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Generalized Modules over Semilattices of Rings

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Abstract

A partial module M over a partial ring K (strong semilattice of rings) is defined so as to obtain a structure theorem of strong sort. Formally, M is proved to be precisely a strong semilattice of modules over K. It is proved that in the category of partial modules over K, all products and coproducts exist. Some generalizations concerning exact sequences including a five Lemma, for partial modules are obtained.

Keywords: Partial modules, Exact sequences, Five Lemma

1.Introduction and preliminaries

In [2] a partial ring is defined and characterized as a strong semilattice of rings, embeddable in a partial ring of partial mappings. Analogous structure theorems and representations by partial mappings have been established, for partial groups in [1] and for partial monoids in [4]. Certain types of partial groups in which the structure maps are epimorphisms have been studied in [5] and [6], and referred to as q partial groups. In the present work, we proceed in the same direction and introduce the notion of a partial module over a partial ring, proving that a partial module M over a partial ring K (i.e. a strong semilattice of rings) is precisely a "strong semilattice of modules" over the maximal subrings of K. We then introduce the notion of a $K-$ partial module homomorphism between $K-$ partial modules, for a fixed partial ring K, proving that the category **PMod-K**, whose objects are all left $K-$ partial modules and whose morphisms are the $K-$ partial module homomorphisms has all products and coproducts (direct sums). We claim that many of the standard notions and constructions in Ring - Module theory can be introduced for partial rings and partial modules. In the present work, we are mainly concerned with the structure theorem of partial modules over a given partial ring, their categorical products, and exact sequences. In a subsequent paper we introduce and study other basic categorical properties of partial modules. In order for this paper to be self contained, we introduce from [1] and [2] the required definitions and results concerning partial groups and partial rings respectively. A *partial group* is a multiplicative semigroup S subject to the axioms:

1. for every $x \in S,$ there is a necessarily unique, element denoted e_x and called the *partial identity* of x such that, (i) $xe_x = e_x x = x,$ and (ii) if $yx = xy = x,$ for some $y \in S,$ then $e_x y = ye_x = e_x.$
2. for every $x \in S,$ there is a necessarily unique, element denoted x^{-1} (or $-x$ if S is commutative) and called the *partial inverse* of x such that, (i) $xx^{-1} = x^{-1}x = e_x,$ and (ii) $e_x x^{-1} = x^{-1}e_x = x^{-1}.$
3. The partial identity map, $x \mapsto e_x$ is a homomorphism, that is $e_{xy} = e_x e_y$, for all $x, y \in S.$ The partial inverse map, $x \mapsto x^{-1}$ is an anti homomorphism, that is $(xy)^{-1} = y^{-1}x^{-1},$ for all $x, y \in S.$

If S is a partial group, the set $\{e_x : x \in S\}$ of all partial identities in S is denoted by E_S, it is precisely the set of all idempotents in S. Every idempotent in S is its own partial identity, whence $e_{e_x} = e_x$ for all $x \in S,$ also $e_{(x^{-1})} = e_x$ for all $x \in S.$ E_S is central in $S,$ and is a semilattice with $e \geq f$ if and only if $ef = f$ (for all $e, f \in E_S$).

Theorem 1.1 *For a semigroup S, the following statements are equivalent*
(a) S is a partial group
(b) S is a Clifford semigroup, i.e. S is regular with central idempotents
(c) S is a semilattice of groups
(d) S is a strong semilattice of groups.

It follows that a partial group S is precisely the strong semilattice of groups $S = \rho[E_S; S_e, \phi_{e,f}]$ where S_e is the maximal subgroup of S with identity e, and for $e \geq f$ in $E_S, \varphi_{e,f} : S_e \to S_f$ is the homomorphism of groups given by $x \mapsto xf$, for all $x \in S_e.$

Notions such as a subpartial group, homomorphism (monomorphism, epimorphism, etc.) of partial groups are defined as usual. In particular. A subpartial group B of a partial group S is called wide (or full) if $E_S \subset B.$ If $f : S \to T$ is a homomorphism of partial groups, then $f(e_x) = e_{f(x)}$ and $f(x^{-1}) = (f(x))^{-1},$ for all $x \in S.$

A *generalized ring* is a set K with two binary operation, addition"+" and multilication "." subject to the axioms:

i. $\langle K, + \rangle$ is an abelian partial group, (ii) $\langle K, . \rangle$ is a semigroup, and (iii) left and right distributive laws hold.

ii. A generalized ring K is *commutative* if $\langle K,.\rangle$ is so, and said to have a necessarily unique *unit*, if $\langle K,.\rangle$ has a one, which is then the unit of K. A mapping $\lambda : K \to K'$, between partial rings, is a *homomorphism* if for all $r,s \in K$, (i) $\lambda(r+s) = \lambda(r)+\lambda(s)$, (ii) $\lambda(rs) = \lambda(r)\lambda(s)$ and

iii. $\lambda(1) = 1'$, whenever K and K' have units 1 and $1'$ respectively. Monomorphisms, epimorphisms and isomorphisms of generalized rings are defined as usual.

Proposition 1.1 *Let K be a generalized ring. For all $r,s \in K$, we have :*
(i) $e_r s = e_{rs} = r e_s$, (ii) $e_{rs} = e_r e_s$, and (iii) $(-r)s = -(rs) = r(-s)$.

In a generalized ring K, the set of all additive idempotents, i.e. the idempotents in $\langle K,+\rangle$, is denoted by E_K^+, whearse $E_K^\times$ denotes the set of all idempotents in $\langle K,.\rangle$. It follows that E_K^+ is precisely the set of all partial identities in the commutative partial group $\langle K,+\rangle$, when no confusion exists, E_K^+ will be denoted simply by E_K . It can be shown that an additive idempotent in K need not be a multiplicative idempotent, and hence that a generalized ring K need not be, even, a union of rings.

A partial ring is a generalized ring K in which $E_K \subset E_K^\times$.(i.e. every idempotent in $\langle K,+\rangle$ is an idempotent in $\langle K,.\rangle$).

The notion of a strong semilattice of algebras of a certain type has been introduced in literature and referred to as a plonka sum. As we are concerned with generalized rings, we formally define a strong semilattice of rings $K = \rho[Y; K_u, \Psi_{u,v}]$ to be a disjoint union of rings $K = \cup\{K_u : u \in Y\}$ indexed by a semilattice Y such that, for every $u,v \in Y$ with $u \geq v$, there exists a ring homomorphism $\Psi_{u,v} : K_u \to K_v$ such that
(i) $\Psi_{u,u}$ is the identical automorphism of K_u, for every $u \in Y$
(ii) $\Psi_{v,w} \cdot \Psi_{u,v} = \Psi_{u,w}$, for all $u,v,w \in Y$ with $u \geq v \geq w$.

There are two associative binary operations, addition and multiplication on K (extending the operations of each K_u) defined as follows:

For any $r,s \in K$, say $r \in K_u$ and $s \in K_v$, $r+s = (\Psi_{u,uv} r) + (\Psi_{v,uv} s)$, $rs = (\Psi_{u,uv} r) \cdot (\Psi_{v,uv} s)$.

Theorem 1.2 *Let $K = \langle K,+,.\rangle$ be a set with two binary operations, such that $\langle K,+\rangle$ is a semigroup and $\langle K,.\rangle$ is a monoid. The following two statements are equivalent:*
(A) K is a partial ring
(B) K is a strong semilattice of rings.

In the proof (A) implies (B) of the above theorem (cf.[2]), it is proved that $uv = u+v$, for all $u,v \in E_K$, . That is addition and multiplication in E_K coincide. The above theorem implies also that a partial ring K is precisely the strong semilattice of rings $K = \rho[E_K; K_u, \Psi_{u,v}]$, where K_u is the maximal subring of K with additive identity u (for every $u \in E_K$), and for $u \geq v$ in E_K, $\Psi_{u,v} : K_u \to K_v$ is ring homomorphism given by $r \mapsto r + v$ for every $r \in K_u$.

If a partial ring K has a multiplicative unit 1, then for every $u \in E_K$, K_u has a (multiplicative) unit $1 + u$, and these units are preserved by the structure maps $\Psi_{u,v}$. These properties of a partial ring will be used freely, when needed, without further references.

2.Partial modules over partial rings

Let $K = \rho[E_K; K_u, \Psi_{u,v}]$ be a partial ring. An abelian partial group $M = \rho[E_M; M_e, \varphi_{e,f}]$ is called a *left* $K-$*generalized module* if there is a function

$$K \times M \to M, (r, x) \mapsto rx$$

called scalar multiplication, such that the following axioms hold

PMP1 $r(x + y) = rx + ry, (r + s)x = rx + sx$ for all $r, s \in K$ and $x, y \in M$

PMP2 $(rs)x = r(sx)$, for all $r, s \in K$ and $x \in M$

PMP3 $1x = x$, for all $x \in M$ whenever K has a unit 1.

Right $K-$*generalized modules* may be defined analogously.

Throughout this section, unless sated otherwise, K denotes an arbitrary, but fixed, partial ring $K = \rho[E_K; K_u, \Psi_{u,v}]$.

Lemma 2.1 *Let* M *be a left* $K-$*generalized module. For all* $r \in K$ *and* $x \in M$, *we have*

(i) $e_{rx} = e_r x = re_x = e_r e_x$,

(ii) $(-r)x = -(rx) = r(-x)$.

Proof. (i) $rx + e_r x = (r + e_r)x = rx$, and if $y + rx = rx$, for some $y \in M$, then $e_r x + y = (-r + r)x + y = (-r)x + (rx + y) = (-r)x + (y + rx) = (-r)x + rx = (-r + r)x = e_r x$. Thus $e_{rx} = e_r x$. Similarly, we can show that $e_{rx} = re_x$. It follows immediately that $e_r x \in E_M$, and hence that $e_r x = e_{(e_r x)} = e_r e_x$. (ii) Follows from (i).

A l*eft* $K-$*partial module* M is a left $K-$generalized module that satisfies the following two additional axioms

PMP4 There is an isomorphism of semilattices $\sigma : E_M \to E_K$ such that $\sigma(e)e = e$, for all $e \in E_M$.

PMP5 $\sigma(ue) = u\sigma(e)$, for all $u \in E_K$ and $e \in E_M$.

Lemma 2.2 *Let M be a left $K-$partial module. Then*
(i) M_e is a left K_u -module for all $u \in E_K$ and $e \in E_M$ with $u \geq \sigma(e)$. In particular, M_e is a left $K_{\sigma(e)}$ module, for every $e \in E_M$.
(ii) $ue = \sigma^{-1}(u) + e$, (in particular $e \geq ue$), for all $u \in E_K$ and $e \in E_M$.

Proof. (i) Let $r \in K_u$ and $x \in M_e$. By using Lemma 2.1 and PMP4, we have $e_{rx} = e_r e_x = e_r e = ue = u(\sigma(e)e) = (u\sigma(e))e = (u + \sigma(e))e = \sigma(e)e = e.$ Thus $K_u M_e \subset M_e$, (ii) For arbitrary $u \in E_K$ and $e \in E_M$, we have by using PMP5, $\sigma(ue) = u\sigma(e) = u + \sigma(e)$. Thus $ue = \sigma^{-1}(u + \sigma(e)) = \sigma^{-1}(u) + e.$

A *strong Semilattice of left K_u -modules* is a disjoint union of abelian groups

$$M = \cup\{M_u : u \in E_K\}$$

such that

SSM1 M_u is a left K_u -module for every $u \in E_K$.

It follows immediately that for every $v \in E_K$, M_v is also a K_u -module for every $u \in E_K$, with $u \geq v$, with action of K_u on M_v given naturally by (cf. Lemma 2.3 below)

$$rx = (\psi_{u,v} r)x$$

for all $r \in K_u$ and $x \in M_v$.

SSM2 For any $u, v \in E_K$, with $u \geq v$, there is a K_u -module homomorphism

$$\phi_{u,v} : M_u \to M_v$$

such that

(i) $\phi_{u,u}$ is $1_{M_u} : M_u \to M_u, a \mapsto a$

(ii) $\phi_{v,w} \circ \phi_{u,v} = \phi_{u,w}$, whenever $u \geq v \geq w$ in E_K.

Let M be defined as above. As in the case of strong semilattices of groups, there is a commutative associative operation (also called addition) on M defined by the rule :

For $a, b \in M$, say $a \in M_u$ and $b \in M_v$, for some $u, v \in E_K$,

$$a + b = \phi_{u,uv}\, a + \phi_{v,uv}\, b.$$

There is an action of K on M extending that of K_u on M_u ($u \in E_K$) defined by the rule:

For any $r \in K$, $a \in M$, say $r \in K_u$ and $a \in M_v$, for some $u, v \in E_K$,
$r.a = \psi_{u,uv}\, r.\phi_{v,uv}\, a.$

Lemma 2.3 *Let*
$$M = \cup\{M_u : u \in E_K\}$$

be a disjoint union of left K_u *-modules. Then*
(i) For any $v \in E_K, M_v$ *is a left* K_u *-module for every* $u \in E_K$ *with* $u \geq v$*. The action of* K_u *on* M_v *is given in terms of the action of* K_v *on* M_v *as follows:*
For any $r \in K_u, a \in M_v$
$$ra = (\psi_{u,v}\, r) \cdot a$$

(ii) If M *is also a strong semilattice of left* K_u *-modules, i.e.* M *satisfies also the axiom SSM2, then the action defined as in (i) coincides with the action defined by the structure maps. Moreover, for any* $u, v, w, o \in E_K$*, with* $u \geq v \geq w$ *and* $o \geq v$*, we have for any* $r \in K_u$ *and* $a \in M_o$*,*
$$\phi_{v,w}(\psi_{u,v}\, r \cdot \phi_{o,v}\, a) = \psi_{u,w}\, r \cdot \phi_{o,w}\, a,$$

and hence,(by using (i))
$$\phi_{v,w}\, (r \cdot \phi_{o,v}\, a) = r \cdot a.$$

Proof. (i) Let $s, r \in K_u$ and $a, b \in M_v$. Then
$$\begin{aligned}(s+r)a &= \psi_{u,v}\,(s+r) \cdot a = (\psi_{u,v}\, s + \psi_{u,v}\, r) \cdot a \\ &= \psi_{u,v}\, s \cdot a + \psi_{u,v}\, r \cdot a = sa + ra. \\ r(a+b) &= (\psi_{u,v}\, r)(a+b) = \psi_{u,v}\, r \cdot a + \psi_{u,v}\, r \cdot b \\ &= ra + rb. \\ (sr) \cdot a &= (\psi_{u,v}\,(sr)) \cdot a = ((\psi_{u,v}\, s)(\psi_{u,v}\, r)) \cdot a \\ &= (\psi_{u,v}\, s)(\psi_{u,v}\, r \cdot a) = s(ra).\end{aligned}$$

(ii) Let $r \in K_u, a \in M_v$, where $u \geq v$ in E_K. Then
$$\psi_{u,uv}\, r \cdot \phi_{v,uv}\, a = \psi_{u,v}\, r \cdot \phi_{v,v}\, a = \psi_{u,v}\, r \cdot a.$$

This proves the first statement of (ii). For the second statement, we have:
$$\begin{aligned}\phi_{v,w}(r \cdot \phi_{o,v}\, a) &= \phi_{v,w}(\psi_{u,v}\, r \cdot \phi_{o,v}\, a) \\ &= (\psi_{u,v}\, r) \cdot (\phi_{v,w}(\phi_{o,v}\, a)) \\ &= \psi_{u,v}\, r \cdot \phi_{o,w}\, a = \psi_{v,vw}(\psi_{u,v}\, r) \cdot \phi_{w,vw}(\phi_{o,w}\, a) \\ &= \psi_{u,w}\, r \cdot \phi_{o,w}\, a = r \cdot \phi_{o,w}\, a.\end{aligned}$$

Theorem 2.1 *Let* $M = \rho[E_M; M_e, \varphi_{e,f}]$ *be an abelian partial group with an operation* $K \times M \to M, (r,a) \mapsto ra.$

The following two statements are equivalent.
(A) M *is a left* $K-$ *partial module.*
(B) M *is a strong semilattice* $\rho[E_K; M_u, \varphi_{u,v}]$ *of left* K_u- *modules.*

Proof. **(A) implies (B)**. By assumption, M is a strong semilattice of abelian groups $\rho[E_M; M_e, \varphi_{e,f}]$ together with an isomorphism of semilattices $\sigma: E_M \to E_K$, and the axioms PMP1-PMP5 are satisfied. By Lemma 2.2, M_e is a left K_u- module for all $u \in E_K$ and $e \in E_M$ with $u \geq \sigma(e)$. Identifying every $e \in E_M$ with its unique image $\sigma(e) = u \in E_K$, then M may be viewed as a strong semilattice $\rho[E_K; M_u, \varphi_{u,v}]$ of left K_u- modules, with M_v is a left K_u- module for every $u,v \in E_K$ such that $u \geq v.$ For these $u,v \in E_K$, the homomorphism of abelian groups $\phi_{u,v}$ is actually given by

$$\phi_{u,v} = \phi_{\sigma^{-1}(u),\sigma^{-1}(v)} = \phi_{e,f} : M_e \to M_f, a \mapsto a + f,$$

where $e = \sigma^{-1}(u)$ and $f = \sigma^{-1}(v).\phi_{u,v}$ is also a K_u- module homomorphism, for if $r \in K_u$ and $a \in M_u$, then

$$\begin{aligned}
\phi_{u,v}(ra) &= ra + \sigma^{-1}(v) = ra + \sigma^{-1}(u+v) \\
&= ra + \sigma^{-1}(uv) = ra + \sigma^{-1}(e_r \cdot v) \\
&= ra + \sigma^{-1}(e_r \sigma(f)) = ra + \sigma^{-1}(\sigma(e_r \cdot f)) && \text{(PMP5)} \\
&= ra + e_r \cdot f \\
&= ra + rf && \text{(Lemma2.1)} \\
&= r(a+f) = r\phi_{u,v}(a).
\end{aligned}$$

Thus M satisfies SSM1 and SSM2. It remains to show that the operation $K \times M \to M, (r,a) \mapsto ra$ coincides with the action given by the structure maps. Let $u,v \in E_K$ be arbitrary and let e, f be the unique elements in E_M such that $e = \sigma^{-1}(u)$ and $f = \sigma^{-1}(v)$. For any $r \in K_u$ and $a \in M_v$ (where $M_v = M_f$), we have

$$\begin{aligned}
\psi_{u,uv} r \cdot \phi_{u,uv} a &= (r+uv)(a + \sigma^{-1}(uv)) \\
&= ra + r\sigma^{-1}(uv) + (uv)a + (uv)\sigma^{-1}(uv) \\
&= ra + r(e+f) + u(va) + u(v(e+f)) \\
&= ra + re + rf + uf + u(ve+f)) \\
&= ra + e + uf + uf + u(ve+f)) && \text{(Lemma2.1)}
\end{aligned}$$

$= \quad ra + e + \sigma^{-1}(u) + f + u(\sigma^{-1}(v) + e + f)$ (Lemma2.2(ii))
$= \quad ra + e + e + f + u(f + e + f)$
$= \quad ra + e + f + u(e + f)$
$= \quad ra + e + f + e + uf$ (Lemma2.2(ii))
$= \quad ra + e + f + e + f$
$= \quad ra + e + f$
$= \quad ra$ (Lemma2.1andLemma2.2(ii))

(B) implies (A). We have a strong semilattice of left K_u modules
$M = \rho[E_K; M_u, \varphi_{u,v}]$.

By hypothesis, M is also a strong semilattice of abelian groups
$M = \rho[E_M; M_e, \varphi_{e,f}]$.

It follows that the correspondence
$\sigma : E_M \to E_K$

given by $\sigma(e) = u$ if and only if $M_e = M_u$ is an isomorphism of semilattices, that M_e is a left $K_{\sigma(e)}-$ module for all $e \in E_M$ and that $\phi_{e,f} = \phi_{u,v}$ if and only if $\sigma(e) = u$ and $\sigma(f) = v$. Let $r \in K_u, a \in M_e$ and $b \in M_f$, for some $u \in E_K$ and $e, f \in E_M$. Then $x + y \in M_{e+f} = M_{\sigma(e+f)}$ and so

$$r(x+y) = \psi_{u,u+\sigma(e+f)}\, r \cdot \phi_{(e+f),\sigma^{-1}(u)+(e+f)}(x+y)$$
$$= \psi_{u,u+\sigma(e+f)}\, r \cdot \phi_{e+f,\sigma^{-1}(u)+(e+f)}(\phi_{e,e+f}\, a + \phi_{f,e+f}\, b)$$
$$= \psi_{u,u+\sigma(e+f)}\, r \cdot (\phi_{e,\sigma^{-1}(u)+e+f}\, a + \phi_{f,\sigma^{-1}(u)+e+f}\, b)$$
$$= \psi_{u,u+\sigma(e+f)}\, r \cdot \phi_{e,\sigma^{-1}(u)+e+f}\, a + \psi_{u,u+\sigma(e+f)}\, r \cdot \phi_{f,\sigma^{-1}(u)+e+f}\, b$$

(since $M_{\sigma^{-1}(u)+e+f}$ is a $K_{u+\sigma(e+f)}-$module). On the other hand,

$$rx + ry = \psi_{u,u+\sigma(e)}\, r \cdot \phi_{e,\sigma^{-1}(u)+e}\, a + \psi_{u,u+\sigma(f)}\, r \cdot \phi_{f,\sigma^{-1}(u)+f}\, b$$
$$= \phi_{\sigma^{-1}(u)+e,\sigma^{-1}(u)+e+f}(\psi_{u,u+\sigma(e)}\, r \cdot \phi_{e,\sigma^{-1}(u)+e}\, a)$$
$$+ \phi_{\sigma^{-1}(u)+f,\sigma^{-1}(u)+e+f}(\psi_{u,u+\sigma(f)}\, r \cdot \phi_{f,\sigma^{-1}(u)+f}\, b)$$
$$= \psi_{u,u+\sigma(e+f))}\, r \cdot \phi_{e,\sigma^{-1}(u)+e+f}\, a + \psi_{u,u+\sigma(e+f))}\, r \cdot \phi_{f,\sigma^{-1}(u)+e+f}\, b \quad \text{(Lemma2.3(i))}.$$

Thus $r(x+y) = rx + ry$ which gives PMP1. Let $r \in K_u, s \in K_v$ and $a \in M_e$, for some $u, v \in E_K$ and $e \in E_M$. Then

$$\begin{aligned}(rs)a &= (\psi_{u,u+v}\, r \cdot \psi_{v,u+v}\, s)a \\ &= (\psi_{u+v,u+v+\sigma(e)}(\psi_{u,u+v}\, r \cdot \psi_{v,u+v}\, s))\phi_{e,\sigma^{-1}(u+v)+e}\, a \\ &= (\psi_{u,u+v+\sigma(e)}\, r \cdot \psi_{v,u+v+\sigma(e)}\, s)\phi_{e,\sigma^{-1}(u+v)+e}\, a \\ &= \psi_{u,u+v+\sigma(e)}\, r \cdot (\psi_{v,u+v+\sigma(e)}\, s \cdot \phi_{e,\sigma^{-1}(u+v)+e}\, a).\end{aligned}$$

On the other hand,

$$\begin{aligned}r(sa) &= r(\psi_{v,v+\sigma(e)}\, s \cdot \phi_{e,\sigma^{-1}(v)+e}\, a) \\ &= \psi_{u,u+v+\sigma(e)}\, r \cdot \phi_{\sigma^{-1}(v)+e,\sigma^{-1}(u+v)+e}(\psi_{v,v+\sigma(e)}\, s \cdot \phi_{e,\sigma^{-1}(v)+e}\, a) \\ &= \psi_{u,u+v+\sigma(e)}\, r(\psi_{v,u+v+\sigma(e)}\, s \cdot \phi_{e,\sigma^{-1}(u+v)+e}\, a). \qquad \text{(Lemma2.3(ii))}\end{aligned}$$

Therefore, $(rs)a = r(sa)$ and PMP2 follows. If K has a unit 1 then, for all $u \in E_K$, K_u has a unit $1+u$ (cf. Sec.1), and for any $r \in K$, $1r = r$ which gives $e_1 e_r = e_r$ (Proposition 1.1 (ii)), or equivalently $e_1 + e_r = e_r$. It follows that $e_1 \geq e_r$ for every $r \in K$. Thus e_1 is an upper bound for E_K. Hence $\sigma^{-1}(e_1)$ is an upper bound for E_M. Now for any $a \in M$, say $a \in M_e$, for some $e \in E_M$, we have

$$\psi_{e_1,e_1+\sigma(e)}\, 1 = 1 + e_1 + \sigma(e) = 1 + \sigma(e),$$

which is the unit of the ring $K_{\sigma(e)}$. Thus,

$$\begin{aligned}1a &= \psi_{e_1,e_1+\sigma(e)}\, 1 \cdot \varphi_{e,\sigma^{-1}(e)+e}\, a \\ &= \psi_{e_1,\sigma(e)}\, 1 \cdot \varphi_{e,e}\, a \\ &= (1+\sigma(e))a = a.\end{aligned}$$

(Since M_e is a $K_{\sigma(e)}$ - module)

This proves PMP3. Axiom PMP4 follows immediately, from the fact that M_e is a left $K_{\sigma(e)}$ – module for every $e \in E_M$. Let $u \in E_K$ and $e \in E_M$. We have

$$\begin{aligned}ue &= \psi_{u,u+\sigma(e)}\, u \cdot \varphi_{e,\sigma^{-1}(u)+e}\, e \\ &= (u+\sigma(e)) \cdot (\sigma^{-1}(u)+e) \\ &= \sigma^{-1}(u)+e\end{aligned}$$

(Since $M_{\sigma^{-1}(u)+e}$ is a $K_{u+\sigma(e)}$ - module)

Therefore,

$\sigma(ue) = \sigma(\sigma^{-1}(u) + e) = u + \sigma(e) = u\sigma(e).$

Thus PMP5 follows and the proof is complete.

3.Categorical products of partial modules

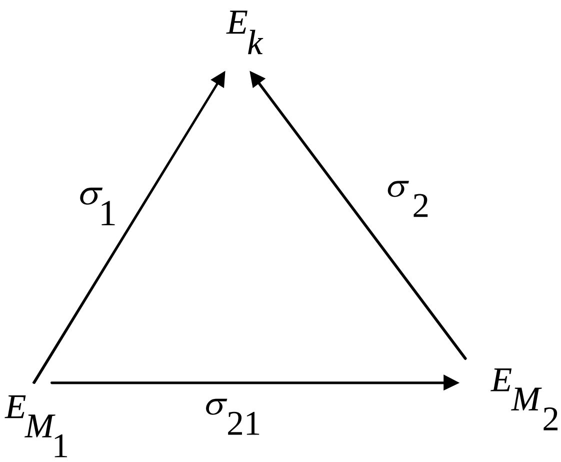
Let M_1 and M_2 be two left $K-$ partial modules. There exist two isomorphisms of semilattices:

$\sigma_1 : E_{M_1} \to E_K$ and $\sigma_2 : E_{M_2} \to E_K,$

which satisfy the axioms PMP4 and PMP5 (Sec.2). Thus there is an isomorphism of semilattices

$\sigma_2^{-1} \circ \sigma_1 : E_{M_1} \to E_{M_2},$

which we denote by σ_{21}, making the following diagram



commutative. To each $e \in E_{M_1}$, there corresponds a unique element in E_{M_2}, which we denote by e', such that $\sigma_1(e) = \sigma_2(e')$ namely,

$e' = \sigma_{21}(e) = \sigma_2^{-1}\sigma_1(e).$

Whence, by Lemma 2.1, for every $u \in E_K$ and $e \in E_{M_1}$, we have $(M_1)_e$ is a K_u- module if and only if $(M_2)_{e'}$ is a K_u- module. In particular, $(M_2)_{e'}$ is a $K_{\sigma_1(e)}-$ module for every $e \in E_{M_1}$.

We call a function

$\alpha : M_1 \to M_2$

a (left) $K-$ partial module homomorphism, if the following three axioms hold,
PMH1 $\alpha(a+b)=\alpha(a)+\alpha(b),$ *for all* $a,b\in M_1.$
PMH2 $\alpha(ra)=r\alpha(a),$ *for all* $r\in K$ *and* $a\in M_1.$
PMH3 $\sigma_1(e)\le\sigma_2(\alpha(e))$, *for all* $e\in E_{M_1}.$

Since α is neccessarily a partial group homomorphism, (PMH1) we have for any $a\in M_1, \alpha(e_a)=e_{\alpha(a)}$ and $\alpha(-a)=-\alpha(a),$ (cf. Sec1). The first of these two properties indicates that α maps E_{M_1} into E_{M_2}.

The homomorphism α is called, a monomorphism if α is one-to- one, an epimorphism if $Im\alpha=\{\alpha(a):a\in M\}=N,$ and an isomorphism, in which case M and N are called isomorphic, if α is both a monomorphism and epimorphism. In the following two lemmas, M_1 and M_2 denote arbitrary left $K-$ partial modules and $\alpha:M_1\to M_2$ is a $K-$ partial module homomorphism.

Lemma 3.1 *(i) For every* $e\in E_{M_1},$ $\sigma_1(e)=\sigma_2(\alpha(e)),$ *that is*

$$\alpha(e)=\sigma_2^{-1}\sigma_1(e)=\sigma_{21}(e)=e'.$$

(ii) The restriction

$$\alpha\Big|_{E_{M_1}}:E_{M_1}\to E_{M_2}$$

is an isomorphism of semilattices that sends each $e\in E_{M_1}$ *to* $e'=\sigma_{21}(e).$

Proof. Since both σ_1 and σ_2 are isomorphisms (ii) follows from (i). To prove (i), let $e\in E_{M_1}$. Then

$$\begin{aligned}\sigma_2(\alpha(e) &= \sigma_2(\alpha(\sigma_1(e)\cdot e)) \qquad \text{(PMP4)}\\ &= \sigma_2(\sigma_1(e)\cdot\alpha(e))\\ &= \sigma_1(e)\cdot\sigma_2(\alpha(e)) \qquad \text{(PMP5)}\\ &= \sigma_1(e)+\sigma_2(\alpha(e)).\end{aligned}$$

Thus $\sigma_2(\alpha(e))\le\sigma_1(e),$ and (i) follows by using PMH3.

Lemma 3.2 *(i)* α *is uniquely determined by a (pairwise disjoint) family* $(\alpha_e)_{e\in E_M}$ *of* $K_{\sigma(e)}-$*left module homomorphisms* $\alpha_e:(M_1)_e\to(M_2)_{e'},(e\in E_{M_1})$ *in the sense that, for any* $a\in M,\alpha(a)=\alpha_e(a),$ *where* e *is the unique element in* E_{M_1} *such that* $e=e_a.$ *In other words,* $\alpha(a)=\alpha_{e_a}(a),$ *for every* $a\in M_1.$
(ii) For any $e, f\in E_M$ *and* $a\in(M_1)_e,$

$\phi_{e^{'},e^{'}+f^{'}}(\alpha_e(a)) = \alpha_{e+f}\ (\phi_{e,e+f}\ a).$

In particular, if $e \geq f$, *then*
$\phi_{e^{'},f^{'}}\ \alpha_e = \alpha_f\ \phi_{e,f}.$

(iii) For any $e, f \in E_M$ *and* $a \in (M_1)_e$ *and* $b \in (M_1)_f$
$\alpha(a+b) = \alpha_{e+f}(a+b) = \alpha_e(a) + \alpha_f(b).$

Proof. (i) For every $e \in E_{M_1}$, let α_e be the restriction of α on the left $K_{\sigma(e)}-$module $(M_1)_e$. The result follows by using Lemma 3.1.

(ii) $\varphi_{e^{'},e^{'}+f^{'}}(\alpha_e(a)) = \alpha_e(a) + e^{'} + f^{'} = \alpha(a) + \alpha(e+f)$
$= \alpha(a+e+f) = \alpha_{e+f}\ (a+e+f)$
$= \alpha_{e+f}\ (\varphi_{e,e+f}\ (a)).$

(iii) Follows by using (i) above and PMH1.

Lemma 3.3 *There is a category, denoted by **PMod-K**, whose objects are all left* $K-$ *partial modules and whose morphisms are all left* $K-$ *partial module homomorphisms.*

Proof. If $L \xrightarrow{\delta} M \xrightarrow{\eta} N$ is a pair of left $K-$ partial module homomorphisms, then for every $e \in E_L$ we have

$$\begin{aligned}\sigma_N(\eta\,\delta(e)) &= \sigma_N(\eta\,(\delta(e)) \\ &= \sigma_N(\eta\,(\sigma_M^{-1}\sigma_L(e)) \\ &= \sigma_N(\sigma_N^{-1}\,\sigma_M\,(\sigma_M^{-1}\,\sigma_L(e))) \\ &= \sigma_L(e).\end{aligned}$$

Whence $\eta\,\delta$ is a left $K-$ partial module homomorphism, and the result follows immediately.

Theorem 3.1 *The category PMod-*K *has*
(i) all products,
(ii) all coproducts.

Proof. Let $\{M_i\}_{i\in I}$ be a family of left $K-$ partial modules indexed by a nonempty set I. For each $i \in I$, there is an isomorphism of semilattices $\sigma_i : E_{M_i} \rightarrow E_K$ that satisfies PMP4 and PMP5. As usual, we denote, for every $i, j \in I,$ the composite isomorphism $\sigma_j^{-1}\sigma_i : E_{M_i} \to E_{M_j},$ by σ_{ji}. For every $u \in E_K$ and every $i \in I,$ the element $\sigma_i^{-1}(u)$ in E_{M_i} is denoted by $u_i^{'},$ whence M_i is a strong semilattice of left $K_u -$ modules,

$$M_i = \rho[E_{M_i}; (M_i)_{u_i^{'}}, \phi_{u_i^{'}, v_i^{'}}]$$

where $(M_i)_{u_i^{'}}$ is the left $K_u -$ module with identity $u_i^{'},$ and for every $i, j \in I,$ we clearly have $\sigma_{ji}(u_i^{'}) = u_j^{'}$. For every $u \in E_K,$ let $M_{u^{'}}$ denote the cartesian product $\prod_{i\in I}(M_i)_{u_i^{'}}$ of the left $K_u -$ modules $(M_i)_{u_i^{'}}, i \in I.$ A typical element in $M_{u^{'}}$ is a collection $(a_{u_i^{'}})_{i\in I}$, with $a_{u_i^{'}} \in (M_i)_{u_i^{'}}$ for all $i \in I.$ We have two operations on $M_{u^{'}}$:

$$(a_{u_i^{'}})_{i\in I} + (b_{u_i^{'}})_{i\in I} = (c_{u_i^{'}})_{i\in I},$$

where $c_{u_i^{'}} = a_{u_i^{'}} + b_{u_i^{'}}$ $(i \in I)$ and

$$r(a_{u_i^{'}})_{i\in I} = (ra_{u_i^{'}})_{i\in I},$$

for any $r \in K_u$. These turn $M_{u^{'}}$ into a left $K_u -$ module with identity $u^{'} = (u_i^{'})_{i\in I}$. Actually, $M_{u^{'}}$ is the categorical product of the family $\left\{(M_i)_{u_i^{'}}, i \in I\right\}$ in the category $K_u -$**Mod** of left $K_u -$ modules. We have a collection of left $K_u -$ module homomorphisms (the universal canonical projections)

$$\pi_{u_j^{'}} : M_{u^{'}} \to (M_j)_{u_j^{'}} \quad , \left(a_{u_i^{'}}\right)_{i\in I} \mapsto a_{u_j^{'}} \qquad (j \in I).$$

Let

$$M = \bigcup_{u\in E_K} M_{u^{'}},$$

and let $E_M = \left\{u^{'} : u \in E_K\right\}$. Then E_M is a semilattice isomorphic to E_K by letting $u^{'} \geq v^{'}$ if and only if $u \geq v$ in E_K. In other words, we have an isomorphism of semilattices

$$\sigma_M : E_M \to E_K, u^{'} \mapsto u.$$

Clearly, $u \geq v$ in E_K if and only if $u^{'} \geq v^{'}$ in E_M if and only if $u_i^{'} \geq v_i^{'}$ in E_{M_i} for every $i \in I.$ Now for $u^{'} \geq v^{'}$ in E_M, define a map

$$\phi_{u',v'} : M_{u'} \to M_{v'}$$

by

$$\phi_{u',v'}(a_{u_i'})_{i\in I} = (\phi_{u_i',v_i'}\, a_{u_i'})_{i\in I}.$$

It is easy to see that $\phi_{u',v'}$ is a left K_u- module homomorphism and that $M = \rho[E_M;(M_{u'},\phi_{u',v'}]$ is a strong semilattice of left K_u- modules, or equivalently, a left $K-$ partial module with the isomorphism σ_M defined above. For each $j \in I,$ we define, the canonical projection,

$$\pi_j : M \to M_j$$

as follows: For any element a in $M = \underset{u\in E_K}{\cup} M_{u'},$ say $a = (a_{u_i'})_{i\in I} \in M_{u'} = \prod_{i\in I}(M_i)_{u_i'}$ (for some $u' \in E_M$), let

$$\pi_j(a) = \pi_{u_j'}(a_{u_i'})_{i\in I} = a_{u_j'} \in (M_j)_{u_j'}.$$

Thus, $(\pi_j)_{u'} = \pi_{u_j'}$ (foe every $u' \in E_M$). We can easily see that π_j satisfies PMH1 and PMH2. Also, PMH3 is satisfied, since

$$\sigma_M(u') = u = \sigma_j(u_j') = \sigma_j(\pi_j((u_i')_{i\in I}) = \sigma_j\pi_j(u')$$

for all $u' \in E_M$. Now let N be a left $K-$ partial module with isomorphism, $\sigma_N : E_N \to E_K$ of semilattices and let $(\alpha_i)_{i\in I}$ be a collection of left $K-$ partial module homomorphisms

$$\alpha_i : N \to M_i.$$

Define

$$\alpha : N \to M$$

as follows: For any $b \in N,$ say $b \in N_{\sigma_N^{-1}(u)},$ for some $u \in E_K,$ let $\alpha(b) = (\alpha_i(b))_{i\in I}.$ This definition implies clearly that α satisfies PMH1 and PMH2. Let $e \in E_N$ and let u be the unique element in E_K such that $e = \sigma_N^{-1}(u).$ We have

$$\begin{aligned}\alpha(e) &= \alpha(\sigma_N^{-1}(u)) = (\alpha_i(\sigma_N^{-1}(u)))_{i\in I}\\ &= (\sigma_i^{-1}\sigma_N(\sigma_N^{-1}(u)))_{i\in I} \qquad \text{(by Lemma3.1)}\\ &= (\sigma_i^{-1}(u))_{i\in I} = (u_i')_{i\in I} = u'\\ &= \sigma_M^{-1}(u) = \sigma_M^{-1}\sigma_N(\sigma_N^{-1}(u)) = \sigma_M^{-1}\sigma_N(e).\end{aligned}$$

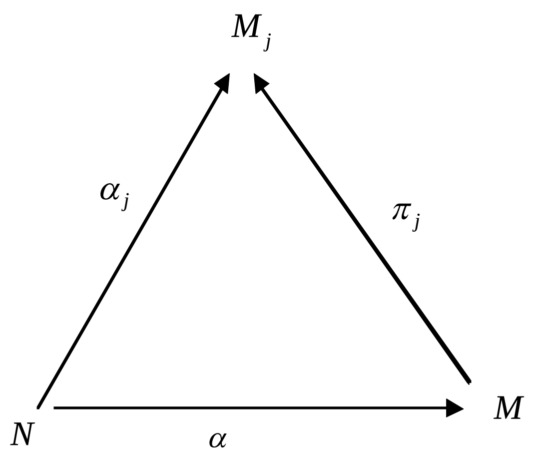
Hence $\sigma_N(e) = \sigma_M(\alpha(e))$ and PMH3 holds. It follows that α is a left $K-$ partial module homomorphism and that

$$(\alpha)_{\sigma_N^{-1}(u)} = ((\alpha_i)_{\sigma_N^{-1}(u)})_{i\in I}.$$

Let $j \in I, a \in N$ and let u be the unique element in E_K such that $\sigma_N(e_a) = u.$ Then

$$(\pi_j\alpha)(a) = \pi_j((\alpha_i(a))_{i\in I}) = \alpha_j(a)$$

and hence $\pi_j\alpha = \alpha_j$, (for every $j \in I$), that is the diagram



commutes for all $j \in I$. Equivalently (by Lemma 3.2), we have

$$(\pi_j\alpha)_{\sigma_N^{-1}(u)} = (\alpha_j)_{\sigma_N^{-1}(u)} \qquad \text{(for every } u \in E_K)$$

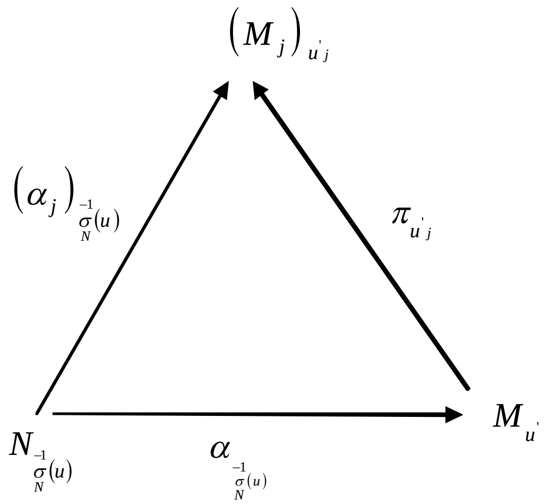
that is

$$(\pi_j)_{u'} \circ \alpha_{\sigma_N^{-1}(u)} = (\alpha_j)_{\sigma_N^{-1}(u)}$$

i.e.

$$\pi_{u_j'} \circ \alpha_{\sigma_N^{-1}(u)} = (\alpha_j)_{\sigma_N^{-1}(u)} \qquad \text{(for every } u \in E_K).$$

That is the diagram



commutes for every $u \in E_K$ and $j \in I$. Suppose that $\beta : N \to M$ is also a left $K-$ partial module homomorphism such that

$\pi_j \circ \beta = \alpha_j$ for all $j \in I$

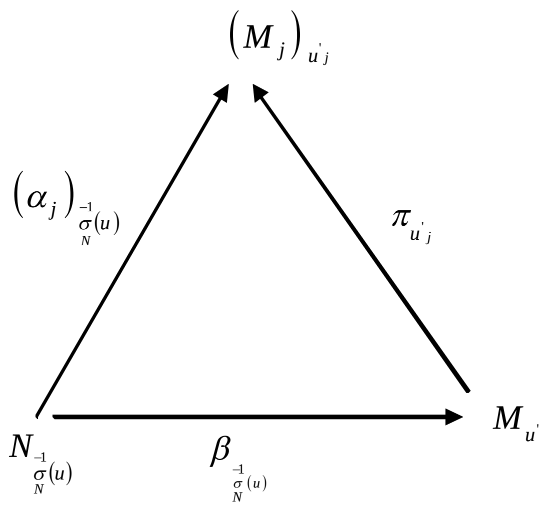
Again by Lemma 3.2, we must have

$(\pi_j \circ \beta)_{\sigma_N^{-1}(u)} = (\alpha_j)_{\sigma_N^{-1}(u)}$ (for every $u \in E_K$).

i.e.

$(\pi_{u'_j} \circ \beta)_{\sigma_N^{-1}(u)} = (\alpha_j)_{\sigma_N^{-1}(u)}$ (for every $u \in E_K$).

That is the diagram



commutes for every $u \in E_K$, and every $j \in I$. It follows by the universal property of the product $M_{u'} = \prod_{i \in I} (M_i)_{u_i'}$, that

$$\beta_{\sigma_N^{-1}(u)} = \alpha_{\sigma_N^{-1}(u)}, \qquad \text{(for every } u \in E_K\text{)}.$$

Whence, by Lemma 3.2, we must have $\beta = \alpha$. Thus $M = \bigcup_{u \in E_K} M_{u'}$, which may be written, $M = \prod_{i \in I} M_i$, is a product in **PMod-K**, and (i) is proved. For each $u \in E_K$, let

$$S_{u'} = \sum_{i \in I} (M_i)_{u_i'},$$

be the (usual) direct sum of the family of left K_u- modules $\left\{(M_i)_{u_i'} : i \in I\right\}$. Observe that $u_i' = \sigma_i^{-1}(u)$ is the zero of the left K_u- module $(M_i)_{u_i'}$. $S_{u'}$ is naturally a left K_u- module that satisfies the usual universal property. Let $E_S = E_M = \left\{u' : u \in E_K\right\}$ and let $\sigma_S : E_S \to E_K$ be the isomorphism defined by $\sigma_S = \sigma_M$. Thus, $u \geq v$ in E_K if and only if $u' \geq v'$ in E_S if and only if $u_i' \geq v_i'$ in E_{M_i} for every $i \in I$. Clearly $S_{u'}$ is a K_u- submodule of $M_{u'}$. For every $u \in E_K$ and if $u' \geq v'$ in E_S we have a K_u- module homomorphism

$$\phi_{u',v'} : M_{u'} \to M_{v'}.$$

Now $S_{u'} \subset M_{u'}$, and (by the definition of $\phi_{u',v'}$) we clearly have

$$\phi_{u',v'}(S_{u'}) \subset S_{v'}.$$

Thus the restriction of $\phi_{u',v'}$ on $S_{u'}$ is a left K_u- module homomorphism $S_{u'} \to S_{v'}$ which we also denote it (when no confusion exists) by $\phi_{u',v'}$. Setting

$$S = \bigcup_{u' \in E_S} S_{u'},$$

we clearly have a strong semilattice of left K_u- modules

$$S = \rho[E_S; S_{u'}, \phi_{u',v'}].$$

For each $u \in E_K$, we have a left K_u- module homomorphism

$$\iota_{u_j'} : (M_j)_{u_j'} \to S_{u'},$$

for every $j \in I,$ that satisfies the universal property of the direct sum $S_{u'}$. For each $j \in I$, we define the *canonical injection*

$$\iota_j : M_j \to S$$

as follows: Given $a \in M_j$, there is a unique $u \in E_K$ such that $a \in (M_j)_{u_j'}$, let $\iota_j(a) = \iota_{u_j'}(a)$. It is easy to see that ι_j satisfies PMH1 and PMH2. For PMH3, observe that $\iota_j(u_j') = \iota_{u_j'}(u_j') = u'$, for every $u \in E_K$, and so

$$\iota_j = \sigma_S^{-1} \circ \sigma_j.$$

Thus for every $j \in I, \iota_j$ is a left $K-$ partial module homomorphism. Finally, let B be any left $K-$ partial module and let

$$\beta_i : M_i \to B$$

be a left $K-$ partial module homomorphism, for every $i \in I$. Define

$$\beta : S \to B,$$

as follows: Given $a \in S,$ say $a = (a_{u_i'})_{i \in I} \in S_{u'} = \sum (M_i)_{u_i'}$, for some $u' = (u_i') \in E_S$, let

$$\beta(a) = \sum_{i \in I} (\beta_i)_{u_i'} (a_{u_i'}).$$

It is easy to see that β is a well-defined left $K-$ partial module homomorphism, that $\beta \circ \iota_i = \beta_i$ for all $i \in I$ and that β is unique with respect to this property. Thus $S = \bigcup_{u' \in E_S} S_{u'}$ is a coproduct of the family $\{M_i\}_{i \in I}$ in **PMod-K** and (ii) follows.

We will use, the more traditional, " direct sums" for " coproducts" in **PMod-K** and also write $\sum_{i \in I} M_i$ to denote (the) direct sum of objects $M_i, i \in I,$ in **PMod-K**.

In case I is a finite set with n elements, say $I = \{1,2,...,n\},$ the product and the direct sum of $M_1, M_2, ..., M_n$ coincide, and denoted by

$$M_1 \oplus M_2 \oplus ... \oplus M_n.$$

Examples of partial rings (cf. [2]) and partial modules may be constructed by using the corresponding structure theorems. But here we extend some special and simple cases. Observe first that if K is a partial ring, then K, viewing as an abelian partial group is a left $K-$ partial module with scalar multiplication the product in K. The following example extends the fact that every abelian group is a module over the ring of integers $\mathbb{Z}$.

Example 3.1 Let A be an abelian partial group, or equivalently, a strong semilattice of abelian groups

$$A = \rho[E; A_e, \phi_{e,f}].$$

For each e in the semilattice E, let $\mathbb{Z}_{e'}$ be a copy of the ring $\mathbb{Z}$ of integers with zero e' and let $i_{e'} : \mathbb{Z}_{e'} \to \mathbb{Z}$ be the natural isomorphism. There is a semilattice $E' = \{e' : e \in E\}$, with $e' \geq f'$ in E' if and only if $e \geq f$ in E, and

$$\sigma : E \to E', \quad e \mapsto e'$$

is a semilattice isomorphism. For each $e' \geq f'$ in E', let

$$\psi_{e',f'} : \mathbb{Z}_{e'} \to \mathbb{Z}_{f'}$$

be defined by

$$\psi_{e',f'} = i_{f'}^{-1} \circ i_{e'}.$$

For each $e \in E$, A_e is naturally a $\mathbb{Z}_{\sigma(e)}$ $(= \mathbb{Z}_{e'})-$ module and

$$\varphi_{e,f} : A_e \to A_f$$

is obviously a $\mathbb{Z}_{e'}-$ module homomorphism. Let $\mathbb{Z}_{E'}$ be the (disjoint) union

$$\mathbb{Z}_{E'} = \bigcup_{e' \in E'} \mathbb{Z}_{e'}.$$

We can easily observe that $\mathbb{Z}_{E'}$ is a strong semilattice of rings

$$\mathbb{Z}_{E'} = \rho[E'; \mathbb{Z}_{e'}, \psi_{e',f'}]$$

and that

$$A = \rho[E; A_e, \phi_{e,f}]$$

is a (left) $\mathbb{Z}_{E'}-$ partial module.

4.Exact sequences

Let M be a left $K-$ partial module. A subset B of M is called a *left* $K-$ *subpartial module* of M if with the induced operations from M, B is a left $K-$ partial module. In which case, B is a strong semilattice of left $K_{\sigma(e)}-$ modules

$$B = \rho[E_M; B_e, \phi_{e,f}],$$

and B_e is a left $K_{\sigma(e)}-$ submodule of M_e for every $e \in E_M$. Let M and N be left $K-$ partial modules, and let

$$\alpha : M \to N$$

be a left $K-$ partial module homomorphism. We define the kernel of α to be the set

$$\ker\alpha = \{a \in M : \alpha(a) \in E_N\},$$

and the image of α is denoted $Im\alpha$, as usual, that is

$$Im\alpha = \{\alpha(a) : a \in M\}.$$

Since

$$\alpha(a) = \alpha_{e_a}(a) \in N_{\sigma_N^{-1}\sigma_M}(e_a)$$
$$= N_{\sigma_N\sigma_M}(e_a),$$

it follows that, $a \in \ker\alpha$ if and only if $\alpha(a) = \sigma_{NM}(e_a)$ if and only if $a \in \ker\alpha_{e_a}$. Also, for any $c \in N$, $c \in Im\alpha$ if and only if $c \in Im\alpha_{\sigma_M^{-1}\sigma_N(e_c)}$. We have proved the following lemma.

Lemma 4.1 *Let M and N be left $K-$ partial modules, and let $\alpha : M \to N$ be a left $K-$ partial module homomorphism. Then*

(i) $\ker\alpha$ *is a left $K-$ subpartial module of M and*

$$\ker\alpha = \bigcup_{e\in E_M} \ker\alpha_e = \bigcup_{u\in E_K} \ker\alpha_{\sigma_M^{-1}(u)}$$

(ii) $Im\alpha$ is a left $K-$ subpartial module of N and

$$Im\alpha = \bigcup_{e\in E_M} Im\alpha_e = \bigcup_{u\in E_K} Im\alpha_{\sigma_M^{-1}(u)}$$

(iii) α is a monomorphism if and only if $\ker\alpha = E_M$, *if and only if α_e is a monomorphism for every $e \in E_M$*

(iv) α is an epimorphism (resp. isomorphism) if α_e is an epimorphism (resp. isomorphism) for every $e \in E_M$.

As defined in module theory, we call a pair of left $K-$ partial module homomorphisms

$$A \xrightarrow{\alpha} B \xrightarrow{\beta} C$$

exact at B if $Im\alpha = \ker\beta$. A finite sequence of $K-$ partial modules homomorphisms

$$A_0 \xrightarrow{\alpha_1} A_1 \xrightarrow{\alpha_2} A_2 \xrightarrow{\alpha_3} \dots \xrightarrow{\alpha_{n-1}} A_{n-1} \xrightarrow{\alpha_n} A_n$$

is *exact* if $Im\alpha_i = \ker\alpha_{i+1}$ for $i = 1,2,...,n-1$. An infinite sequence of $K-$ partial module homomorphisms

$$... \overset{\alpha_{i-1}}{\to} A_{i-1} \overset{\alpha_i}{\to} A_i \overset{\alpha_{i+1}}{\to} A_{i+1} \overset{\alpha_{i+2}}{\to} ...$$

is *exact* if $Im\alpha_i = \ker\alpha_{i+1}$ for all $i \in$.

We observe that, if

$$... \overset{\alpha_i}{\to} A_i \overset{\alpha_{i+1}}{\to} A_{i+1} \overset{\alpha_{i+2}}{\to} ...$$

is any sequence of left $K-$ partial modules, then for each $u \in E_K$, there exists a sequence of left K_u- modules

$$... \to (A_i)_{\sigma_i^{-1}(u)} \overset{(\alpha_{i+1})_{\sigma_i^{-1}(u)}}{\to} (A_{i+1})_{\sigma_{i+1}^{-1}(u)} \to ...$$

Lemma 4.2 *A finite, or infinite, sequence*

$$... \overset{\alpha_i}{\to} A_i \overset{\alpha_{i+1}}{\to} A_{i+1} \overset{\alpha_{i+2}}{\to} ...$$

of left $K-$ partial module homomorphisms is exact if and only if, the induced sequence of left K_u- module homomorphisms

$$... \overset{(\alpha_i)_{\sigma_{i-1}^{-1}(u)}}{\to} (A_i)_{\sigma_i^{-1}(u)} \overset{(\alpha_{i+1})_{\sigma_i^{-1}(u)}}{\to} (A_{i+1})_{\sigma_{i+1}^{-1}(u)} \overset{(\alpha_{i+2})_{\sigma_{i+1}^{-1}(u)}}{\to} ...$$

is exact for every $u \in E_K$.

Proof. Follows immediately, by applying Lemma 4.1.

Let E be any semilattice isomorphic to E_K. Then, clearly, E is a left $K-$ partial module. For each $e \in E$, E_e is the zero left $K_{\sigma(e)}-$ module $\{e\}$. It follows that there exists (up to an isomorphism) a unique left $K-$ partial module isomorphic to E_K. We call this $K-$ partial module the zero (left) $K-$ partial module and denote it (as usual) by 0. For any left $K-$ partial module A, we then have unique left $K-$ partial module homomorphisms

$0 \to A$ and $A \to 0$.

It follows that a sequence of left $K-$ partial module homomorphisms

$$0 \to A \overset{\alpha}{\to} B \overset{\beta}{\to} C \to 0$$

is exact if and only if α is a monomorphism, β is an epimorphism, and $Im\alpha = \ker\beta$. Many of the results concerning exact sequences of modules over rings may be extended to partial modules. In the rest of this paper we give extensions of some results concerning exact sequences of module homomorphisms.

Example 4.1 *Let A and B be any pair of left $K-$ partial modules. The direct sum sequences*

$$0 \to A \overset{i}{\to} A \oplus B \overset{\pi}{\to} B \to 0$$

and

$$0 \to B \overset{i}{\to} A \oplus B \overset{\pi}{\to} A \to 0$$

are (short) exact, where the " $i^{,s}$ and $\pi^{,s}$ " are the canonical injections and projections respectively.

Example 4.2 *Quotients partial groups in terms of normal subpartial groups, or equivalently idempotent separating congruences have been studied in [3]. If A is a partial group and N a normal subpartial group of A, there is a quotient partial group A/N which is a strong semilattice of quotient groups*

$$A/N = \rho[E_{A/N}; A_e/N_e, \phi_{e,f}]$$

where $\varphi_{e,f}$ is defined in terms of the structure maps of A, namely,

$$\phi_{e,f} : A_e/N_e \to A_f/N_f, aN_e \mapsto (af)N_e,$$

for all $a \in A_e$ and all $e, f \in E_A$ such that $e \geq f$. There exists a canonical epimorphism

$$p: A \to A/N, a \mapsto aN_{e_a}.$$

If A is a left $K-$ partial module and C is a $K-$ subpartial module of A, then C is a normal subpartial group of the additive partial group A and there is an additive quotient partial group A/C. Let $r \in K$, and $a \in A$, say $r \in K_u$ and $a \in A_e$, for some $u \in E_K$ and $e \in E_A$. We define the action of r on the element $a + C_e$ of A/C as follows:

$$r(a + C_e) = ra + C_e + \sigma_A^{-1}(u).$$

This turns A/C into a left $K-$ partial module with isomorphism $\sigma_{A/C}$ of semilattices

$$\sigma_{A/C} : E_{A/C} \to E_K,$$

given by

$$\sigma_{A/C}(C_e)=\sigma_{A/C}(e+C_e)=\sigma_A(e)\text{, forevery } e\in E_A,$$

or equivalently for every $C_e \in E_{A/C}$. The canonical epimorphism

$$p: A \to A/C, a \mapsto a + C_{e_a}$$

can be easily shown to be a left $K-$ partial module homomorphism.

We conclude that the sequence

$$0 \to C \overset{i}{\to} A \overset{p}{\to} A/C \to 0$$

is exact, where i is the inclusion map.

Lemma 4.3 *(The short Five Lemma for partial modules). Let*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \overset{g}{\longrightarrow} & B & \overset{h}{\longrightarrow} & C & \longrightarrow & 0 \\ & & \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma & & \\ 0 & \longrightarrow & A' & \overset{g'}{\longrightarrow} & B & \overset{h'}{\longrightarrow} & C' & \longrightarrow & 0 \end{array}$$

be a commutative diagram of left $K-$ partial modules and $K-$ partial module homomorphisms such that each row is a short exact sequence. Then

(i) α,γ monomorphisms $\Rightarrow \beta$ is a monomorphism;

(ii) α,γ epimorphisms $\Rightarrow \beta$ is an epimorphism;

(iii) α,γ isomorphisms $\Rightarrow \beta$ is an isomorphism.

Proof. We have shown in Lemma 3.3 that if

$$L \overset{\delta}{\to} M \overset{\eta}{\to} N$$

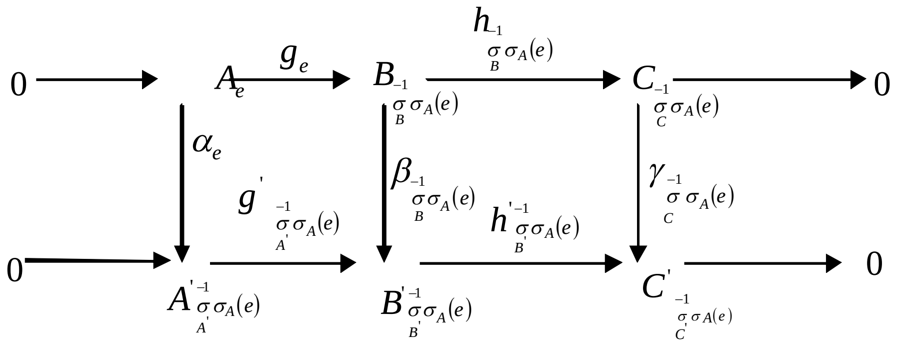
is a pair of left $K-$ partial module homomorphisms, then the composition $\eta\,\delta : L \to N$ is also a left $K-$ partial module homomorphism. Thus, by Lemma 3.2, we have for such a pair

$$\eta\,\delta(e) = \eta(\delta(e)) = \eta(\delta_e(e)) = (\eta_{\delta_e(e)}\,\delta_e)(e)$$

for each $e \in E_L$. Therefore,

$$(\eta\,\delta) = \eta_{\delta_e(e)}\,\delta_e = \eta_{\sigma_M^{-1}\sigma_L(e)}\delta_e$$

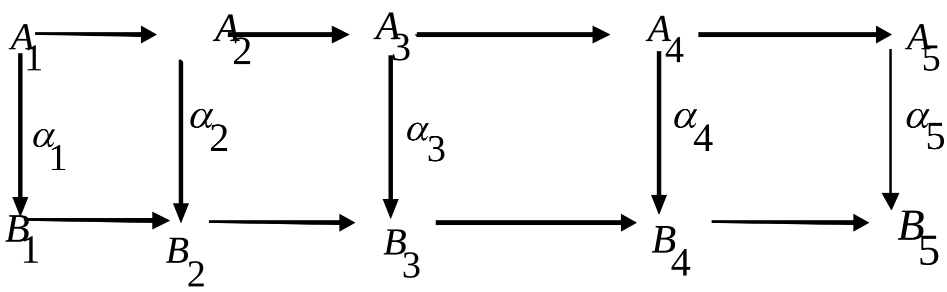
for every $e \in E_L$. Applying this result for the given diagram, we obtain for each $e \in E_A$, a commutative diagram



of left $K_u -$ module and $K_u -$ module homomorphisms, where $u = \sigma_A(e)$. The result now follows by applying Lemma 4.2, Lemma 4.1 and the short five Lemma for modules and module homomorphisms ([8], IV, Lemma 1.17).

By the technique used in the proof of the above lemma, and the five Lemma for modules we obtain the following Lemma.

Lemma 4.4 *(The Five Lemma for partial modules). Let*



be a commutative diagram of left $K-$ partial modules and $K-$ partial module homomorphisms, with exact rows.Then

(a) α_1 an epimorphism and α_2, α_4 monomorphisms $\Rightarrow \alpha_3$ is a monomorphism;

(b) α_5 a monomorphism and α_2, α_4 epimorphisms $\Rightarrow$ α_3 is an epimorphism.

The short Five Lemma for module theory is an important tool in algebra and algebraic topology. Here we use the partial module analogue, Lemma 4.3, to extend a known result in module theory concerning split exact sequences ([8],IV Theorem 1.18). We begin by extending some definitions.

Two short exact sequences of left $K-$ partial modules and $K-$ partial module homomorphisms are called *isomorphic* if there is a commutative diagram of $K-$ partial module homomorphisms

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow f & & \downarrow g & & \downarrow h & & \\ 0 & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & 0 \end{array}$$

such that $f, g,$ and h are isomorphisms.

A short exact sequence of left $K-$ partial modules and $K-$ partial module homomorphisms is *split* or a *split exact sequence* if it satisfies the equivalent conditions of the following theorem

Theorem 4.1 *Let*

$$0 \to A_1 \xrightarrow{f} B \xrightarrow{g} A_2 \to 0$$

be a short exact sequence of left $K-$ partial module homomorphisms. Then the following conditions are equivalent

(i) There is a $K-$ partial module homomorphism $h: A_2 \to B,$ with $gh = 1_{A_2},$

(ii) There is a $K-$ partial module homomorphism $k: B \to A_1$ with $kf = 1_{A_1},$

(iii) The given sequence is isomorphic (with identity maps on A_1 and A_2) to the direct sum short exact sequence

$$0 \to A_1 \xrightarrow{i_1} A_1 \oplus A_2 \xrightarrow{\pi_2} A_2 \to 0$$

Proof. **(i) implies (iii)**. There is a left $K-$ partial module homomorphism $\phi: A_1 \oplus A_2 \to B$ that satisfies the universal property of the direct sum (Theorem 3.1 (ii)). Actually, φ is defined explicitely in terms of f and h as follows: For any $a \in A_1 \oplus A_2,$ say $a = (a_1, a_2) \in (A_1 \oplus A_2)_{\sigma^{-1}(u)} = (A_1)_{\sigma_{A_1}^{-1}(u)} \times (A_2)_{\sigma_{A_2}^{-1}(u)},$ for some $u \in E_K,$ where $\sigma: E_{(A_1 \oplus A_2)} \to E_K$ is the isomorphism

$$(\sigma_1^{-1}(u), \sigma_2^{-1}(u)) \mapsto u,$$

let

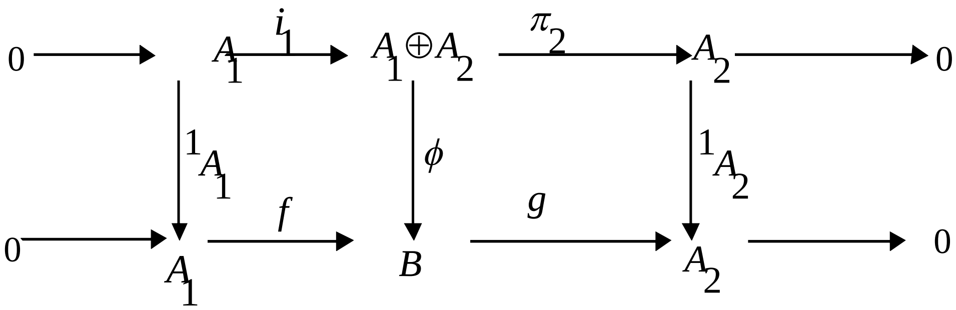
$$\varphi(a) = \varphi(a_1, a_2) = f_{\sigma_{A_1}^{-1}}(a_1) + h_{\sigma_{A_2}^{-1}}(a_2) \in B_{\sigma_B^{-1}(u)}.$$

φ is clearly well-defined, satisfies PMH1-PMH3, and $\varphi i_1 = f$. By exactness of the given sequence, we must have $gf = E_{A_2}$.

Thus for any $(a_1, a_2) \in A_1 \oplus A_2$, say $a_1 \in (A_1)_{\sigma_{A_1}^{-1}(u)}$, $a_2 \in (A_2)_{\sigma_{A_2}^{-1}(u)}$, for some $u \in E_K$, we have

$$\begin{aligned}(g\varphi)(a_1, a_2) &= g(f_{\sigma_{A_1}^{-1}(u)}(a_1) + h_{\sigma_{A_2}^{-1}(u)}(a_2)) \\ &= (gf)(a_1) + gh_{\sigma_{A_2}^{-1}(u)}(a_2) \\ &= \sigma_{A_2}^{-1}(u) + i_{A_2}(a_2) = a_2 = (1_{A_2}\pi_2)((a_1, a_2)).\end{aligned}$$

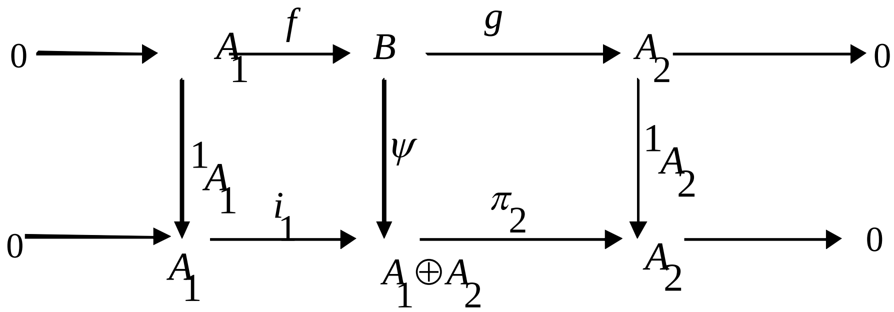
Therefore, the following diagram is commutative.



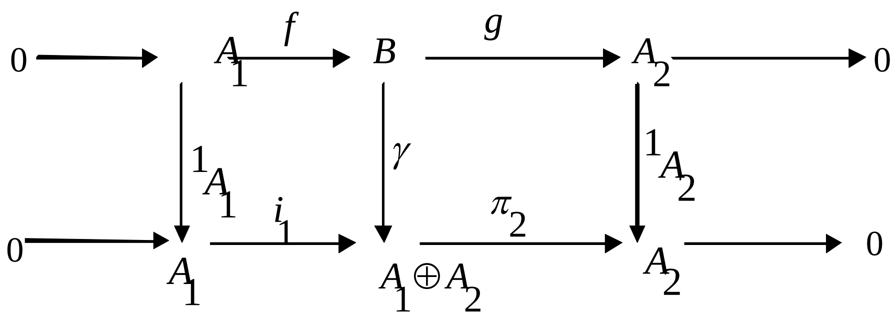
By the short Five Lemma for $K-$ partial modules (Lemma 4.3), φ is an isomorphism. **(ii) implies (iii)**. By Theorem 3.1 (i), there exists a left $K-$ partial module homomorphism $\psi : B \to A_1 \oplus A_2$ given by $b \mapsto (k(b), g(b))$, with $\pi_2\psi = g$ and for any $a \in A_1$

$$\begin{aligned}(\psi f)(a) &= \psi(f(a)) = (k\ f(a), g\ f(a)) \\ &= (1_{A_1}(a), \sigma_{A_2}^{-1}\sigma_{A_1}(e_a)) \quad \text{(by exactness)} \\ &= (a, \sigma_{A_2}^{-1}\sigma_{A_1}(e_a)) = (i_1 1_{A_1})(a).\end{aligned}$$

Therefore, the diagram



is commutative and (iii) follows by the short Five Lemma (4.3).**(iii) implies (i) and (ii)**. By hypothesis, there exists an isomorphism $\gamma : B \to A_1 \oplus A_2$ that makes the following diagram commutative



Define $h : A_2 \to B$ and $k : B \to A_1$ by

$$h = \gamma^{-1} i_2$$

and

$$k = \pi_1 \gamma.$$

By Lemma 3.3, h and k are left $K-$ partial module homomorphisms. By the definitions of the projections and injections maps and commutativity of the right square, we obtain

$$\begin{aligned} gh &= g\gamma^{-1} i_2 = (1_{A_2}^{-1} \pi_2 \gamma)\gamma^{-1} i_2 \\ &= 1_{A_2} \pi_2 i_2 = 1_{A_2} 1_{A_2} = 1_{A_2}. \end{aligned}$$

Similarly, $kf = 1_{A_1}$.

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