Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 9, Number 4 (2013), pp. 369–378

http://www.ripublication.com/gjpam.htm

Common fixed point theorems of bivariate and self maps in $d-$metric spaces

A. Muraliraj
Department of Mathematics, Urumu Dhanalakshmi College, Kattur, Tiruchirappalli-620 019, TamilNadu, India.

R. Jahir Hussain
P.G. and Research Department of Mathematics, Jamal Mohamed College (Autonomous), Tiruchirappalli-620 020, TamilNadu, India.

Abstract

In this paper, weakly compatible and occasionally weakly compatible mappings are discussed in detail and generalize common fixed point theorems of bivariate and self maps, $T : X \times X \to X$ and $S : X \to X$ are proved in Dislocated metric space with E.A property. Our results extends and generalists many well known results.

AMS subject classification: Primary 54H25; Secondary 47H10, 47H09.
Keywords: Fixed point, coincidence point, bivariate map, self map, dislocated metric linear space.

1. Introduction

In the last two decades many fixed point theorems for contractions in dislocated metric spaces appeared (see [3]–[18]). Hitzler and Seda introduced the concept of dislocated metric space and studied the dislocated topologies which is a generalization of the conventional topologies and can be thought of as underlying the notion of dislocated metrics. They also proved a generalized version of Banach contraction mapping theorem which was applied to obtain fixed point semantics for logic programs. In this paper we have discussed the dislocated topologies associated with a Dislocated metric space and also proved a common fixed point theorem of bivariate and self maps with E. A property which extends and generalizes the well known Banach contraction principle and known results.

2. Preliminaries

Definition 1. Let X be a nonempty set, let $d : X \times X \to [0, \infty)$ be a function satisfying following conditions.

$$\text{(i) } d(x, y) = d(y, x) = 0 \ \ implies \ \ x = y,$$
$$\text{(ii) } d(x, y) \leq d(x, z) + d(z, y) \ \ forall \ \ x, y, z \in X.$$
$$\text{(iii) } d(x, y) = d(y, x) \ \ forall \ \ x, y \in X.$$

Then d is called a dislocated metric spaces or d-metric on X.

Definition 2. [3] A sequence $\{x_n\}$ in dislocated metric spaces (X, d) is said to be a Cauchy sequence if for given $\epsilon > 0$, there exists $n_0 \in N$ such that $\forall m, n \geq n_0$, implies $d(x_m, x_n) < \epsilon$.

Definition 3. [3] A sequence $\{x_n\}$ in dislocated metric spaces (X, d) is said to be a Convergent to x if $<d(x_n, x) = 0$.

In this case, x is called limit of $\{x_n\}$ and we write $x_n \to x$.

Definition 4. [3] A dislocated metric spaces (X, d) is said to be a Complete if every cauchy sequence is convergent in X.

Definition 5. Let T and S be self maps of a set X. Maps T and S are said to be commuting if $STx = TSx$ for all $x \in X$.

Definition 6. Let T and S be self maps of a set X. If $w = Tx = Sx$ for some $x \in X$, then x is called a coincidence point of T and S, and w is called a point of coincidence of T and S.

Example 7. Take $X = [0, 1]$, $Sx = x^2$, $Tx = \frac{x}{2}$. It is clear that $\{0, \frac{1}{2}\}$ is the set of coincidence points of S and T and 0 is the unique common fixed point.

Definition 8. The mappings S and T are said to be weakly compatible if and only if they commute at their coincidence points.

Definition 9. The mappings S and T are said to be occasionally weakly compatible if and only if they commute at some coincidence point of S and T, i.e. $STu = TSu$ for some coincidence point u.

Lemma 10. [6] If a weakly compatible pair (S, T) of self maps has a unique point of coincidence, then the point of coincidence is a unique common fixed point of S and T.

Example 11. Take $X = [0, 1]$, $Sx = x^2$, $Tx = \frac{x}{2}$. It is clear that $\{0, \frac{1}{2}\}$ is the set of coincidence points of S and T, $ST0 = TS0$ but $ST\frac{1}{2} \neq TS\frac{1}{2}$ and so S and T are occasionally weakly compatible but not weakly compatible.

Lemma 12. [6]Let S and T be occasionally weakly compatible mappings of X. If S and T have a unique point of coincidence then S and T are weakly compatible.

Definition 13. Let (X, d) be a dislocated metric Space (or $d-$metric Space), $T : X \times X \to X$ and $S : X \to X$ be mappings. A point $z \in X$ is said to be a coincidence point of S and T if $T(z, z) = Sz$. z is said to be a common fixed point of S and T if $T(z, z) = Sz = z$.

3. Main results

Consider a function $\phi : [0, 1]^2 \to [0, 1]$ such that

(a) ϕ is an increasing function, i.e $x_1 \leq y_1$, $x_2 \leq y_2$ implies $\phi(x_1, x_2) \leq \phi(y_1, y_2)$.

(b) $\phi(t, t) \leq t$, for all $t \in [0, 1]$

(c) ϕ is continuous in both variables.

Theorem 1. Let (X, d) be a dislocated metric space. Let $S : X \to X$ and $T : X \times X \to X$ be mappings Such that $T(X \times X) \subseteq S(X)$, $S(X)$ is complete and $d(T(x_1, x_2), T(x_2, x_3)) \leq \alpha\phi[d(Sx_1, Sx_2), d(Sx_2, Sx_3)]$ where x_1, x_2, x_3 are arbitrary elements in X, $0 < \alpha < \frac{1}{2}$. Then T and S have a coincidence point in X.

Proof. Let x_1, x_2 be arbitrary points in X, and a sequence $\{y_n\} \in X$ such that

$$y_{n+2} = Sx_{n+2} = T(x_n, x_{n+1})$$

for $n = 0, 1, \cdots$

Let $d_n = d(y_n, y_{n+1})$

$$\begin{aligned} d_n &= d(Sx_n, Sx_{n+1}) \\ &= d(T(x_{n-1}, x_{n-2}), T(x_n, x_{n-1})) \\ &\leq \alpha\ \phi[d(Sx_{n-2}, Sx_{n-1}), d(Sx_{n-1}, Sx_n)] \\ &= \alpha\ \phi[d(y_{n-2}, y_{n-1}), d(y_{n-1}, y_n)] \end{aligned}$$

By the method of mathematical induction we will prove that,

$$d_n \leq \left(\frac{L - \delta^n}{L + \delta^n}\right)^2 \tag{1}$$

where

$$\delta = \frac{1}{\alpha},$$

$$L = \min\left\{\frac{\delta(1+\sqrt{d_1})}{1-\sqrt{d_1}}, \frac{\delta^2(1+\sqrt{d_2})}{1-\sqrt{d_2}}\right\}.$$

It is clear that (1) is true for $n = 1, 2$. there fore

$$d_{n+1} \leq \left(\frac{L-\delta^{n+1}}{L+\delta^{n+1}}\right)^2$$

be the induction hypothesis. Then we have

$$\begin{aligned}
d_{n+2} &= d(y_{n+2}, y_{n+3}) \\
&= d(T(x_{n+1}, x_n), T(x_{n+2}, x_{n+1})) \\
&\leq \alpha\ \phi[d(Sx_n, Sx_{n+1}), d(Sx_{n+1}, Sx_{n+2})] \\
&\leq \phi[d_n, d_{n+1}] \\
&\leq \phi\left\{\left(\frac{L-\delta^n}{L+\delta^n}\right)^2, \left(\frac{L-\delta^{n+1}}{L+\delta^{n+1}}\right)^2\right\} \\
&\leq \phi\left\{\left(\frac{L-\delta^{n+1}}{L+\delta^{n+1}}\right)^2, \left(\frac{L-\delta^{n+1}}{L+\delta^{n+1}}\right)^2\right\} \\
&\leq \left(\frac{L-\delta^{n+1}}{L+\delta^{n+1}}\right)^2 \leq \left(\frac{L-\delta^{n+2}}{L+\delta^{n+2}}\right)^2.
\end{aligned}$$

Now, for $p \in N$ it follows

$$\begin{aligned}
d(y_n, y_{n+p}) &\leq d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + \cdots + d(y_{n+p-1}, y_{n+p}) \\
&\leq \left(\frac{L-\delta^n}{L+\delta^n}\right)^2 + \left(\frac{L-\delta^{n+1}}{L+\delta^{n+1}}\right)^2 + \cdots + \left(\frac{L-\delta^{n+p-1}}{L+\delta^{n+p-1}}\right)^2.
\end{aligned}$$

$\therefore d(y_n, y_{n+p}) \to 0$ as $n \to \infty$. Hence $\{y_n\}$ is a Cauchy sequence in $S(X)$, and since $S(X)$ is complete, there exist $z \in S(X)$ such that $\lim_{n\to\infty} y_n = z$. Let $z = Su$ for some $u \in X$. Then we have

$$\begin{aligned}
d(T(u,u), Su) &= \lim_{n\to\infty} d(T(u,u), y_{n+2}) \\
&= \lim_{n\to\infty} d(T(u,u), T(x_n, x_{n+1})) \\
&\leq \lim_{n\to\infty} [d(T(u,u), T(u,x_n)) + d(T(u,x_n), T(x_n, x_{n+1}))] \\
&\leq \lim_{n\to\infty} \alpha[\phi(d(Su,Su), d(Su,Sx_n)) + \phi(d(Su,Sx_n), d(Sx_n, Sx_{n+1}))] \\
&\to 0
\end{aligned}$$

Similarly it can be shown that $d(T(u,u), Su) \to 0$. Hence z is the coincidence point of T and S. ■

Theorem 2. Let (X, d) be a dislocated metric space. Let $S : X \to X$ and $T : X \times X \to X$ be mappings Such that $T(X \times X) \subseteq S(X)$, $S(X)$ is complete and $d(T(x_1, x_2), T(x_3, x_1)) \leq \alpha\phi[d(Sx_1, Sx_3), d(Sx_2, Sx_1)]$ where x_1, x_2, x_3 are arbitrary elements in X, $0 < \alpha < \frac{1}{2}$. Then T and S have a coincidence point in X.

Proof. Let x_1, x_2 be arbitrary points in X, and a sequence $\{y_n\} \in X$ such that $y_{n+2} = Sx_{n+2} = T(x_n, x_{n+1})$ for $n = 0, 1, \cdots$

Let $d_n = d(y_{n+1}, y_n)$. As proceed in above theorem. By the method of mathematical induction we will prove that,

$$d_n \leq \left(\frac{L - \delta^n}{L + \delta^n}\right)^2 \tag{1}$$

where

$$\delta = \frac{1}{\alpha},$$

$$L = \min\left\{\frac{\delta(1 + \sqrt{d_1})}{1 - \sqrt{d_1}}, \frac{\delta^2(1 + \sqrt{d_2})}{1 - \sqrt{d_2}}\right\}.$$

It is clear that (1) is true for $n = 1, 2$. there fore

$$d_{n+1} \leq \left(\frac{L - \delta^{n+1}}{L + \delta^{n+1}}\right)^2$$

be the induction hypothesis. Then we have

$$\begin{aligned}
d_{n+2} &= d(y_{n+3}, y_{n+2}) \\
&= d(T(x_{n+1}, x_{n+2}), T(x_n, x_{n+1})) \\
&\leq \alpha\ \phi[d(Sx_{n+1}, Sx_n), d(Sx_{n+2}, Sx_{n+1})] \\
&\leq \phi[d_n, d_{n+1}] \\
&\leq \phi\left\{\left(\frac{L - \delta^n}{L + \delta^n}\right)^2, \left(\frac{L - \delta^{n+1}}{L + \delta^{n+1}}\right)^2\right\} \\
&\leq \phi\left\{\left(\frac{L - \delta^{n+1}}{L + \delta^{n+1}}\right)^2, \left(\frac{L - \delta^{n+1}}{L + \delta^{n+1}}\right)^2\right\} \\
&\leq \left(\frac{L - \delta^{n+1}}{L + \delta^{n+1}}\right)^2 \leq \left(\frac{L - \delta^{n+2}}{L + \delta^{n+2}}\right)^2.
\end{aligned}$$

Now, for $p \in N$ it follows

$$\begin{aligned}
d(y_{n+p}, y_n) &\leq d(y_{n+p}, y_{n+p-1}) + d(y_{n+p-1}, y_{n+p-2}) + \cdots + d(y_{n+1}, y_n) \\
&\leq \left(\frac{L - \delta^{n+p-1}}{L + \delta^{n+p-1}}\right)^2 + \left(\frac{L - \delta^{n+p-2}}{L + \delta^{n+p-2}}\right)^2 + \cdots + \left(\frac{L - \delta^{n+1}}{L + \delta^{n+1}}\right)^2.
\end{aligned}$$

$\therefore d(y_{n+p}, y_n) \to 0$ as $n \to \infty$. Hence $\{y_n\}$ is a Cauchy sequence in $S(X)$, and since $S(X)$ is complete, there exist $z \in S(X)$ such that $\lim_{n\to\infty} y_n = z$. Let $z = Su$ for some $u \in X$. Then we have

$$\begin{aligned} d(T(u,u), Su) &= \lim_{n\to\infty} d(T(u,u), y_{n+2}) \\ &= \lim_{n\to\infty} d(T(u,u), T(x_n, x_{n+1})) \\ &\leq \lim_{n\to\infty} [d(T(u,u), T(u, x_{n+1})) + d(T(u, x_{n+1}), T(x_n, x_{n+1}))] \\ &\leq \lim_{n\to\infty} \alpha[\phi(d(Su, Sx_{n+1}), d(Su, Su)) + \phi(d(Sx_n, Sx_{n+1}), d(Su, Sx_{n+1}))] \to 0 \end{aligned}$$

Similarly it can be shown that $d(T(u,u), Su) \to 0$. Hence z is the coincidence point of T and S. ■

Theorem 3. Let (X, d) be a dislocated metric space. Let $S : X \to X$ and $T : X \times X \to X$ be mappings Such that $T(X \times X) \subseteq S(X)$, $S(X)$ is complete and

$$d(T(x_1, x_2), T(x_3, x_1)) \leq \alpha\phi[d(Sx_1, Sx_3), d(Sx_2, Sx_1)]$$

where x_1, x_2, x_3 are arbitrary elements in X, $0 < \alpha < \frac{1}{2}$. Then the sequence $\{y_n\} \in X$ defined by $y_{n+2} = Sx_{n+2} = T(x_n, x_{n+1})$, $n = 0, 1, \cdots$ converges to a unique common fixed point in X.

Proof. Proceeding as same as in the proof of above Theorem [3.2], we see that sequence $\{y_n\}$ converges to z which is a point of coincidence of S and T. And suppose there exists $z' \in X$ such that $S(u') = T(u', u') = z'$ for some coincidence point u'.

$$\begin{aligned} d(z, z') &= d(T(u,u), T(u', u')) \\ &\leq d(T(u,u), T(u, u')) + d(T(u, u'), T(u', u')) \\ &\leq \alpha[\phi(d(Su, Su), d(Su, Su')) + \phi(d(Su, Su'), d(Su', Su'))] \\ &\leq 2\alpha[\phi(d(Su, Su'), d(Su, Su')) + \phi(d(Su, Su'), d(Su, Su'))] \\ &\leq 2\alpha[d(Su, Su') + d(Su, Su')] \\ &= 2^2\alpha d(Su, Su') \\ &= 2^2\alpha d(T(u,u), T(u', u')) \\ &\leq 2^2\alpha[d(T(u,u), T(u, u')) + d(T(u, u'), T(u', u'))] \\ &\leq 2^2\alpha^2[\phi(d(Su, Su), d(Su, Su')) + \phi(d(Su, Su'), d(Su', Su'))] \\ &\leq 2^3\alpha^2[\phi(d(Su, Su'), d(Su, Su')) + \phi(d(Su, Su'), d(Su, Su'))] \\ &\leq 2^4\alpha^2 d(Su, Su') \end{aligned}$$

Repeating the above process n times we get

$$d(z, z') \leq 2^{n+1}\alpha^{n-1} d(Su, Su')$$

Taking the limit as $n \to \infty$ we get $d(z, z') \to 0$, and so $z = z'$, i.e. z is the unique point of coincidence of S and T. Hence by above lemmas, z is a unique common fixed point of S and T. ■

Definition 4. An element $(x, y) \in X \times X$, is called a coupled fixed point of mapping $S : X \times X \to X$ if $S(x, y) = x$ and $S(y, x) = y$.

Definition 5. An element $(x, y) \in X \times X$, is called a coupled coincident point of mapping $S : X \times X \to X$ and $T : X \to X$ if $S(x, y) = T(x)$ and $S(y, x) = T(y)$.

Definition 6. The mappings S and T where $S : X \times X \to X$ and $T : X \to X$, are said to be compatible if

$$\lim_{n\to\infty} d(T(S(x_n, y_n)), S(T(x_n), T(y_n))) = 0$$

and $\lim_{n\to\infty} d(T(S(y_n, x_n)), S(T(y_n), T(x_n))) = 0$. whenever $\{x_n\}$ and $\{y_n\}$ are sequences in X, such that

$$\lim_{n\to\infty} S(x_n, y_n) = T(x_n) = x$$

and $\lim_{n\to\infty} S(y_n, x_n) = T(y_n) = y$, for all $x, y \in X$ are satisfied.

Definition 7. The bivariate self mapping, i.e., $S : X \times X \to X$ and self mapping $T : X \to X$ of a dislocated metric space (X, d) are said to be weakly compatible if they commute at there coincidence points, that is, if for all $x, y \in X$

$$S(x, y) = T(x) \text{ for some } x \in X, \text{ then } S(T(x), T(y)) = T(S(x, y)).$$

and

$$S(y, x) = T(y) \text{ for some } y \in X, \text{ then } S(T(y), T(x)) = T(S(y, x)).$$

Definition 8. The mappings S and T where $S : X \times X \to X$ and $T : X \to X$, of an dislocated metric space (X, d) satisfy E.A. property, if there exist sequences $\{x_n\}$ and $\{y_n\}$ in X, such that

$$S(x_n, y_n) = T(x_n) = T(u)$$

and $S(y_n, x_n) = T(y_n) = T(v)$ for $u, v \in X$.

Theorem 9. Let (X, d) be a dislocated metric space. Let $S : X \to X$ and $T : X \times X \to X$ be weakly compatible mappings of X Such that, for some $x, y, u, v \in X$,

$$d(S(x, y), S(u, v)) \leq \min\{d(T(x), T(u)), d(S(x, y), T(u)), d(S(u, v), T(u)),$$
$$d(S(y, x), T(v)), d(S(x, y), T(u)\}.$$

If S and T satisfy E. A. property and T is a closed subspace of X, then S and T have a unique common fixed point.

Proof. Since S and T satisfy E. A. property, therefore, we can find sequences $\{x_n\}$ and $\{y_n\}$ in X and the point $u, v \in X$ such that

$$\lim_{n\to\infty} S(x_n, y_n) = \lim_{n\to\infty} T(x_n) = T(u)$$

and

$$\lim_{n\to\infty} S(y_n, x_n) = \lim_{n\to\infty} T(y_n) = T(v).$$

Then,

$$d(S(x_n, y_n), S(u, v)) \le \min\{d(T(x_n), T(u)), d(S(x_n, y_n), T(u)), d(S(u, v), T(u)), d(S(y_n, x_n), T(v)), d(S(x_n, y_n), T(u))\}$$

Taking the limit as n tends to infinity in the above inequality,

$$\begin{aligned} d(T(u), S(u, v)) &\le \min\{d(T(u), T(u)), d(T(u), T(u)), d(S(u, v), T(u)), d(T(v), T(v)), d(T(u), T(u))\} \\ &\le \min\{0, 0, d(S(u, v), T(u)), 0, 0\} \\ &\le d(S(u, v), T(u)) \end{aligned}$$

Now, if $S(u, v) \neq T(u)$, then $d(S(u, v), T(u)) > d(S(u, v), T(u))$, contradicting the above inequality. This proves that $d(T(u), S(u, v)) = 0$, which implies $S(u, v) = T(u)$. Similarly, it can be proved that $S(v, u) = T(v)$. By denoting $S(u, v) = T(u) = z_1$ and $S(v, u) = T(v) = z_2$, Since S and T are weakly compatible then one obtain that $S(z_1, z_2) = Tz_1$ and $S(z_2, z_1) = Tz_2$. Let us prove that $z_1 = F(z_1, z_2)$.

Indeed, we obtain by (1)

$$\begin{aligned} d(S(z_1, z_2), z_1) &= d(S(z_1, z_2), S(u, v)) \\ &\le \min\{d(T(z_1), T(u)), d(S(z_1, z_2), T(u)), d(S(u, v), T(u)), d(S(v, u), T(v)), d(S(z_1, z_2), T(u))\}). \\ &= \min\{d(S(z_1, z_2), z_1), d(S(z_1, z_2), z_1), d(z_1, z_1), d(z_2, z_2), d(S(z_1, z_1), z_1)\}). \\ &= d(S(z_1, z_2), z_1)). \end{aligned}$$

This implies that $S(z_1, z_2) = z_1$. Hence z_1 is a common fixed point of S and T. Similarly, it can be proved that z_2 is common fixed point of S and T.

Finally, we prove that common fixed point is unique i.e., $z_1 = z_2$.

Suppose that it not true. Then

$$\begin{aligned} d(z_1, z_2) &= d(S(z_1, z_2), S(z_2, z_1)) \\ &\le \min\{d(T(z_1), T(z_2)), d(S(z_1, z_2), T(z_2)), d(S(z_2, z_1), T(z_2)), d(S(z_1, z_2), T(z_1)), d(S(z_1, z_2), T(z_2))\} \\ . &= \min\{d(z_1, z_2), d(z_1, z_2), d(z_2, z_2), d(z_1, z_1), d(z_1, z_2)\}. \\ &= d(z_1, z_2). \end{aligned}$$

which is a contradiction this concludes the proof. ■

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