Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 9, Number 4 (2013), pp. 353-362

http://www.ripublication.com/gjpam.htm

Characterization of Random Packing by Some Disconnected Graphs

Elachini V. Lal and Karakkattu S. Parvathy

Dept. of Higher Secondary, Govt. HSS Kuttippuram, Kerala – 679571
lamathematics@gmail.com
Dept. of Mathematics, St. Mary' s College ,Thrissur, Kerala – 680020
parvathymaths@gmail.com

Abstract

Graph G is said to be packable by the graph F if its edges can be partitioned into copies of F.

It is called randomly F – packable if what remains after deletion of the edges of a proper subgraph that is F – packable is also F – packable. In this paper we characterize the class of graphs that are randomly F – packable where F = $C_4 \cup K_{1,2}$ / $C_5 \cup P_5$ / $C_4 \cup C_3$ / $C_4 \cup P_{2n+1}$ ($n \geq 2$) / $C_r \cup P_{2n+1}$ ($r \geq 5$, $n \geq 2$) / $C_r \cup P_{2n}$ ($r \geq 4$, $n \geq 2$).

Key Words: F – packable graphs, Randomly F – packable graphs.

1. Introduction :

By a graph we mean finite, undirected graph without loops and without multiple edges

The concept of random packability was introduced by Sergio Ruiz [1] under the name of 'random decomposable graphs'. Ruiz [1] obtained a characterization of randomly *F-* packable graphs, when *F* is P_3 or K_2. Later, Barrientos et al extended the result to all star graphs. Lowell W Beineke [2] and Peter Hamburger [2] and Wayne D Goddard [2] characterized *F*-packable graphs where *F* is K_n, P_4, P_5, or P_6. S.Arumugham [3] and S.Meena [3] extended a characterization of the random packability of two or more disconnected graphs like $K_n \cup K_{1,n}$, $C_4 \cup P_2$, $3K_2$.

A graph *G* is said to be packable by the graph *F* if its edges can be partitioned into copies of *F* and is called randomly *F* – packable if for every proper *F* – packable subgraph *H* of *G*, G – *E (H)* is also *F* – packable. If *H* is a subgraph of *G* which is F – packable but not randomly *F* – Packable then *H* is called a *F* – forbidden subgraph of

G. The union $G_1 \cup G_2$ of G_1 and G_2 is the subgraph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$ and the subdivided star $S_{2k}^{(r)}$ is the graph from r paths of length $2k$ by identifying their center vertex. $2C_3$ is the union of 2 disjoint 3-cycles and $2tC_3$ is the union of t copies of $2C_3$. A connected graph consisting of t blocks each is isomorphic to C_n and having exactly one cut vertex is denoted by $C_3^{(t)}$. For any integer $r \geq 2$, let B_r denote the class of all bipartite graphs with bipartition (X, Y) such that the degree of every vertex in X is a multiple of r and the degree of every vertex in Y is less than r.

We also use the following results.[3]

Theorem 1.1
A graph is randomly $2K_2$ – packable if and only if it is one of the following:
$C_4, K_4, 2K_3, K_3 \cup K_{1,3}, 2K_{1,n}$ *or* $2nK_2$ $(n \geq 1)$.

Theorem 1.2
A graph G is randomly $K_{1,2}$ – Packable if and only if each component of G is isomorphic to C_4 or $K_{1,2r}$

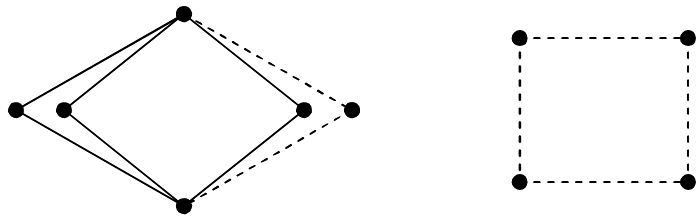
Theorem 1.3
A graph is randomly K_n – Packable if and only if every edge lies in precisely one copy of K_n in G.

Theorem 1.4
For, $r \geq 2$, a connected graph G is randomly $K_{1,r}$ – packable, if and only if either it is $K_{r,r}$ or it belongs to B_r.

2. Main Results :

Theorem : 2.1
A graph G is $C_4 \cup K_{1,2}$ *-packable iff G is isomorphic to* $K_{2,2r} \cup J$, *where J is any* C_4*-free graph with any one of the following:* $rK_{1,2}$ *or* $S_2^{(r)}$, $(r \geq 1)$.

Proof:
Suppose G is randomly $C_4 \cup K_{1,2}$-packable. Let J_1 be any subgraph of G induced by the edges of two edge disjoint copies of $C_4 \cup K_{1,2}$. Let J_2 be any subgraph induced by the edges of two copies of C_4 in J_1. Then J_2 is isomorphic to $K_{2,4}$. If J_1 contains a C_4, then G contains a subgraph isomorphic to $K_{2,4} \cup C_4$ - which is $C_4 \cup K_{1,2}$ - forbidden.



$K_{2,4}$ C_4

Fig. 2.1

If G is packable by r-copies of $C_4 \cup K_{1,2}$, then the subgraph G_1 induced by r copies of C_4 is isomorphic to $K_{2,2r}$ and the subgraph J induced by r copies of $K_{1,2}$ is disjoint from G_1 is isomorphic to $rK_{1,2}$ or $S_2^{(r)}$.

Then G is isomorphic to $K_{2,2r} \cup J$, where J is a C_4 - free graph with any one of the following: $rK_{1,2}$ or

$S_2^{(r)}, (r \geq 1)$ **(2)**

Lemma: 2.1

The graph $tC_4 .C_3$ is randomly $C_4 \cup C_3$ – packable if and only if $t = 2$

Proof:

Let $t = 2$

Fig. 2.2

We can easily see that $2C_4 .C_3$ is randomly $C_4 \cup C_3$ – packable

Conversely, suppose that t □ 2.

Let $t = 3$

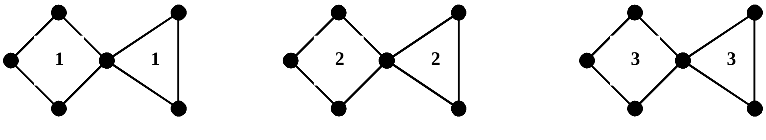


Fig. 2.3

Then $3C_4 .C_3$ is $C_4 \cup C_3$ – packable as in the following

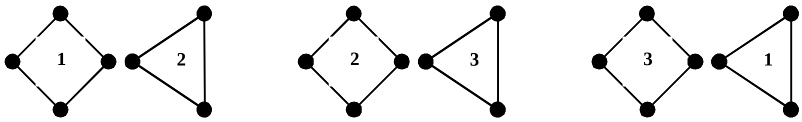


Fig. 2.4

But from the following figure, it follows that $3C_4.C_3$ is $C_4 \cup C_3$ - forbidden

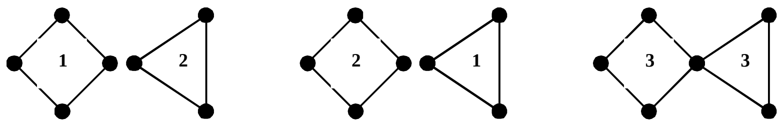


Fig. 2.5

Therefore, if we consider $tC_4 .C_3$ that are $C_4 \cup C_3$ – packable for $t \geq 3$, then it will contain a $C_4 \cup C_3$ – forbidden subgraph $3C_4 .C_3$. Hence the lemma is over.

Theorem: 2.2
The only connected randomly $C_4 \cup C_3$ packable graphs are A_i $(1 \leq i \leq 7)$ in Fig 2.6 *B_j $(1 \leq j \leq 4)$ in Fig* 2.7, *C_k $(1 \leq k \leq 4)$ in Fig* 2.8, *D_l $(1 \leq l \leq 4)$ in Fig* 2.9, *E_m $(1 \leq m \leq 3)$ in Fig* 2.10.

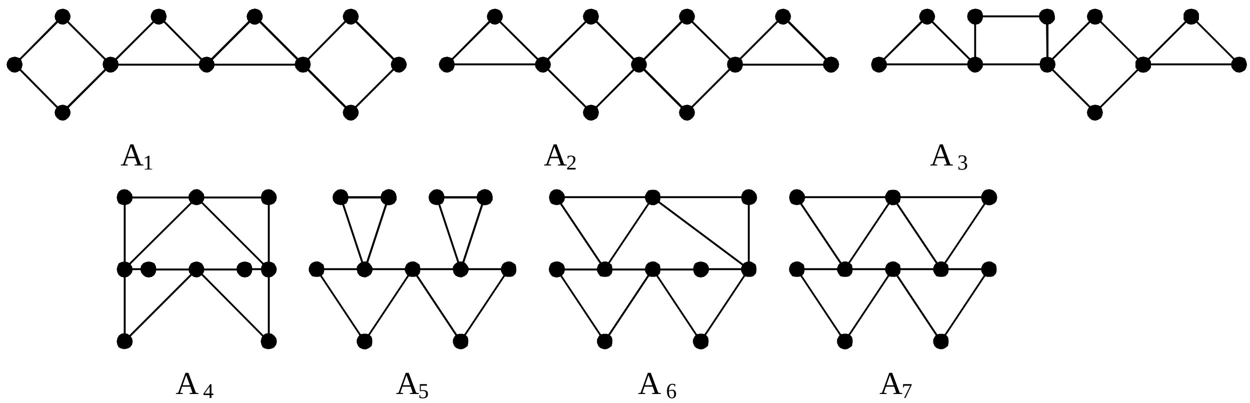


Fig. 2.6

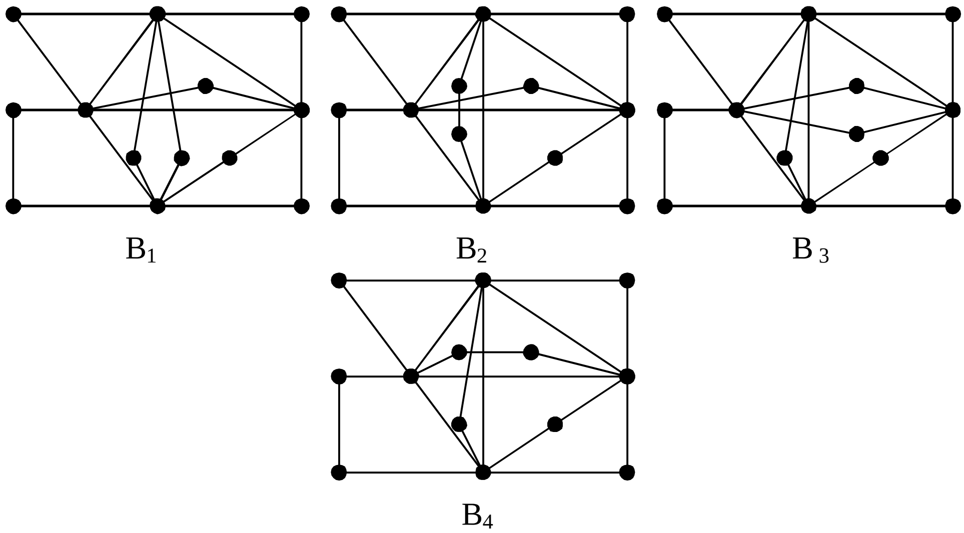


Fig. 2.7

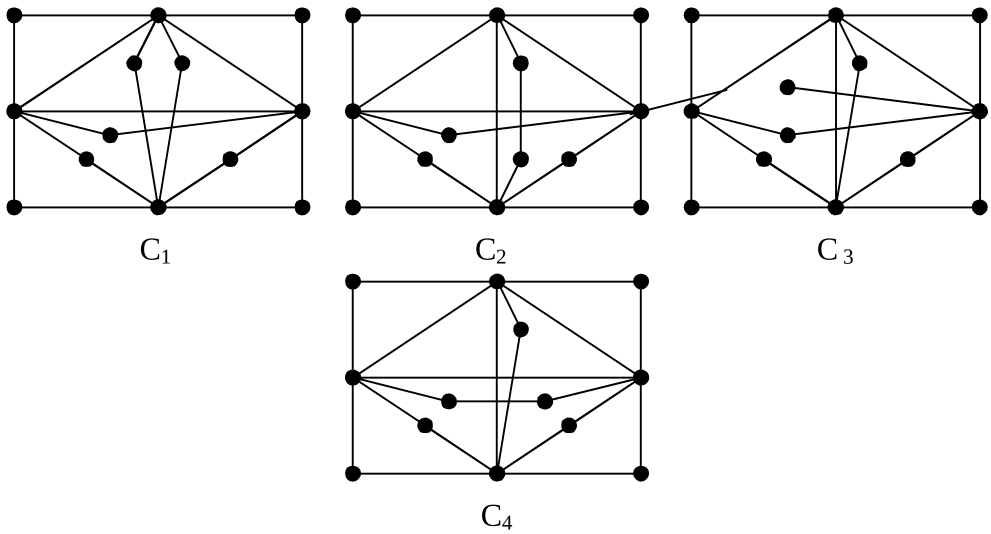
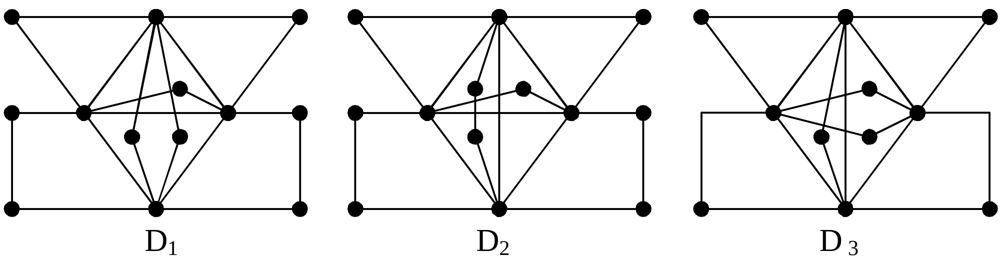


Fig. 2.8



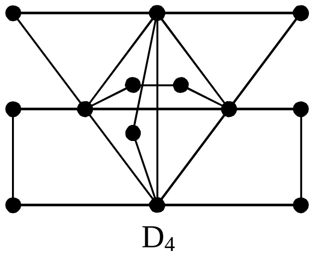


Fig. 2.9

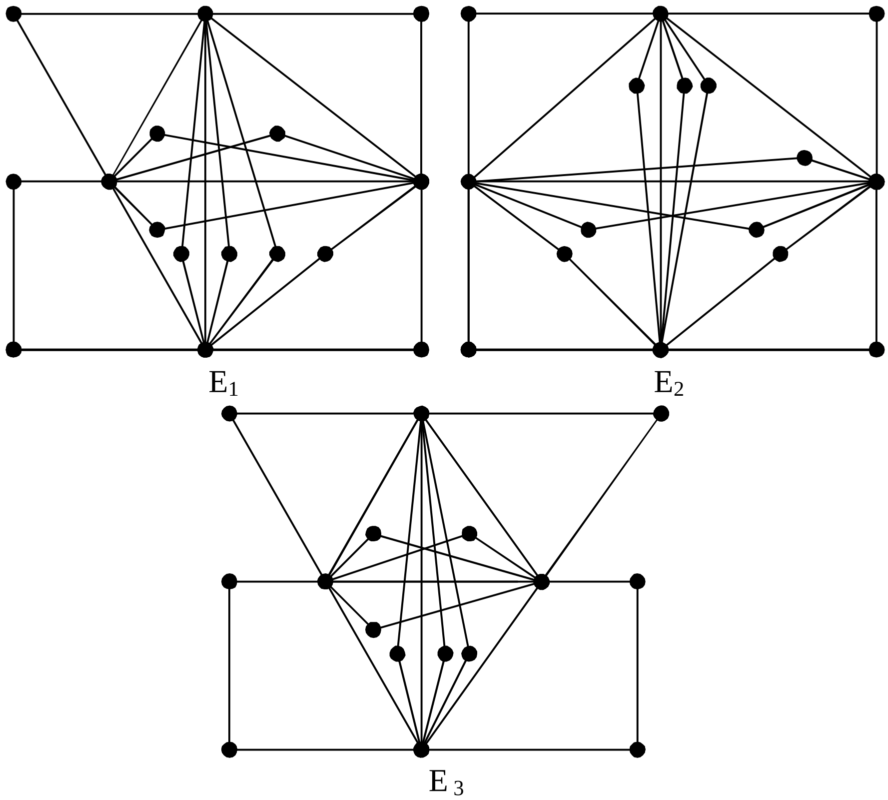


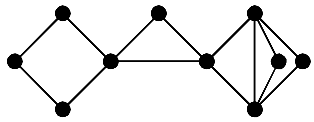
Fig. 2.10

Proof :

Let G be a randomly $C_4 \cup C_3$ – packable graph.

Suppose G is randomly $C_4 \cup C_3$ – packable with exactly two copies of $C_4 \cup C_3$. Let J and K be the two copies of $C_4 \cup C_3$ in G.

If a copy of C_4 in J is disjoint from a copy of C_4 in K, we get the following $C_4 \cup C_3$ – forbidden graph F_1 as given in fig. 2.11.



F_1

Fig. 2.11

Therefore, in this case, the only randomly $C_4 \cup C_3$ – packable graph is A_1. So, G is isomorphic to A_1.

If a copy of C_4 in J is not disjoint from a copy of C_4 in K, we get $C_4 \cup C_3$ – forbidden graphs F_i $(2 \leq j \leq 8)$ as given in fig. 2.12.

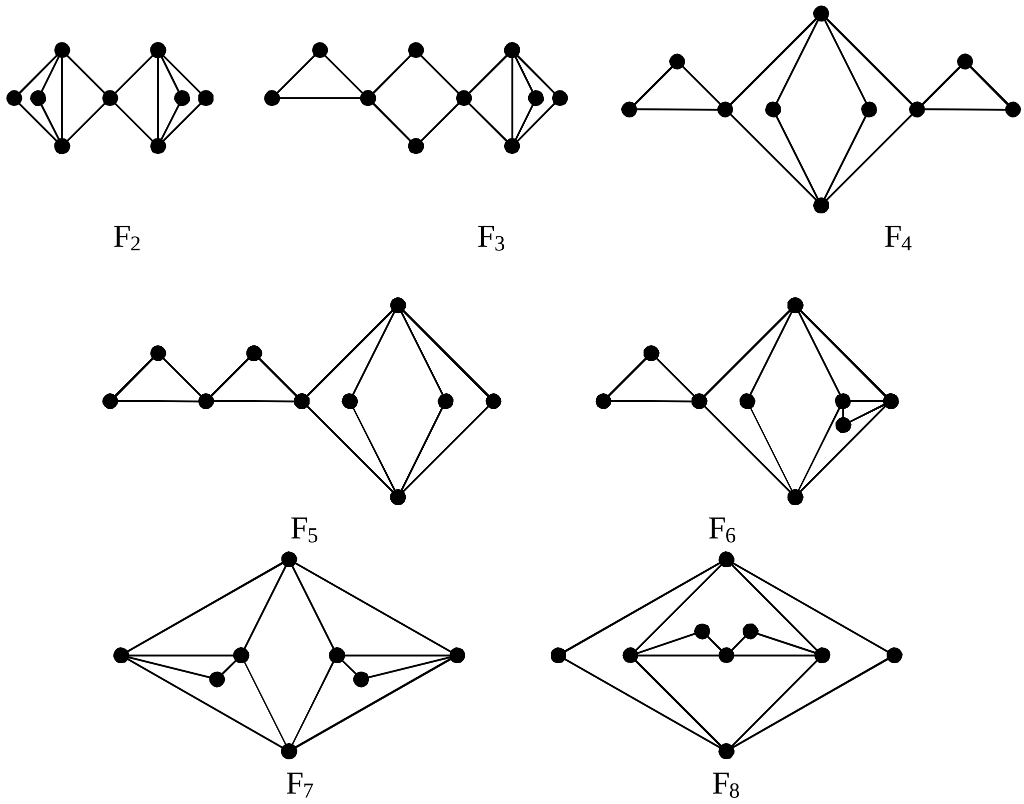


Fig. 2.12

In this case, the only randomly $C_4 \cup C_3$ – packable graphs are A_i $(2 \leq i \leq 7)$. Therefore, if G is randomly $C_4 \cup C_3$ – packable with exactly two copies of $C_4 \cup C_3$, then G is isomorphic to A_i $(1 \leq i \leq 7)$.

Now suppose G contains more than two copies of $C_4 \cup C_3$. Let H be a subgraph of G induced by two edge – disjoint copies of $C_4 \cup C_3$. If H is disconnected, then it will be either $2C_4 .C_3$, $C_4^{(2)} \cup C_3^{(2)}$ or $2C_4 \cup 2C_3$. Hence G contains a subgraph isomorphic to either L_1 , L_2 , M_1 , M_2 , N_1 , N_2 or N_3 , (given in fig. 2.13), which are not randomly $C_4 \cup C_3$ – packable.

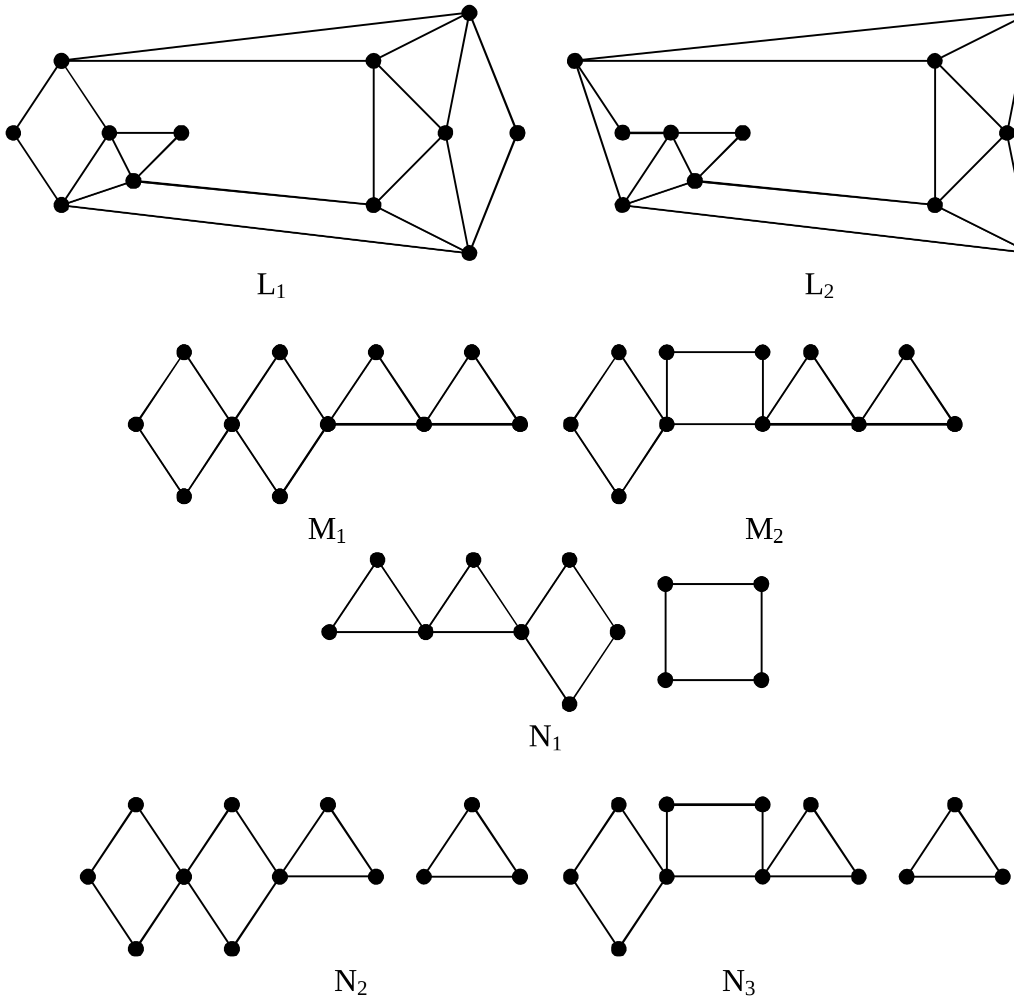


Fig. 2.13

Hence, *H* is connected and hence *H* is isomorphic to $A_i (1 \le i \le 7)$.

Thus the subgraph of *G* induced by any two edge – disjoint copies of $C_4 \cup C_3$ is isomorphic to
$A_i (1 \le i \le 7)$.

Let G_1 be a subgraph induced by *H* and *J*, where *J* is one copy of $C_4 \cup C_3$. If either $C_4 \cup C_3$ of *J* is disjoint from any C_4 or C_3 of *H* in G_1 , then G_1 will contain $C_4 \cup C_3$ – forbidden sub graph M_1 , M_2 , N_1 , N_2 or N_3 .

Therefore, if *G* is randomly $C_4 \cup C_3$ – packable with three edge disjoint copies of

$C_4 \cup C_3$, then G is isomorphic to $B_j\ (1 \le j \le 4)$, $C_k\ (1 \le k \le 4)$, $D_l\ (1 \le l \le 4)$ and it follows that G cannot have more than four edge – disjoint copies of $C_4 \cup C_3$. Hence if G is randomly $C_4 \cup C_3$ – packable with exactly four copies of $C_4 \cup C_3$, then it is isomorphic to $E_m\ (1 \le m \le 3)$. Hence, the theorem is proved.

Theorem: 2.3
The only randomly $C_5 \cup P_5$-packable graphs are $C_5 \cup P_5$ and $C_5^{(t)} \cup S_4^{(t)}\ (t \ge 2)$.

Proof:
Let G be a randomly $C_5 \cup P_5$ -packable graph. Let H_1 be any subgraph of G induced by two copies of $C_5 \cup P_5$. Let H_2 be any subgraph induced by the edges of two copies of P_5 in H_1. Then H_2 is isomorphic to C_8, $C_4^{(2)}$, $K_{2,4}$, $S_4^{(2)}$. Hence G is isomorphic to $C_5^{(2)} \cup C_8$, $C_5^{(2)} \cup C_4^{(2)}$, $C_5^{(2)} \cup K_{2,4}$ or $C_5^{(2)} \cup S_4^{(2)}$.

Suppose that G is packable by three copies of $C_5 \cup P_5$ .Then G is isomorphic to $C_5^{(3)} \cup C_{12}$, $C_5^{(3)} \cup C_4^{(3)}$, $C_5^{(3)} \cup K_{3,4}$ or $C_5^{(3)} \cup S_4^{(3)}$. But the first three cases are $C_5 \cup P_5$ - forbidden graphs. Hence, if G is isomorphic to $C_5^{(3)} \cup S_4^{(3)}$, G is randomly $C_5 \cup P_5$ -packable.

Hence, if G is packable by t copies of $C_5 \cup P_5$, it is isomorphic to $C_5^{(t)} \cup S_4^{(t)}$, $(t \ge 2)$.

The converse is obvious.

Theorem : 2.4
A graph G is randomly $C_4 \cup P_{2n+1}$- packable iff G is isomorphic to $K_{2,2t} \cup S_{2n}^{(t)}$ $(t \ge 2,\ n \ge 2)$.

Proof:
Suppose G is randomly $C_4 \cup P_{2n+1}$ -packable $(n \ge 2)$. Let J be any subgraph of G induced by the edges of two edge disjoint copies of $C_4 \cup P_{2n+1}$. Let K be any subgraph of G induced by two copies of P_{2n+1}. Then K is isomorphic to $S_{2n}^{(2)}$.

If G is packable by t copies of $C_4 \cup P_{2n+1}$, then the subgraph G_1 induced by t copies of C_4 is isomorphic to $K_{2,2t}$ and the subgraph induced by t copies of P_{2n+1} is disjoint from G_1 is isomorphic to $S_{2n}^{(t)}$ $(t \ge 2)$. Thus G is isomorphic to $K_{2,2t} \cup S_{2n}^{(t)}$ $(t \ge 2,\ n \ge 2)$.

The converse follows.

Theorem: 2.5
The only randomly $C_r \cup P_{2n+1}$ - packable graph is $C_r \cup P_{2n+1}$ and $C_r^{(t)} \cup S_{2n}^{(t)}$ $(r \ge 5,\ n \ge 2,\ t \ge 1,\ r \le 2n+1)$.

Proof:
Let G be a randomly $C_r \cup P_{2n+1}$ -packable graph $(r \ge 5,\ n \ge 2)$. Let F_1 be any subgraph of G induced by two copies of $C_r \cup P_{2n+1}$. Also let F_2 be any subgraph of F_1 induced by two edge disjoint copies of P_{2n+1}. It is clear that all graphs except C_{4n}, $P_{2n+1}^{(2)}$, $S_{2n}^{(2)}$ are $C_r \cup P_{2n+1}$ – forbidden.

Hence, F_2 is isomorphic to C_{4n}, $P_{2n+1}^{(2)}$, $S_{2n}^{(2)}$.

Assume that G is packable by t $(t \geq 3)$ copies of $C_r \cup P_{2n+1}$. Then G can occur the following ways $C_r^{(t)} \cup C_{2nt}$, $C_r^{(t)} \cup K_{t, 2n}$ or $C_r^{(t)} \cup S_{2n}^{(t)}$ of which the first two are not randomly $C_r \cup P_{2n+1}$-packable. Thus G is isomorphic to $C_r^{(t)} \cup S_{2n}^{(t)}$ $(r \geq 5$, $n \geq 2$, $t \geq 1)$.

Let us suppose that the graph is $C_r^{(t)} \cup S_{2n}^{(t)}$ $(r \geq 5$, $n \geq 2$, $t \geq 1)$. Then, it is very clear that the graph is $C_r \cup P_{2n+1}$ - packable randomly.

Theorem: 2.6

There is no randomly $C_r \cup P_{2n}$ - packable graph of more than two copies of it $(r \geq 4$, $n \geq 2$, $r \leq 2n)$.

Proof:

It is clear that $C_r \cup P_{2n}$ is randomly $C_r \cup P_{2n}$-packable graph. If G is isomorphic t $C_r^{(t)} \cup C_{2n-1}^{(t)}$ or $C_r^{(t)} \cup C_{2(2n-1)}$ if $t = 2$ and if $t > 2$, G contains a $C_r \cup P_{2n}$ – forbidden subgraph $(n \geq 2)$.

3. Conclusion:

In this paper, we obtained a characterization of random packing by two different graphs simultaneously and by some disconnected graphs.

4. References:

[1] S. Ruiz Discrete Math.57 (1985) 123 – 28.

[2] Lowell W. Beineke, and Peter Hamburger and D Wayne Goddard, Random Packing of Graphs, Discrete Mathematics 125 (1994) 45 – 54.

[3] S. Arumugham and S. Meena, Graphs that are randomly packable by some common disconnected graphs : Indian J. Pure and Applied Mathematics 29 (1998)1129 – 1136.

[4] R. Balakrishnan $ K. Ranganathan: A text book of Graph theory , Springer (2000) Publication.

[5] J. A. Bondy $ U. S. R. Murthy : Graph Theory with Application, Macmillan (1976) Publication.

[6] Hararay: Graph Theory, Addition – Wesley (1969) Publication.