_{(f,f')}(c,d).$$

On the other hand

$$\bar{\lambda}(a\otimes b)+\bar{\lambda}(c\otimes d) = \bar{\lambda}_{(e,e')}\xi_{(e,e')}(a,b)+\bar{\lambda}_{(f,f')}\xi_{(f,f')}(c,d)$$

$$= \lambda_{(e,e')}(a,b)+\lambda_{(f,f')}(c,d).$$

Hence $\bar{\lambda}$ is a homomorphism (in Set E_K) of partial groups. That $\bar{\lambda}\xi=\lambda$ follows from the construction of $\bar{\lambda}$ and (3), and that $\bar{\lambda}$ is unique with respect to this property follows from the uniqueness of each $\bar{\lambda}_{(e,e')}, e\in E_A,$ with respect to the corresponding property.

We write A_K (resp. $_KA$) to mean A is a right $K-$partial module (resp. A is a left $K-$partial module).

Corollary 4.2 *If* $A_K, A'_K, {}_KB$ *and* ${}_KB'$ *are* $K-$*partial modules and* $\alpha: A\to A', \beta: B\to B'$ *are* $K-$*partial module homomorphisms, then there is a unique* $\square_{E_K}$ *–homomorphism* $A\otimes_K B\to A'\otimes_K B'$ *such that* $a\otimes b\mapsto\alpha(a)\otimes\beta(b),$ *for all* $a\in A, b\in B$ *with* $\sigma_{BA}(e_a)=e_{b'}.$

Proof. For each $e\in E_A, A_e$ and $A'_{\sigma_{A'A}(e)}$ are right $K_{u_e}-$ modules, whereas $B_{\sigma_{BA}(e)}$

and $B'_{\sigma_{B'A}(e)}$ are left $K_{u_e}-$ modules where $u_e = \sigma_A(e)$. By Lemma 4.1 (i), $\alpha(e) = \sigma_{A'A}(e)$ and

$\beta\left(\sigma_{BA}(e)\right) = \sigma_{B'B}\left(\sigma_{BA}(e)\right) = \sigma_{B'}^{-1}\sigma_B\left(\sigma_B^{-1}\sigma_A(e)\right) = \sigma_{B'}^{-1}\sigma_A(e) = \sigma_{B'A}(e).$ Now, α and β induce (for each $e \in E_A$, $K_{u_e}-$module homomorphism $\alpha_e : A_e \to A'_{\sigma_{A'A}(e)}$ and $\beta_{\sigma_{BA}(e)} : B_{\sigma_{BA}(e)} \to B'_{\sigma_{B'A}(e)}$. By Corollary 5.3, Chapter IV in [6] , we have for each $e \in E_A$ a unique group homomorphism

$$\gamma_{(e,\sigma_{BA}(e))} = \alpha_e \otimes \beta_{\sigma_{BA}(e)} : A_e \otimes_{K_{u_e}} B_{\sigma_{BA}(e)} \to A'_{\sigma_{A'A}(e)} \otimes_{K_{u_e}} B'_{\sigma_{B'A}(e)}$$

such that

$$a \otimes b \mapsto \alpha_e(a) \otimes \beta_{\sigma_{BA}(e)}(b),$$

for all $a \in A_e$ and $b \in B_{\sigma_{BA}(e)}$. Define

$$\gamma : A \times B \to A' \otimes_K B'$$

as follows; for any $(a,b) \in A \times B,$ say $a \in A_e$ and $b \in B_{\sigma_{BA}(e)}$ (for some $e \in E_A$), let

$$\gamma(a,b) = \alpha_e \otimes \beta_{\sigma_{BA}(e)}(a \otimes b) = \alpha_e(a) \otimes \beta_{\sigma_{BA}(e)}(b).$$

Thus γ may be viewed as a collection $\gamma = (\gamma_{(e,\sigma_{BA}(e))})$. We show that γ is in Set $_{E_K}$. Let $(e,e') \ge (f,f')$ in $E_{A\times B}$, and consider the diagram

$$\begin{array}{ccc}
A_e \otimes B_{e'} & \xrightarrow{\quad\varphi\quad} & A_f \otimes B_{f'} \\
\Big\downarrow{\scriptstyle \gamma_{(e,e')}} & & \Big\downarrow{\scriptstyle \gamma_{(f,f')}} \\
A'_{\sigma_{AA'}(e)} \times B'_{\sigma_{BB'}(e')} & \xrightarrow{\quad\varphi'\quad} & A'_{\sigma_{A'A}(f)} \times B'_{\sigma_{B'B}(f')}
\end{array} \qquad (1)$$

Let $(a,b)\in A_e\times B_{e'}$, then

$$\gamma_{(f,f')}\cdot\phi(a,b) = \gamma_{(f,f')}(a+f,b+f')$$
$$= \alpha_f(a+f)\otimes\beta_{f'}(b+f')$$
$$= (\alpha_e(a)+\alpha_f(f))\otimes(\beta_{e'}(b)+\beta_{f'}(f)) \qquad \text{(Lemma3.2(iii),[1])}$$
$$=(\alpha_e(a)+\sigma_{A'A}(f))\otimes(\beta_{e'}(b)+\sigma_{B'B}(f')). \tag{2}$$

On the other hand

$$\phi'\circ\gamma_{(e,e')}(a,b)=\phi'(\alpha_e(a)\otimes\beta_{e'}(b))$$
$$=(\alpha_e(a)+\sigma_{A'A}(f))\otimes(\beta_{e'}(b)+\sigma_{B'B}(f')). \tag{3}$$

from (2) and (3), the diagram (1) commutes, and hence γ is a morphism in Set $_{E_K}$. Now, we show that γ is middle linear. Let $a_1,a_2\in A_e, b\in B_{e'}$, for arbitrary $e\in E_A$, where $e'=\sigma_{BA}(e)$. Then,

$$\gamma(a_1+a_2,b) = \alpha_e(a_1+a_2)\otimes\beta_{e'}(b)$$
$$= (\alpha_e(a_1)+\alpha_e(a_2))\otimes\beta_{e'}(b)$$
$$= (\alpha_e(a_1)\otimes\beta_{e'}(b))+(\alpha_e(a_2)\otimes\beta_{e'}(b)) \qquad (\text{since} A_e \text{ and } \beta_{e'} \text{ are} K_{u_e}-\text{modules})$$
$$= \gamma(a_1,b)+\gamma(a_2,b).$$

Similarly, $\gamma(a,b_1+b_2)=\gamma(a,b_1)+\gamma(a,b_2)$, for any $a\in A_e$, say and $b_1,b_2\in\sigma_{BA}(e)$. If $a\in A_e, r\in K_{u_e}, b\in\beta_{e'}$, where $e'=\sigma_{BA}(e)$ and $u_e=\sigma_A(e)$, then

$$\gamma(ar,b) = \alpha_e(ar)\otimes\beta_{e'}(b)=\alpha_e(a)r\otimes\beta_{e'}(b)$$
$$= \alpha_e(a)\otimes r\beta_{e'}(b)=\alpha_e(a)\otimes\beta_{e'}(rb)$$
$$= \gamma(a,rb).$$

Therefore, γ is middle linear. By Theorem 4.2, there is a unique $\square_{E_K}-$ homomorphism $\bar{\gamma}:A\otimes_K B\to A'\otimes_K B'$ such that $\bar{\gamma}\,\xi=\gamma$. That is $\bar{\gamma}(a\otimes b)=\bar{\gamma}\,\xi(a,b)=\gamma(a,b) =\alpha(a)\otimes\beta(b)$ for all $a\otimes b\in A\otimes B$.

Also following the notations used in module theory, the unique $\square_{E_K}-$ homomorphism $\bar{\gamma}$ of Corollary 4.2 may be denoted

$$\alpha\otimes\beta : A\otimes_K B\to A^{'}\otimes_K B^{'}.$$

Corollary 4.3 *Let α and β be as in Corollary 4.2. If $\alpha^{'} : A_K^{'}\to A_K^{''}$ and $\beta^{'} : B_K^{'}\to B_K^{''}$ are also $K-$ partial module homomorphisms, then*

$$\left(\alpha^{'}\otimes\beta^{'}\right)\left(\alpha\otimes\beta\right)=\alpha^{'}\alpha\otimes\beta^{'}\beta : A\otimes_K B\to A^{''}\otimes_K B^{''}.$$

Consequently, if α and β are $K-$ partial module homomorphisms, then $\alpha\otimes\beta$ is a partial group isomorphism with inverse $\alpha^{-1}\otimes\beta^{-1}$.

Proof. There are composite $K-$ partial module homomorphisms $\alpha^{'}\alpha : A\to A^{''}$ and $\beta^{'}\beta : B\to B^{''}$. By Lemma 3.1 (i) [1], we have for each $e\in E_A$

$$\left(\alpha^{'}\alpha\right)(e)=\alpha^{'}\left(\alpha(e)\right)=\alpha^{'}\left(\sigma_{A^{'}A}(e)\right)=\sigma_{A^{''}A^{'}}\left(\sigma_{A^{'}A}(e)\right)=\sigma_{A^{''}A}(e).$$

Similarly,

$$\left(\beta^{'}\beta\right)\left(\sigma_{BA}(e)\right)=\beta^{'}\left(\sigma_{B^{'}B}\sigma_{BA}(e)\right)=\beta^{'}\left(\sigma_{B^{'}A}(e)\right)=\sigma_{B^{''}B^{'}}\left(\sigma_{B^{'}A}(e)\right)=\sigma_{B^{''}B}(e).$$

Whence,

$$\left(\alpha^{'}\alpha\otimes\beta^{'}\beta\right)_{\left(e\otimes\sigma_{BA}(e)\right)} : A_e\otimes_{K_{u_e}} B_{\sigma_{BA}(e)}\to A^{''}_{\sigma_{A''A}(e)}\otimes_{K_{u_e}} B^{''}_{\sigma_{B''A}(e)}.$$

On the other hand

$$\begin{aligned}
\left(\left(\alpha^{'}\otimes\beta^{'}\right)\left(\alpha\otimes\beta\right)\right)\left(e\otimes\sigma_{BA}(e)\right) &= \left(\alpha^{'}\otimes\beta^{'}\right)\left(\alpha(e)\otimes\beta\left(\sigma_{BA}(e)\right)\right)\\
&= \left(\alpha^{'}\otimes\beta^{'}\right)\left(\sigma_{A^{'}A}(e)\otimes\sigma_{B^{'}B}\left(\sigma_{BA}(e)\right)\right)\\
&= \left(\alpha^{'}\otimes\beta^{'}\right)\left(\sigma_{A^{'}A}(e)\otimes\sigma_{B^{'}A}(e)\right)\\
&= \alpha^{'}\left(\sigma_{A^{'}A}(e)\otimes\sigma_{B^{'}A}(e)\right)\\
&= \sigma_{A^{''}A}\left(\sigma_{A^{'}A}(e)\right)\otimes\sigma_{B^{''}B^{'}}\left(\sigma_{B^{'}A}(e)\right)\\
&= \sigma_{A^{''}A}(e)\otimes\sigma_{B^{''}A}(e).
\end{aligned}$$

It follows that

$$\left(\left(\alpha^{'}\otimes\beta^{'}\right)\left(\alpha\otimes\beta\right)\right)_{e\otimes\sigma_{BA}(e)} : A_e\otimes_{K_{u_e}} B_{\sigma_{BA}(e)}\to A^{''}_{\sigma_{A^{''}A}(e)}\otimes_{K_{u_e}} B^{''}_{\sigma_{B^{''}A}(e)}.$$

By using the known result for module homomorphisms (cf. [6]) we then have for each $e \in E_A$

$$\left(\left(\alpha' \otimes \beta'\right)\left(\alpha \otimes \beta\right)\right)_{e \otimes \sigma_{BA}(e)} = \left(\left(\alpha'\alpha\right)_e \otimes \left(\beta'\beta\right)_{\sigma_{BA}(e)}\right).$$

The result follows by using Lemma 3.2 (i), [1].

Theorem 4.3 *If* $A \overset{\alpha}{\rightarrow} B \overset{\beta}{\rightarrow} C \rightarrow 0$ *is exact sequence of left* $K-$ *partial modules and* D *is a right* $K-$ *partial module, then*

$$D \otimes_K A \overset{1_D \otimes \alpha}{\rightarrow} D \otimes_K B \overset{1_D \otimes \beta}{\rightarrow} D \otimes_K C \rightarrow 0$$

is exact of abelian partial groups. Analogous statement holds for right $K-$ partial modules A, B and $C,$ and left $K-$ partial module $D.$

Proof. For each $e \in E_D,$ we have

$$\left(D \otimes_K A\right)_{e \otimes \sigma_{AD}(e)} = D_e \otimes_{K_{u_e}} A_{\sigma_{AD}(e)},$$

$$\left(D \otimes_K B\right)_{e \otimes \sigma_{BD}(e)} = D_e \otimes_{K_{u_e}} B_{\sigma_{BD}(e)},$$

$$\left(1_D \otimes \alpha\right)_{e \otimes \sigma_{AD}(e)} = 1_{D_e} \otimes \alpha_{\sigma_{AD}(e)},$$

etc. Thus, by using Lemma 4.2, [1], we have for every $e \in E_D,$ exact sequence of left $K_{u_e} -$ modules

$$A_{\sigma_{AD}(e)} \overset{\alpha_{\sigma_{AD}(e)}}{\rightarrow} B_{\sigma_{BD}(e)} \overset{\beta_{\sigma_{BD}(e)}}{\rightarrow} C_{\sigma_{CD}(e)} \rightarrow 0.$$

By using Proposition 5.4, Chapter 1V, [6], we have for every $e \in E_D,$ exact sequence of abelian partial groups

$$D_e \otimes_{K_{u_e}} A_{\sigma_{AD}(e)} \overset{1_{D_e} \otimes \alpha_{\sigma_{AD}(e)}}{\rightarrow} D_e \otimes_{K_{u_e}} B_{\sigma_{BD}(e)} \overset{1_{D_e} \otimes \beta_{\sigma_{BD}(e)}}{\rightarrow} D_e \otimes_{K_{u_e}} C_{\sigma_{CD}(e)} \rightarrow 0.$$

Again, by Lemma 4.2, [1], the sequence of abelian partial groups

$$D \otimes_K A \overset{1_D \otimes \alpha}{\rightarrow} D \otimes_K B \overset{1_D \otimes \beta}{\rightarrow} D \otimes_K C \rightarrow 0$$

is exact.

From the above theorem and Corollary 4.2 we can conclude that, if $\gamma : A_K \rightarrow A'_K$ and $\delta : {}_K B \rightarrow {}_K B'$ are partial module epimorphisms, so is $\gamma \otimes \delta : A \otimes_K B \rightarrow A' \otimes_K B'.$

Theorem 4.4 *Let T and S be partial rings and ${}_SA_T$, ${}_TB$ (bi) partial modules*

(i) $A\otimes_T B$ is a left $S-$ partial module such that $s(a\otimes b)=sa\otimes b^{'}$, for all $s\in S$ and $a\otimes b\in A\otimes_T B$, where $b^{'}=b+\left(\sigma_B^S\right)^{-1}(e_s)$,

(ii) If $\alpha : A\to A^{'}$ is a homomorphism of $S-T$ bipartial modules and $\beta : B\to B^{'}$ is a *T $-$partial module homomorphism, then the induced map $\alpha\otimes\beta : A\otimes_T B\to A^{'}\otimes_T B^{'}$ is a homomorphism of left S $-$partial modules.*

Dual statements hold for (bi) partial modules C_T, ${}_TD_S$.

Proof. (i) To avoid any confusion that may exist, we denote isomorphisms of semilattices as follows: $\sigma_A^T : E_A\to E_T, \sigma_A^S : E_A\to E_S, \sigma_{BA}^T=\left(\sigma_B^T\right)^{-1}\sigma_A^T$, etc. For each $e\in E_A$, let $v_e=\sigma_A^S(e)$ and $u_e=\sigma_A^T(e)$. The correspondence $v_e\mapsto u_e$ is an isomorphism of semilattices $E_S\to E_T$, namely $\sigma_A^T\left(\sigma_A^S\right)^{-1}$ which may be denoted by σ_A^{TS}. For any $e, f\in E_A, u_e\geq u_f$ iff $\sigma_A^T(e)\geq\sigma_A^T(f)$ iff $e\geq f$ iff $\sigma_A^S(e)\geq\sigma_A^S(f)$ iff $v_e\geq v_f$. Now for each $e\in E_A$ and $e^{'}=\sigma_{BA}^T(e)$ $\left(=\sigma_{BA}^S(e)\right)$, ${}_{S_{v_e}}(A_e)_{T_{u_e}}$ is $S_{v_e}-T_{u_e}$ bimodule, and $B_{e^{'}}$ is a left $T_{u_e}-$module. By Theorem 5.5 (i), Ch. 1V,[6], $A_e\otimes_{T_{u_e}} B_{e^{'}}$ is a left $S_{v_e}-$ module such that $s(a\otimes b)=sa\otimes b$ for all $s\in S_{v_e}, a\in A_e, b\in B_{e^{'}}$. Since A is a left $S-$ partial module, A_e is a left S_v- module for every $v\in E_S$ with $v\geq\sigma_A^S(e)=v_e$. (Lemma 2.3 (i), [1]). Let $\left(e\otimes e^{'}\right)\geq\left(f\otimes f^{'}\right)$ in $E_{A\otimes_T B}$. Then

$$\phi_{e\otimes e^{'},f\otimes f^{'}} : A_e\otimes_{T_{u_e}} B_{e^{'}}\to A_f\otimes_{T_{u_f}} B_{f^{'}}$$

is a homomorphism of abelian groups. Both $A_e\otimes_{T_{u_e}} B_{e^{'}}$, and $A_f\otimes_{T_{u_f}} B_{f^{'}}$ are left $S_{v_e}-$ modules, where $v_e=\sigma_A^S(e)$. Let $s\in S_{v_e}$ and $a\otimes b$ a generator in $A_e\otimes_{T_{u_e}} B_{e^{'}}$. Then

$$\begin{aligned}
\phi_{e\otimes e^{'},f\otimes f^{'}}\left(s(a\otimes b)\right) &= \phi_{e\otimes e^{'},f\otimes f^{'}}(sa\otimes b)\\
&= (sa+f)\otimes\left(b+f^{'}\right)=(sa+sf)\otimes\left(b+f^{'}\right)\\
&= s(a+f)\otimes\left(b+f^{'}\right)\ (\text{since } A \text{ is a left } S-\text{partial module})\\
&= s\left((a+f)\otimes\left(b+f^{'}\right)\right)\\
&= s\phi_{e\otimes e^{'},f\otimes f^{'}}(a\otimes b).
\end{aligned}$$

Hence $\phi_{e\otimes e^{'},f\otimes f^{'}}$ is a left $S_{v_e}-$ module homomorphism and (i) follows.

(ii) By Theorem 5.5 (ii) , Ch. 1V,[6], we have for each $e\in E_A, e^{'}=\sigma_{BA}^T(e)$ a homomorphism of left $S_{v_e}-$ modules

$$(\alpha\otimes\beta)_{e\otimes e'} \quad : \quad A_e\otimes_{T_{u_e}} B_{e'} \to A'_{\sigma_{A'A}(e)}\otimes_{T_{u_e}} B'_{\sigma_{B'B}(e)},$$
$$a\otimes b \quad \mapsto \quad \alpha_e(a)\otimes\beta_{e'}(b)$$

where, as usual, $u_e=\sigma_A^T(e), v_e=\sigma_A^S(e)$. Let $s\in S$ and $a\otimes b\in A\otimes_T B$, say $s\in S_v$, for some $v\in E_S$ and $a\otimes b$ a generator in $A_e\otimes_{T_{u_e}} B_{e'}$, for some $e\in E_A$. Th

$$(\alpha\otimes\beta)(s(a\otimes b)) = (\alpha\otimes\beta)\cdot\left(\psi_{v,v+(\sigma_A^S)(e)}\ s\cdot\phi_{e\otimes e',e\otimes e'+(\sigma_A^S)^{-1}(v)\otimes(\sigma_B^S)^{-1}(v)}\ (a\otimes b)\right)$$
$$= (\alpha\otimes\beta)\left(s+(\sigma_A^S)(e)\right)\left(\left(a+(\sigma_A^S)^{-1}(v)\right)\otimes\left(b+(\sigma_B^S)^{-1}(v)\right)\right)$$
$$= \alpha_{e+(\sigma_A^S)^{-1}(v)}\left(\left(s+(\sigma_A^S)(e)\right)\left(a+(\sigma_A^S)^{-1}(v)\right)\otimes\beta_{e'+(\sigma_B^S)^{-1}(v)}\left(b+(\sigma_B^S)^{-1}(v)\right)\right)$$
$$= \left(s+(\sigma_A^S)(e)\right)\cdot\alpha_{e+(\sigma_B^S)^{-1}(v)}\left(a+(\sigma_A^S)^{-1}(v)\right)\otimes\beta_{e'+(\sigma_B^S)^{-1}(v)}\left(b+(\sigma_B^S)^{-1}(v)\right)$$
$$= \left(s+(\sigma_A^S)(e)\right)\cdot\left(\alpha_{e+(\sigma_B^S)^{-1}(v)}\ a+(\sigma_A^S)^{-1}(v)\otimes\beta_{e'+(\sigma_B^S)^{-1}(v)}\left(b+(\sigma_B^S)^{-1}(v)\right)\right)$$
$$= \left(s+(\sigma_A^S)(e)\right)\alpha_e(a)+\alpha_{(\sigma_A^S)^{-1}(v)}\left(\left(\sigma_A^S\right)^{-1}(v)\right)\otimes\beta_{e'}(b)+\beta_{e(\sigma_B^S)^{-1}(v)}\left(\left(\sigma_B^S\right)^{-1}(v)\right),$$

Lemma3.2(iii),[1]

$$= \left(s+(\sigma_A^S)(e)\right)\left(\alpha_e(a)+\left(\sigma_{A'}^S\right)^{-1}(v)\right)\otimes\left(\left(\beta(b)+\left(\sigma_{B'}^S\right)^{-1}(v)\right)\right)$$
$$= \psi_{v,v+(\sigma_A^S)(e)}\ s\cdot\phi_{\alpha(e)\otimes\beta(e'),(\alpha(e)\otimes\beta(e'))+(\sigma_{A'}^S)^{-1}(v)\otimes(\sigma_{B'}^S)^{-1}(v)}\cdot\left((\alpha\otimes\beta)(a\otimes b)\right)$$
$$= \psi_{v,v+(\sigma_A^S)(e)}\ s\cdot\phi_{\sigma_{A'A}^S(e)\otimes\sigma_{B'B}^S(e'),(\sigma_{A'A}^S(e)+(\sigma_{A'}^S)^{-1}(v))\otimes(\sigma_{B'B}^S(e')+(\sigma_{B'}^S)^{-1}(v))}\cdot\left((\alpha\otimes\beta)(a\otimes b)\right),$$

(Lemma3.1(i)),[1],(Corollary4.3)

$$= s\cdot(\alpha\otimes\beta)(a\otimes b).$$

This proves (ii)

If K is a commutative partial ring, a left $K-$ partial module A is then a right $K-$ partial module via the action $ar=ra,(r\in K, a\in A)$, whence A is called a $K-K$ (bi)partial module, or just a $K-$ partial module. A middle linear map $\eta: A\times B\to C$ of $K-$ partial modules, is then called bilinear, if it also satisfies

$$\eta(ra,b)=r\eta(a,b)=\eta(a,rb)\quad ((a,b)\in A\times B, r\in K_{\sigma_A(e_a)}).$$

A version of Theorem 4.2 can be easily obtained, by using Th.3.6, Ch.1V, [6], when K is commutative and $\lambda: A\times B\to C$ is bilinear. The technique and constructions developed so far allow to extend other familiar results. Here we give one more application.

Again , K is an arbitrary partial ring.

Theorem 4.5 *If* K *has an identity* 1, *and* $A_K, {}_K B$ *are* $K-$ *partial modules, there exist* $K-$ *partial module isomorphisms,*

$A\otimes_K K \cong A$ and $K\otimes_K B\cong B.$

Proof. By PMP3, Sec. 2, [1], $x1 = x$ (resp. $1y = y$) for all $x\in A$ (resp. for all $y\in B$). By Theorem 3.3, (A) implies (B), [3], $1+u$ is identity for the maximal subring K_u, for every $u\in E_K$. Thus, for each $e\in E_A, e' = \sigma_{BA}(e)$ and $u=\sigma_A(e)=\sigma_B(e')$, we have for every $b\in B_{e'}$,

$$\begin{aligned}(1+u)b &= 1b+ub = b+\sigma_B(e')b\\ &= b+\sigma_B(e')e' && \text{(Lemma2.1(i),[1])}\\ &= b+e' && \text{(PMP4,[1])}\\ &= b.\end{aligned}$$

Thus $B_{e'}$ is a unitary left K_u- module. Dually, A_e is a unitary right K_u- module $(u=\sigma_A(e)=\sigma_B(e'))$. By theorem 5.7, Ch.1V, [6], there exist K_u- module isomorphisms $\alpha_e : A_e\otimes_{K_u} K_u \to A_e$ and $\beta_{e'} : K_u\otimes_{K_u} B_{e'} \to B_{e'}$ for every $e\in E_A, e' = \sigma_{BA}(e), u=\sigma_A(e)$. Viewing as isomorphisms of abelian groups, it follows by Lemma 5.2, [4], that (α_e) and $(\beta_{e'})$ induce isomorphisms of partial groups

$$\alpha : A\otimes_K K\to A \text{ and } \beta : K\otimes_K B\to B$$

given by

$$\begin{aligned}\alpha(a\otimes r) &= \alpha_e(a\otimes r)\\ \beta(s\otimes b) &= \beta_f(s\otimes b)\end{aligned}$$

for $a\otimes r\in A\otimes_K K,$ say $a\in A_e, r\in K_u, \sigma_A(e)=u,$ and $b\in B_f$, $s\in K_v, \sigma_B(f)=v.$ We show that β is a left $K-$ partial module isomorphism the proof that α is a right $K-$ partial module isomorphism is similar, since β is isomorphism of partial groups, it is sufficient to show that it satisfies PMH2 and PMH3 (cf. Sec. 3, [1]). By its very definition, β satisfies PMH3. Let $s\in K$, say $s\in K_v$, for some $v\in E_K$, and let $r\otimes b\in K\otimes_K B$, say $r\in K_u, b\in B_f$, for some $f\in E_B, u=\sigma_B(f).$ Then

$$\beta\left(s\left(r\otimes b\right)\right) = \beta\left(\psi_{v,v+u}\, s\,\phi_{u\otimes f,\left(\left(u\otimes f\right)+\left(u+v\right)\otimes f+\sigma_B^{-1}(v)\right)}\left(r\otimes b\right)\right)$$

$$= \beta\left(\left(s+u\right)\left(r+v\right)\otimes\left(\left(b+\sigma_B^{-1}\left(v\right)\right)\right)\right)$$

$$= \beta_{f+\sigma_B^{-1}(v)}\left(\left(s+u\right)\left(r+v\right)\otimes\left(b+\sigma_B^{-1}\left(v\right)\right)\right) \text{ (Theorem4.4(i))}$$

$$= \left(s+u\right)\beta_{f+\sigma_B^{-1}(v)}\left(\left(r+v\right)\otimes\left(b+\sigma_B^{-1}\left(v\right)\right)\right)$$

$$= \left(s+u\right)\beta_{f+\sigma_B^{-1}(v)}\left(\left(r\otimes b\right)+\left(v\otimes\sigma_B^{-1}\left(v\right)\right)\right) \text{ (Corollary4.1)}$$

$$= \left(s+u\right)\left(\beta\left(r\otimes b\right)+\left(f+\sigma_B^{-1}\left(v\right)\right)\right)$$

$$= \psi_{v,v+u}\, s\,\phi_{f,f+\sigma_B^{-1}(v)}\left(\beta\left(r\otimes b\right)\right)$$

$$= s\,\beta\left(r\otimes b\right).$$

Hence PMH2 holds and the proof is complete.

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