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Y – Domatic Number of a Bipartite Graph

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Abstract

Let G= (X, Y, E) be a bipartite graph with no isolated vertex in Y. A subset S of X is called a Y-dominating set if every y in Y is adjacent to a vertex of S. The minimum cardinality of a Y-dominating set, denoted by $\gamma_Y(G)$ is called the Y-domination number of the graph G. A Y-domatic partition of G is a partition of X, all of whose elements are Y-dominating sets in G. The Y-domatic number of G is the maximum number of classes of a Y-domatic partition of G and is denoted by dY (G) . We study these parameters and characterize graphs with $\gamma_Y(G)$ +dY (G) =2p.

Keywords: Bipartite graph, Y-domatic partition, Y-dominating set.

AMS Classification: 05C69.

1 Introduction

The graphs considered here are finite, undirected simple graphs with p vertices and q edges. A subset D of V of vertices of a simple graph G= (V, E) is a dominating set if for every vertex u in V-D, there exists v in D such that u and v are adjacent. The minimum cardinality of a dominating set in G is called the domination number [1] of G and is denoted by $\gamma(G)$. A domatic partition is a partition of V into dominating sets, and the domatic number d (G) is the largest number of sets in a domatic partition of G. Various types of domination have been defined and studied by various authors and is given in [1]. Corresponding to each domination parameter, a parameter called domatic number is defined and studied. A survey of results on domatic number and its invariants are studied in [2].

The bipartite theory of graphs[3] was introduced by Stephen Hedietniemi and Renulaskar, in which concepts for an arbitrary graphs can be represented as concepts for bipartite graphs. One such reformulation is the Y-dominating set of a bipartite graph. Hereafter, we consider only bipartite graph G= (X, Y, E) with $|X|$=p ; $|Y|$=q.

2 Y-dominating sets

Definition: 1 [3] A subset S of X is called a Y-dominating set if every y in Y is adjacent to a vertex of S. The minimum cardinality of a Y-dominating set, denoted by $\gamma_Y(G)$ is called the Y-domination number of the graph G.

Observation: 1 Existence of Y-dominating set is guaranteed in any bipartite graph if and only if Y does not contain any isolated vertex.

Hence, by a graph G= (X, Y, E) with $|X|$=p ; $|Y|$=q, we mean a bipartite graph with no isolated vertex in Y.

Observation: 2 In an graph G, $1 \le \gamma_Y(G) \le$ p.

A vertex x in X is a full degree vertex if x is adjacent to all the vertices of Y. Let D be a subset of X and u belong to D, then the private neighbor of u with respect to D is defined as pn[u, D]={y:N (y) $\cap$ D={u}}.

Observation: 3 In a graph G, $\gamma_Y(G)$ =1 if and only if there exists a full degree vertex in X.

Corollary: 1 $\gamma_Y(K_{p,q}) = 1$.

Observation: 4 In a graph G, $\gamma_Y(G)$ =p if and only if every vertex in X has a private neighbor with respect to X.

We proceed to obtain a characterization of minimal Y-dominating sets.

Theorem: 1 Let D be a Y-dominating set of a graph G. Then D is a minimal Y-dominating set if and only if for each vertex u in D, pn[u, D]$\neq \phi$.

Proof: Suppose D is a minimal Y-dominating set of G. Then, D-{u} is not a Y-dominating set. There exists y in Y adjacent to a vertex in D but not adjacent to a vertex in D-{u}. Hence, u has a private neighbor with respect to D.

Conversely, assume u in D has a private neighbor. Suppose D is not minimal Y-dominating set. Then D-{u} is a Y-dominating set. A contradiction, to the fact that u in D has a private neighbor. □

3 Y-domatic partition of bipartite graph

A Y-domatic partition of G is a partition of X, all of whose elements are Y-dominating sets in G. The Y-domatic number of G is the maximum number of classes of a Y-domatic partition of G and is denoted by dY (G) .

Obervation: 5 Existence of a Y-domatic partition is guaranteed in any bipartite graph if and only if Y does not contain any isolated vertex.

Observation: 6 $1 \leq$ dY (G) $\leq$ p.

Theorem: 2 In a graph G, dY (G) =p if and only if $G \cong K_{p,q}$.

Proof: Let dY (G) =p. Every vertex in X is a Y-dominating set. Equivalently, every vertex in X is adjacent to all the vertices of Y. Hence, G is the complete bipartite graph $K_{p,q}$.

Conversely, if $G \cong K_{p,q}$, then every vertex in X is adjacent to all the vertices of Y. Therefore, dY (G) =p. □

Theorem: 3 In a graph dY (G) =1 if and only if at least one vertex in X has a private neighbor with respect to X.

Proof: Let dY (G) =1. Then X is a Y-dominating set. Therefore, there exists a vertex y in Y adjacent to a vertex x in X. Hence, at least one vertex in X has a private neighbor with respect to X. If at least one vertex in X has a private neighbor, then X is the only Y-dominating set. Hence, dY (G) =1. □

Given an arbitrary graph G, we can construct bipartite graph which represents the given graph G. We give the construction of the bipartite graph VV (G) and VV+ (G) from a given graph as given in [3, 4].

Definition: 2 Let V^1 be a copy of the vertices of G. The graph VV (G) = (V, V^1, E^1) is defined to be the bipartite graph with bipartition V, V^1 and E^1={ (u, v^1) : (u, v) is in E}.

Definition: 3 The graph VV+ (G) = (V, V^1, E^a) contains the edges of $E^1$of the graph VV (G) together with the edges { (u, u^1) :u in V}.

Theorem: 4 For a connected graph G, d (G) =dY (VV+ (G)) .

Proof: Let $V_1, V_2, \ldots, V_n$ be the domatic partition of G. Each V_i is a dominating set in G. Therefore, V_i is a Y-dominating set in the bipartite graph VV+ (G) . Hence, $V_1, V_2, \ldots, V_n$ is a Y-domatic partition of VV+ (G) . Hence, dY (VV+ (G)) $\geq$ d (G) .

Let $X_1, X_2, \ldots, X_n$ be a Y-domatic partition of VV+ (G) . Each X_i is a Y-domating set in VV+ (G) . Therefore, X_i is a dominating set in the graph G. Hence, $X_1, X_2, \ldots, X_n$ is a domatic partition of G. Hence, d (G) $\geq$ dY (VV+ (G)) . Therefore, d (G) =dY (VV+ (G)) . □

Similarly, one can prove the theorem

Theorem: 5 For a connected graph G, d_t (G) =dY (VV (G)) .

4 Gallai type theorems

Results of the form, $\alpha(G)+\beta(G)=n$, where $\alpha(G)$ is a minimum and $\beta(G)$ is a maximum parameter of G are called Gallai theorems. Here, we give Gallai type theorem involving the Y-domination number $\gamma_Y(G)$ and Y-domatic number dY (G) and characterize the graphs attaining the bound.

Theorem: 6 In a graph G, $\gamma_Y(G)+dY(G)\le 2p$. Equality holds if and only if G $\cong K_{1,q}$.

Proof: We know $\gamma_Y(G)\le p$ and $dY(G)\le p$. Therefore, $\gamma_Y(G)+dY(G)\le 2p$. If $\gamma_Y(G)+dY(G)=2p$, then $\gamma_Y(G)$ =p and dY (G) =p. If dY (G) =p then G is $K_{p,q}$. In the graph $K_{p,q}$, $\gamma_Y(K_{p,q})=1$. Hence, p=1. Therefore, G $\cong K_{1,q}$. □

Let Λ_1 be the family of graph G= (X, Y, E) , $|X|=2;|Y|=q$ and every vertex in X has a private neighbor in Y.

Theorem: 7 In a graph G, $\gamma_Y(G)+dY(G)=2p-1$ if and only if G is one of the graph $K_{2,q}$ or G belongs to Λ_1.

Proof: Let $\gamma_Y(G)+dY(G)=2p-1$. The possible cases are (i) dY (G) =p, $\gamma_Y(G)=p-1$ and (ii) dY (G) =p-1, $\gamma_Y(G)=p$.

Case (i) : dY (G) =p, $\gamma_Y(G)=p-1$.
Since dY (G) =p, G is $K_{p,q}$. We have $\gamma_Y(K_{p,q})=1$. Therefore, p=2. Hence, G=$K_{2,q}$.

Case (ii) : dY (G) =p-1, $\gamma_Y(G)=p$.
Since, $\gamma_Y(G)=p$, every vertex in X has a private neighbor in Y. Then dY (G) =1 and therefore, p=2 and hence, G belongs to Λ_1.

Converse is obvious. □

Let Λ_2 be the family of graphs G= (X, Y, E) , $|X|=2;|Y|=q$ and every vertex in X is a full degree vertex.

Let Λ_3 be the family of graphs G= (X, Y, E) , $|X|=3;|Y|=q$ and every vertex in X has a private neighbor in Y.

Theorem: 8 In a graph G, $\gamma_Y(G)+dY(G)=2p-2$. Equality holds if and only if G is one of the graphs $K_{3,q}$ or G belongs to Λ_2 or G belongs to Λ_3.

Proof: Let $\gamma_Y(G)+dY(G)=2p-2$. The possible cases are

- dY (G) =p, $\gamma_Y(G)=p-2$.
- dY (G) =p-1, $\gamma_Y(G)=p-1$.
- dY (G) =p-2, $\gamma_Y(G)=p$.

Case (i) : dY (G) =p, $\gamma_Y(G)=p-2$**.**
Since dY (G) =p, G is $K_{p,q}$. We have $\gamma_Y(K_{p,q})=1$. Therefore, p=3. Hence G =$K_{3,q}$.

Case (ii) : dY (G) =p-1, $\gamma_Y(G)=p-1$**.**
Since, dY (G) =p-1, we have two subcases.

Subcase (i) : Here p-1 vertices are of full degree and one vertex is not of full degree. Then $\gamma_Y(G)=p-1=1$. Therefore, p=2. Hence, G belongs to Λ_2.

Subcase (ii) : In this case p-2 vertices are of full degree and remaining two vertices dominate Y. We get $\gamma_Y(G)=p-1=1$. Therefore, p=2. Then no vertex in X is of full degree, a contradiction to $\gamma_Y(G)=1$. Hence, no graphs exists.

Case (iii) : dY (G) =p-2, $\gamma_Y(G)=p$**.**
Since, $\gamma_Y(G)=p$, every vertex in X has a private neighbor in Y. Hence, dY (G) =1. Therefore, p=3. Therefore, G belongs to Λ_3.

Converse is obvious. □

Let Λ_4 be the family of graphs G= (X, Y, E) , $|X|=3;|Y|=q$. Two vertices in X are of full degree vertices and other vertices is not a full degree vertex.

Let Λ_5 be the family of graphs G= (X, Y, E) , $|X|=3;|Y|=q$. One vertex in X is of full degree and two vertices dominate Y.

Let Λ_6 be the family of graphs G= (X, Y, E) , $|X|=4;|Y|=q$ and every vertex in X has a private neighbor in Y.

Theorem: 9 In a graph G, $\gamma_Y(G)+dY(G)=2p-3$. Equality holds if and only if G is one of the graph $K_{4,q}$ or G belongs to Λ_4 or G belongs to Λ_5 or G belongs to Λ_6.

Proof: Let $\gamma_Y(G)+dY(G)=2p-3$. The possible cases are

- dY (G) =p, $\gamma_Y(G)=p-3$.
- dY (G) =p-1, $\gamma_Y(G)=p-2$.

- dY (G) =p-2, $\gamma_Y(G) = p-1$.
- dY (G) =p-3, $\gamma_Y(G) = p$.

Case (i) : dY (G) =p, $\gamma_Y(G) = p-3$**.**
Since dY (G) =p, G is is $K_{p,q}$. We have $\gamma_Y(K_{p,q}) = 1$. Therefore, p=4. Hence, G=$K_{4,q}$.

Case (ii) : dY (G) =p-1, $\gamma_Y(G) = p-2$**.**
Subcase (i) : Here (p-1) vertices are of full degree and one vertex is not a full degree vertex. Then, $\gamma_Y(G) = p-2 = 1$. Therefore, p=3. Therefore, G belongs to Λ_4.

Subcase (ii) : In this case (p-2) vertices are of full degree and remaining two vertices dominate Y. Then $\gamma_Y(G) = 1$. Therefore, p=3. Hence, G belongs to Λ_5.

Case (iii) : dY (G) =p-2, $\gamma_Y(G) = p-1$**.**
Subcase (i) : (p-2) vertices in X are of full degree and two vertices are not full degree vertices and they do not dominate Y. Then, $\gamma_Y(G) = p-1 = 1$. Hence, p=2. We get, dY (G) =0, a contradiction. Hence, no such graphs exists.

Subcase (ii) : (p-3) vertices in X are full degree vertices and the remaining three vertices dominate Y. In this case, $\gamma_Y(G) = 1$. Therefore, p=2. Hence dY (G) =0, a contradiction. Hence no such graphs exists.

Case (iv) : dY (G) =p-3, $\gamma_Y(G) = p$**.**
Since $\gamma_Y(G) = p$, every vertex in X has a private neighbor in Y. Then, dY (G) =1=p-3. Therefore, p=4. Hence, G belongs to Λ_6.

Converse is obvious. □

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