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Existence and nonexistence of positive weak solutions for a class of weighted (p, q)-Laplacian nonlinear system

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Abstract

In this article, we study the existence and nonexistence of positive weak solutions for the nonlinear system

$$\left.\begin{array}{rl} -\Delta_{P,p} u = \lambda a(x) u^{\gamma} v^{\beta} & \text{in } \Omega, \\ -\Delta_{Q,q} v = \lambda b(x) u^{\alpha} v^{\delta} & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega. \end{array}\right\}$$

where $\Delta_{R,r}$ with $r > 1$ and $R = R(x)$ is a weight function, denotes the weighted r-Laplacian defined by $\Delta_{R,r} u \equiv div[R(x)|\nabla u|^{r-2}\nabla u]$, λ is a positive parameter, $a(x)$ and $b(x)$ are bounded positive functions, $\gamma, \delta \geq 0, \alpha, \beta > 0, \alpha+\gamma < p-1, \beta+\delta < q-1$ and $\Omega \subset \Re^N$ is a bounded domain with smooth boundary $\partial\Omega$. We use the method of sub–supersolutions to establish ou results.

AMS subject classification: 35J60, 35B40.
Keywords: Positive weak solution, p-Laplacian, sub-supersolutions.

1. Introduction

Recently, many results concerning existence and nonexistence of positive weak solutions for nonlinear systems were obtained by various authors with the help of the sub-supersolutions method (see [2, 4, 7, 8]).

On the other hand, the existence and nonexistence of positive weak solutions for nonlinear systems involving weighted p-Laplacian operators have been studied by many

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authors using an approximation method [16], the theory of nonlinear monotone operators method (see [13, 15]) and the Browder theorem method (see [1, 9]).

Khafagy proved the existence, nonexistence and uniqueness of positive weak solution for the nonlinear system

$$\begin{cases} -\Delta_{P,p} u = f(\lambda, x, u) & \text{in } \Omega \\ u > k & \text{in } \Omega \\ u = k & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

when $f(\lambda, x, u) = \lambda a(x) u^{\alpha}$ with $0 < \alpha < p - 1$ in [12], and when $f(\lambda, x, u) = a(x)(\lambda u^{\alpha} + u^{\beta})$ with $0 < \beta \leq \alpha < p - 1$ in [11], using the sub-supersolutions method.

In [10], using the sub–supersolutions method, the author studied the existence and nonexistence of positive weak solutions for the weighted $(p, q)-$Laplacian nonlinear system

$$\left.\begin{array}{ll} -\Delta_{P,p} u = \lambda a(x) v^{\beta} & \text{in } \Omega, \\ -\Delta_{Q,q} v = \lambda b(x) u^{\alpha} & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{array}\right\} \tag{1.2}$$

where $0 < \alpha < p - 1, 0 < \beta < q - 1$.

In this work, we extend the system (1.2) to the following nonlinear system

$$\left.\begin{array}{ll} -\Delta_{P,p} u = \lambda a(x) u^{\gamma} v^{\beta} & \text{in } \Omega, \\ -\Delta_{Q,q} v = \lambda b(x) u^{\alpha} v^{\delta} & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega. \end{array}\right\} \tag{1.3}$$

where $\Delta_{R,r}$ with $1 < r = p, q$ and $R = R(x)$ is a weight function, $R(x) = P(x)$ when $r = p$ and $R(x) = Q(x)$ when $r = q$, denotes the weighted r-Laplacian defined by $\Delta_{R,r} u \equiv div[R(x)|\nabla u|^{r-2}\nabla u]$, λ is a positive parameter, $a(x)$ and $b(x)$ are bounded positive functions, $\alpha, \beta > 0, \gamma, \delta \geq 0, \alpha + \gamma < p - 1, \beta + \delta < q - 1$ and $\Omega \subset \Re^N$ is a bounded domain with smooth boundary $\partial\Omega$.

This paper is organized as follows:

In section 2, we introduce some technical results which are established in [6]. In section 3, we prove the existence of a positive weak solutions for system (1.3) by using the method of sub–supersolutions. In section 4, we consider the nonexistence results.

2. Technical Results

Now, we introduce some technical results to the weighted homogeneous eigenvalue problem (see [6])

$$\left.\begin{array}{ll} -\Delta_{R,r} u = div[R(x)|\nabla u|^{r-2}\nabla u] = \lambda S(x)|u|^{r-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{array}\right\} \tag{2.1}$$

with $r = p, q$ and $R(x) = P(x)$ when $r = p$ and $R(x) = Q(x)$ when $r = q$.

The function $R(x)$ is a weight function (measurable and positive a.e. in Ω), satisfying the conditions

$$R(x),\ (R(x))^{-\frac{1}{r-1}} \in L^1_{Loc}(\Omega),\ \text{ with } r > 1, (R(x))^{-s} \in L^1(\Omega),$$
$$\text{with } s \in \left(\frac{N}{r}, \infty\right) \cap \left[\frac{1}{r-1}, \infty\right), \tag{2.2}$$

and $S(x)$ is a measurable function satisfies

$$S(x) \in L^{\frac{k}{k-r}}(\Omega), \tag{2.3}$$

with some k satisfies $r < k < r_s^*$ where $r_s^* = \dfrac{Nr_s}{N - r_s}$ with $r_s = \dfrac{rs}{s+1} < r < r_s^*$ and *meas* $\{x \in \Omega : a(x) > 0\} > 0$. Examples of functions satisfying (2.2) are mentioned in [6].

Lemma 2.1. [6] There exists the first eigenvalue $\lambda_1^{(r)} > 0$ and at least one corresponding eigenfunction $\phi_{1,r} \geq 0$ a.e. in Ω of the eigenvalue problem (2.1).

Theorem 2.2. [6] Let $R(x)$ satisfies (2.2) and $S(x)$ satisfies (2.3), then (2.1) admits a positive eigenvalue $\lambda_1^{(r)}$. Moreover, it is characterized by

$$\lambda_1^{(r)} \int_\Omega S(x)|\phi_{1,r}|^r \leq \int_\Omega R(x)|\nabla\phi_{1,r}|^r. \tag{2.4}$$

Moreover, let us consider the weighted Sobolev space $W^{1,p}(P, \Omega)$ which is the set of all real valued functions u defined in Ω with the norm

$$\|u\|_{W^{1,p}(P,\Omega)} = \left(\int_\Omega |u|^p + \int_\Omega P(x)|\nabla u|^p\right)^{\frac{1}{p}} < \infty, \tag{2.5}$$

and the space $W_0^{1,p}(P, \Omega)$ which is the closure of $C_0^\infty(\Omega)$ in $W^{1,p}(P, \Omega)$ with respect to the norm

$$\|u\|_{W_0^{1,p}(P,\Omega)} = \left(\int_\Omega P(x)|\nabla u|^p\right)^{\frac{1}{p}} < \infty, \tag{2.6}$$

which is equivalent to the norm given by (2.5). The two spaces $W^{1,p}(P, \Omega)$ and $W_0^{1,p}(P, \Omega)$ are well defined reflexive Banach Spaces.

3. Existence Results

In this section, by using a method of [5] we shall establish our existence result by constructing a subsolution $(\psi_1, \psi_2) \in W_0^{1,p}(P, \Omega) \times W_0^{1,q}(Q, \Omega)$ and a supersolution $(z_1, z_2) \in W_0^{1,p}(P, \Omega) \times W_0^{1,q}(Q, \Omega)$ of (1.3) such that $\psi_i \le z_i$ for $i = 1, 2$.

Definition 3.1. A pair of nonnegative functions $(\psi_1, \psi_2), (z_1, z_2)$ are called a subsolution and supersolution of (1.3) if they satisfy $(\psi_1, \psi_2) = (0, 0) = (z_1, z_2)$ on $\partial\Omega$ and

$$\begin{aligned}
\int_\Omega P(x)|\nabla\psi_1|^{p-2}\nabla\psi_1\nabla\zeta dx &\le \lambda\int_\Omega \psi_1^\gamma\psi_2^\beta\zeta dx\\
\int_\Omega Q(x)|\nabla\psi_2|^{q-2}\nabla\psi_2\nabla\eta dx &\le \lambda\int_\Omega \psi_1^\alpha\psi_2^\delta\eta dx\\
\int_\Omega P(x)|\nabla z_1|^{p-2}\nabla z_1\nabla\zeta dx &\ge \lambda\int_\Omega z_1^\gamma z_2^\beta\zeta dx\\
\int_\Omega Q(x)|\nabla z_2|^{q-2}\nabla z_2\nabla\eta dx &\ge \lambda\int_\Omega z_1^\alpha z_2^\delta\eta dx
\end{aligned}$$

for all test functions $\zeta \in W_0^{1,p}(P, \Omega)$ and $\eta \in W_0^{1,q}(Q, \Omega)$ with $\zeta, \eta \ge 0$.

Then the following result holds:

Lemma 3.2. (see [3, 14]) Suppose there exist sub and supersolutions (ψ_1, ψ_2) and (z_1, z_2) respectively of (1.3) such that $(\psi_1, \psi_2) \le (z_1, z_2)$. Then (1.3) has a solution (u, v) such that $(u, v) \in [(\psi_1, \psi_2), (z_1, z_2)]$.

Now we shall establish:

Theorem 3.3. Let $\alpha + \gamma < p - 1$, $\beta + \delta < q - 1$. Then system (1.3) has a positive weak solution (u, v) for each $\lambda > 0$.

Proof of Theorem 3.3

Let $\lambda_1^{(r)}$ be the first eigenvalue of the eigenvalue problem (2.1) and $\phi_{1,r}$ the corresponding positive eigenfunction satisfying $\phi_{1,r} > 0$ in Ω and $|\nabla\phi_{1,r}| > 0$ on $\partial\Omega$ with $\|\phi_{1,r}\|_\infty = 1$, for $r = p, q$. Then we have

$$\left.\begin{aligned}
-\Delta_{P,p}\phi_{1,p} &= \lambda_1^{(p)}a(x)|\phi_{1,p}|^{p-2}\phi_{1,p} \quad \text{in}\,\Omega, \quad \phi_{1,p} = 0 \quad \text{on } \partial\Omega,\\
-\Delta_{Q,q}\phi_{1,q} &= \lambda_1^{(q)}b(x)|\phi_{1,q}|^{q-2}\phi_{1,q} \quad \text{in}\,\Omega, \quad \phi_{1,q} = 0 \quad \text{on } \partial\Omega.
\end{aligned}\right\} \tag{3.1}$$

Since $\alpha + \gamma < p - 1, \beta + \delta < q - 1$, we can take k such that

$$\frac{\alpha}{q - 1 - \delta} < k < \frac{p - 1 - \gamma}{\beta}. \tag{3.2}$$

We shall verify that $(\psi_1, \psi_2) = (\xi \phi_{1,p}^{\frac{p}{p-1}}, \xi^k \phi_{1,q}^{\frac{q}{q-1}})$ is a subsolution of (1.3), where $\xi > 0$ is small and specified later. Let $\zeta \in W_0^{1,p}(P, \Omega)$ with $\zeta \geq 0$. A calculation shows that

$$\begin{aligned}\int_\Omega P(x)|\nabla\psi_1|^{p-2}\nabla\psi_1 \cdot \nabla\zeta dx &= \left(\frac{p}{p-1}\xi\right)^{p-1}\int_\Omega P(x)\phi_{1,p}|\nabla\phi_{1,p}|^{p-2}\nabla\phi_{1,p}\cdot\nabla\zeta dx \\ &= \left(\frac{p}{p-1}\xi\right)^{p-1}\int_\Omega [P(x)|\nabla\phi_{1,p}|^{p-2}\nabla\phi_{1,p}\nabla(\phi_{1,p}\zeta) \\ &\quad -P(x)|\nabla\phi_{1,p}|^p]\zeta dx \\ &= \left(\frac{p}{p-1}\xi\right)^{p-1}\int_\Omega [\lambda_1^{(p)}a(x)\phi_{1,p}^p - P(x)|\nabla\phi_{1,p}|^p]\zeta dx.\end{aligned}$$

Similarly, for $\eta \in W_0^{1,q}(Q, \Omega)$ with $\eta \geq 0$, we have

$$\begin{aligned}&\int_\Omega Q(x)|\nabla\psi_2|^{q-2}\nabla\psi_2\cdot\nabla\eta dx \\ &= \left(\frac{q}{q-1}\xi^k\right)^{q-1}\int_\Omega [\lambda_1^{(q)}b(x)\phi_{1,q}^q - Q(x)|\nabla\phi_{1,q}|^q]\eta dx.\end{aligned}$$

Since $\phi_{1,r} = 0$ and $|\nabla\phi_{1,r}| > 0$ on $\partial\Omega$ for $r = p, q$, there is $\epsilon > 0$ such that

$$\lambda_1^{(p)}a(x)\phi_{1,p}^p - P(x)|\nabla\phi_{1,p}|^p \leq 0 \quad \text{and} \quad \lambda_1^{(q)}b(x)\phi_{1,q}^q - Q(x)|\nabla\phi_{1,q}|^q \leq 0 \quad \text{on } \overline{\Omega}_\epsilon,$$

with $\overline{\Omega}_\epsilon = \{x \in \Omega : d(x, \partial\Omega) \leq \epsilon\}$. This shows that

$$\left(\frac{p}{p-1}\xi\right)^{p-1}\int_{\overline{\Omega}_\epsilon}(\lambda_1^{(p)}a(x)\phi_{1,p}^p - P(x)|\nabla\phi_{1,p}|^p)\zeta dx \leq 0 \leq \lambda\int_{\overline{\Omega}_\epsilon} a(x)\psi_1^\gamma\psi_2^\beta\zeta dx,$$

and

$$\left(\frac{q}{q-1}\xi^k\right)^{q-1}\int_{\overline{\Omega}_\epsilon}(\lambda_1^{(q)}b(x)\phi_{1,q}^q - Q(x)|\nabla\phi_{1,q}|^q)\eta dx \leq 0 \leq \lambda\int_{\overline{\Omega}_\epsilon} b(x)\psi_1^\alpha\psi_2^\delta\eta dx.$$

Furthermore, we note that $\phi_{1,p}$, $\phi_{1,q} \geq \sigma$ in $\Omega - \overline{\Omega}_\epsilon$ for some $\sigma > 0$. Then from (3.2) there is $\xi_0 > 0$ such that if $\xi \in (0, \xi_0)$, the following inequalities hold:

$$\xi^{p-1-\gamma-k\beta}\left(\frac{p}{p-1}\right)^{p-1}\lambda_1^{(p)}\phi_{1,p}^p \leq \lambda\sigma^{\frac{\gamma p}{p-1}+\frac{\beta q}{q-1}} \leq \lambda\phi_{1,p}^{\frac{\gamma p}{p-1}}\phi_{1,q}^{\frac{\beta q}{q-1}} \qquad \text{in } \Omega - \overline{\Omega}_\epsilon,$$

and

$$\xi^{k(q-1-\delta)-\alpha}\left(\frac{q}{q-1}\right)^{q-1}\lambda_1^{(q)}\phi_{1,q}^{q}\ \leq \lambda\sigma^{\frac{\alpha p}{p-1}+\frac{\delta q}{q-1}} \leq \lambda\phi_{1,p}^{\frac{\alpha p}{p-1}}\ \phi_{1,q}^{\frac{\delta q}{q-1}}\ \text{ in } \Omega-\overline{\Omega}_\epsilon.$$

Then, we have

$$\begin{aligned}&\int\limits_{\Omega-\overline{\Omega}_\epsilon} P(x)|\nabla\psi_1|^{p-2}\nabla\psi_1\cdot\nabla\zeta dx\\ &=\left(\frac{p}{p-1}\xi\right)^{p-1}\int\limits_{\Omega-\overline{\Omega}_\epsilon}(\lambda_1^{(p)}a(x)\phi_{1,p}^{p}-P(x)|\nabla\phi_{1,p}|^{p})\zeta dx\\ &\leq\lambda\int\limits_{\Omega-\overline{\Omega}_\epsilon} a(x)\xi^{\gamma+k\beta}\phi_{1,p}^{\frac{\gamma p}{p-1}}\ \phi_{1,q}^{\frac{\beta q}{q-1}}\zeta dx=\lambda\int\limits_{\Omega-\overline{\Omega}_\epsilon} a(x)\psi_1^{\gamma}\psi_2^{\beta}\zeta dx.\end{aligned}$$

Similarly,

$$\begin{aligned}&\int\limits_{\Omega-\overline{\Omega}_\epsilon} Q(x)|\nabla\psi_2|^{q-2}\nabla\psi_2\cdot\nabla\eta dx\\ &=\left(\frac{q}{q-1}\xi^{k}\right)^{q-1}\int\limits_{\Omega-\overline{\Omega}_\epsilon}(\lambda_1^{(q)}b(x)\phi_{1,q}^{q}-Q(x)|\nabla\phi_{1,q}|^{q})\eta dx\\ &\leq\lambda\int\limits_{\Omega-\overline{\Omega}_\epsilon} b(x)\xi^{\alpha+k\delta}\phi_{1,p}^{\frac{\alpha p}{p-1}}\ \phi_{1,q}^{\frac{\delta q}{q-1}}\eta dx=\lambda\int\limits_{\Omega-\overline{\Omega}\epsilon} b(x)\psi_1^{\alpha}\psi_2^{\delta}\eta dx,\end{aligned}$$

i.e. (ψ_1,ψ_2) is a subsolution of (1.3).

Next, we construct a supersolution (z_1,z_2) of (1.3). Let $e_r(x)$ be the positive weak solution of (see [16])

$$-\Delta_{R,r}e_r=1\quad\text{ in }\ \Omega,\qquad e_r=0\qquad\text{on }\partial\Omega\qquad\text{for } r=p,q.$$

We denote $z_1(x)=Ae_p$, $z_2(x)=Be_q$, where the constants A, $B>0$ are large and to be chosen later. We shall verify that (z_1,z_2) is the supersolution of (1.3). To do this, let $\zeta\in W_0^{1,p}(P,\Omega)$ with $\zeta\geq 0$. Then we have

$$\int\limits_{\Omega} P(x)|\nabla z_1|^{p-2}\nabla z_1\cdot\nabla\zeta dx=A^{p-1}\int\limits_{\Omega} P(x)|\nabla e_p|^{p-2}\nabla e_p\cdot\nabla\zeta dx=A^{p-1}\int\limits_{\Omega}\zeta dx.$$

Similarly, for $\eta \in W_0^{1,q}(Q,\Omega)$ with $\eta \geq 0$, we have

$$\int_\Omega Q(x)|\nabla z_2|^{q-2}\nabla z_2 \cdot \nabla\eta dx = B^{q-1}\int_\Omega Q(x)|\nabla e_q|^{q-2}\nabla e_q \cdot \nabla\eta dx = B^{q-1}\int_\Omega \eta dx.$$

Since $p-1 > \gamma, q-1 > \delta$, it is easy to prove that there exist positive large constants A, B such that

$$A^{p-1} \geqslant \lambda l_a A^\gamma \mu_p^\gamma B^\beta \mu_q^\beta, \quad B^{q-1} \geqslant \lambda l_b A^\alpha \mu_p^\alpha B^\delta \mu_q^\delta,$$

where $l_a = |a(x)|, l_b = |b(x)|$ and $\mu_r = \|e_r\|_\infty\,; r = p, q$. These imply that

$$\int_\Omega P(x)|\nabla z_1|^{p-2}\nabla z_1 \cdot \nabla\zeta dx \geq \lambda \int_\Omega a(x) z_1^\gamma z_2^\beta \zeta dx,$$

and

$$\int_\Omega Q(x)|\nabla z_2|^{q-2}\nabla z_2 \cdot \nabla\eta dx \geq \lambda \int_\Omega b(x) z_1^\alpha z_2^\delta \eta dx,$$

i.e. (z_1, z_2) is a supersolution of (1.3) with $z_i \geq \psi_i$ with large A, B, for $i = 1, 2$. Thus, there exists a solution (u, v) of (1.3) with $\psi_1 \leq u \leq z_1,\ \psi_2 \leq v \leq z_2$. This completes the proof of Theorem 3.3. ■

4. Nonexistence Results

In this section, under some conditions we prove that system (1.3) has no positive weak solution.

Theorem 4.1. Suppose that $\alpha + \gamma = p - 1, \beta + \delta = q - 1, p\beta = q\alpha$ and $a(x) = b(x)$. Then there exists $\lambda_* > 0$ such that for $0 < \lambda < \lambda_*$, system (1.1) has no positive weak solution.

Proof. Let us assume that $(u, v) \in W_0^{1,p}(P,\Omega) \times W_0^{1,p}(Q,\Omega)$ be a positive weak solution of (1.3). We prove Theorem 4.1 by arriving at a contradiction.

Multiplying the first equation of (1.3) by u, we have from Young inequality that

$$\int_\Omega P(x)|\nabla u|^p dx \leq \lambda \int_\Omega a(x)\left(\frac{u^p}{\mu_1} + \frac{v^q}{\mu_2}\right) dx, \tag{4.1}$$

with $\mu_1 = \dfrac{p}{\gamma+1} > 1$ and $\mu_2 = \dfrac{p}{p-1-\gamma} > 1$.

Similarly, we have

$$\int_\Omega Q(x)|\nabla v|^q dx \le \lambda \int_\Omega b(x)\left(\frac{u^p}{\theta_1}+\frac{v^q}{\theta_2}\right)dx, \tag{4.2}$$

with $\theta_1 = \dfrac{q}{q-1-\delta} > 1$ and $\theta_2 = \dfrac{q}{\delta+1} > 1$.

Note that

$$\lambda_1^{(p)}\int_\Omega a(x)u^p dx \le \int_\Omega P(x)|\nabla u|^p dx, \quad \lambda_1^{(q)}\int_\Omega b(x)v^q dx \le \int_\Omega Q(x)|\nabla v|^q dx. \tag{4.3}$$

Combining (4.1)–(4.3), we obtain

$$\begin{aligned}\lambda_1^{(p)}&\int_\Omega a(x)u^p dx + \lambda_1^{(q)}\int_\Omega b(x)v^q dx\\ &\le \lambda\int_\Omega \left(\frac{a(x)}{\mu_1}+\frac{b(x)}{\theta_1}\right)u^p dx + \lambda\int_\Omega\left(\frac{a(x)}{\mu_2}+\frac{b(x)}{\theta_2}\right)v^q dx.\end{aligned}$$

Now, if $a(x) = b(x)$, we have

$$(\lambda_1^{(p)}-\lambda)\int_\Omega a(x)u^p dx + (\lambda_1^{(q)}-\lambda)\int_\Omega a(x)v^q dx \le 0,$$

which is a contradiction if $0 < \lambda < \lambda_* = \min\{\lambda_1^{(p)}, \lambda_1^{(q)}\}$. Thus system (1.3) has no positive weak solution if $a(x) = b(x)$ and $\lambda \in (0, \lambda_*)$. ■

Remark 4.2. When $\gamma = \delta = 0$ in system (1.3), we have some results presented in [10].

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References

[1] Afrouzi, G.A., Mahdavi, S., and Naghizadeh, Z., 2009, "Existence and uniqueness of solution for p-Laplacian Dirichlet problem", International Journal of Nonlinear Science, 8 (3), pp. 274–278.

[2] Alimohammady, M. and Koozegar, M, 2008, "Some remark on the nonexistence of positive solutions for some a, P-Laplacian system", J. Nonlinear Sci. Appl. 1 (2), pp. 56–60.

[3] Canada, A., Dravek, P. and Gamez, J., 1997, "Existence of positive solutions for some problems with nonlinear diffusion", Trans. Amer. Math. Soc., 349 (10), pp. 4231–4249.

[4] Chin, C., 2005, "On positive weak solutions for a class of quasilinear elliptic systems", Nonlinear analysis, 62, pp. 751–756.

[5] Drabek, P. and Hernandez, J., 2001, "Existence and uniqueness of positive solutions for some quasilinear elliptic problems", Nonlinear Anal. 44, pp. 189–204.

[6] Drabek, P., Kufner, A., and Nicolosi, F., 1997, "Quasilinear elliptic equation with degenerations and singularities", Walter de Gruyter, Bertin, New York.

[7] Ghaemi, M., et al., 2011, "On critical exponent for the existence and multiplicity of positive weak solutions for a class of (p, q)-Laplacian nonlinear system", The Journal of Mathematics and Computer Science, 3 (4), pp. 432–439.

[8] Hai, D. and Shivaji, R., 2004, "An existence result on positive solutions for a class of p-Laplacian systems", Nonlinear Anal. 56, pp. 1007–1010.

[9] Khafagy, S., 2011, "Existence and uniqueness of weak solution for weighted p-Laplacian Dirichlet problem", Journal of Advanced Research in Dynamical and Control Systems, 3 (3), pp. 41–49.

[10] Khafagy, S., 2011, "Existence and Nonexistence of Positive Weak Solutions for a Class of (p, q)-Laplacian with Different Weights", Int. J. Contemp. Math. Sciences, 6 (48), pp. 2391–2400.

[11] khafagy, S., 2012, "Existence, nonexistence and uniqueness of positive weak solution for a nonlinear system involving weighted p-Laplacian", Global Journal of Pure and Applied Mathematics, 8 (2), pp. 205–214.

[12] Khafagy, S., 2012, "On positive weak solution for a nonlinear system involving weighted p-Laplacian", Journal of Advanced Research in Dynamical and Control Systems, 4 (4), pp. 50–58.

[13] Khafagy, S. and Serag, H., 2009, "Existence of Weak Solutions for $n \times n$ Nonlinear Systems Involving Different p-Laplacian Operators", Electron. J. Diff. Eqns., 2009 (81), pp. 1–14.

[14] Lee, E., Shivaji, R. and Ye, J., 2009, "Positive solutions for elliptic equations involving nonlinearities with falling zeroes", Applied Mathematics Letters, 22, pp. 846–851.

[15] Serag, H. and Khafagy, S., 2008, "Existence of Weak Solutions for $n \times n$ Nonlinear Systems Involving Different Degenerated p-Laplacian Operators", New Zealand Journal of Mathematics, 38, pp. 75–86.

[16] Serag, H. and Khafagy, S., 2010, "On Maximum Principle and Existence of Positive Weak Solutions for $n \times n$ Nonlinear Systems Involving Degenerated p-Laplacian Operator", Turk J. M., 34, pp. 59–71.