Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 9, Number 4 (2013), pp. 345-351

http://www.ripublication.com/gjpam.htm

A New Class of Entire Sequences of Interval Numbers

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Abstract

In this paper we introduce the new concept of interval valued sequence space G_λ^i where (λ_k) is a fixed sequence of positive real numbers. We study the different properties like completeness, solidness, AB property etc.

2000 Mathematics subject classification 46A 45

Keywords and Phrases: Banach space, AB space, AK property

1. Introduction

Interval arithmetic was first suggested by Dwyer [11] in 1951. Development of interval arithmetic as a formal system and evidence of its value as a computational device was provided by Moore [8] in 1959 and Moore and Yang [7] 1962. Furthermore, Moore and others[9], [10]and [11] have developed applications to differential equations.

Chiao in [5] introduced sequence of interval numbers and defined usual convergence of sequences of interval number. Sengönül and Eryilmax [3] in 2010 introduced and studied bounded and convergent sequence space of interval numbers and showed that these spaces are complete metric space. Recently Esi [2] introduced a new class of interval numbers.

A set consisting of a closed interval of real numbers x such that $a \le x \le b$ is called an interval number. A real interval can also be considered as a set. Thus we can investigate some properties of interval numbers, for instance arithmetic properties or analysis properties. We denote the set of all real valued closed intervals by $I\Re$. Any elements of $I\Re$ is called closed interval and denoted by $\bar{x}$ .That is $\bar{x} = \{x \in \Re : a \le x \le b\}$. An interval number $\bar{x}$ is a closed subset of real numbers. Let

x_l *and* x_r be respectively first and last points of the interval number $\bar{x}$.

For $\bar{x}_1, \bar{x}_2 \in I\Re$, we define $\bar{x}_1 = \bar{x}_2$ if and only if $x_{1l} = x_{2l}$ *and* $x_{1r} = x_{2r}$

$$\bar{x}_1 + \bar{x}_2 = \{x \in \Re : x_{1l} + x_{2l} \le x \le x_{1r} + x_{2r})\}$$

$$\bar{x}_1 \times \bar{x}_2 = \{x \in \Re : \min(x_{1l}x_{2l}, x_{1l}x_{2r}, x_{1r}x_{2l}, x_{1r}x_{2r}) \le x \le \max(x_{1l}x_{2l}, x_{1l}x_{2r}, x_{1r}x_{2l}, x_{1r}x_{2r})\}$$

The set of all interval numbers $I\Re$ is a complete metric space defined by

$$d(\bar{x}_1, \bar{x}_2) = \max\{|x_{1l} - x_{2l}|, |x_{1r} - x_{2r}|\}$$

In the special case $\bar{x}_1 = [a,a]$ and $\bar{x}_2 = [b,b]$, we obtain usual metric of $\Re$.

Let us define transformation f:N$\rightarrow \Re$, k$\rightarrow$f(k)= $\bar{x}_k$, then $\bar{x} = (\bar{x}_k)$ is called sequence of interval numbers. $\bar{x}_k$ is called k[th] term of sequence $\bar{x} = (\bar{x}_k)$. ω^i denotes the set of all interval numbers with real terms and the algebraic properties of ω^i in [4].

A sequence $\bar{x} = (\bar{x}_k)$ of interval numbers is said to be convergent to the interval number $\bar{x}_0$ if for each ε>0 there exists a positive integer k$_0$ such that $d(\bar{x}_k, \bar{x}_0) < \varepsilon$ for all k$\ge$k$_0$ and we denote it by $\lim_k \bar{x}_k = \bar{x}_0$.Equivalently $\lim_k \bar{x}_k = \bar{x}_0$ iff $\lim_k x_{kl} = x_{0l}$ *and* $\lim_k x_{kr} = x_{0r}$.

Let us denote the space of all entire sequence of interval numbers by Γ^i. That is,

$$\Gamma^i = \{\bar{x} = (\bar{x}_k) \in \omega^i : \lim_{k\to\infty} \rho(\bar{x}_k, \bar{0}) = \bar{0}\}$$, where ρ is defined by

$$\rho(\bar{x}_k, \bar{y}_k) = \max\{|x_{kl} - y_{kl}|^{1/k}, |x_{kr} - y_{kr}|^{1/k}\} = [d(\bar{x}_k, \bar{y}_k)]^{1/k}$$, for each fixed k.

Let G^i_λ denote the class of all entire sequences of interval numbers $\bar{x} = (\bar{x}_k)$ such that $\sum_{k=1}^{\infty} \lambda_k d(\bar{x}_k, \bar{0})^2$ converges, where (λ_k) is a fixed sequence of positive real numbers such that $\lambda_k^{1/k} \to \infty$ as $k \to \infty$.That is,

$$G^i_\lambda = \{\bar{x} = (\bar{x}_k) : \sum_{k=1}^{\infty} \lambda_k d(\bar{x}_k, \bar{0})^2 < \infty,\ \lambda_k > 0, \lambda_k^{1/k} \to \infty \text{ as } k \to \infty\}$$

2. Main Results:

Theorem 2.1. The sequence space G^i_λ is a complete metric space with respect to the metric defined by

$$\bar{d}(\bar{x}, \bar{y}) = \sum_{k=1}^{\infty} \lambda_k d(\bar{x}_k, \bar{y}_k)^2 \qquad (2.1)$$

Proof: Let $(\bar{x}^n)$ be a Cauchy sequence in G^i_λ. Then for a given $\varepsilon > 0$ there exists n$_0$ $\in$N such that

$\bar{d}(\bar{x}^n, \bar{x}^m) < \varepsilon$ for all n, m $\geq$ n_0

then $\sum_{k=1}^{\infty} \lambda_k d(\bar{x}_k{}^n, \bar{x}_k{}^m)^2 < \varepsilon$ for all n, m $\geq$ n_0 (2.2)

$d(\bar{x}_k{}^n, \bar{x}_k{}^m)^2 \lambda_k < \varepsilon$ for all n, m $\geq$ n_0

$d(\bar{x}_k{}^n, \bar{x}_k{}^m)^2 < \varepsilon / \lambda_k$ for all n, m $\geq$ n_0 and for all k $\in$ N

$d(\bar{x}_k{}^n, \bar{x}_k{}^m) < \left(\frac{\varepsilon}{\lambda_k}\right)^{1/2} < \varepsilon$ for all n, m $\geq$ n_0 and for all k $\in$ N

This means that $(\bar{x}_k{}^n)$ is a Cauchy sequence in $I\Re$. Since $I\Re$ is a Banach space, $(\bar{x}_k{}^n)$ is convergent. Now, let $\lim_n \bar{x}_k{}^n = \bar{x}_k$ for each k $\in$ N and $\bar{x} = (\bar{x}_k)$

Taking limit as m $\to \infty$ in (2.2) we have

$\sum_{k=1}^{\infty} \lambda_k d(\bar{x}_k{}^n, \bar{x})^2 < \varepsilon$ for all n $\geq$ n_0

$\bar{d}(\bar{x}^n, \bar{x}) < \varepsilon$ for all n $\geq$ n_0

Now for all n $\geq$ n_0,

$\bar{d}(\bar{x}, 0) \leq \bar{d}(\bar{x}^n, \bar{x}) + \bar{d}(\bar{x}^n, 0) < \varepsilon + \infty = \infty$

Thus $\bar{x} = (\bar{x}_k) \in G_\lambda^i$ and so G_λ^i is complete. This completes the proof.

Theorem 2.2. G_λ^i is a subset of Γ^i

Proof: Let $\bar{x} \in G_\lambda^i$, then $\sum_{k=1}^{\infty} \lambda_k d(\bar{x}_k, \bar{0})^2 < \infty,\ \lambda_k{}^{1/k} \to \infty\ as\ k \to \infty$

Suppose $[d(\bar{x}_k, \bar{0})]^{1/k}$ does not converge to zero as $k \to \infty$. Then there exists a positive integer N_1 such that $0 < \frac{1}{\eta} < [d(\bar{x}_k, \bar{0})]^{1/k}$ for all k $\geq$ N_1 and for some $\eta > 0$

That is, $\frac{1}{\eta^{2k}} < [d(\bar{x}_k, \bar{0})]^2$ for all k $\geq$ N_1 and for some $\eta > 0$ (2.1)

Since $\lambda_k{}^{1/k} \to \infty$ as $k \to \infty$, there exists an integer N_2 such that

$\eta^{2k} \leq \lambda_k$ for all k $\geq$ N_2 and for any $\eta > 0$ (2.2)

Hence from (2.1) and (2.2)

$1 \leq \lambda_k [d(\bar{x}_k, \bar{0})]^2$ for all k $\geq$ max(N_1, N_2)

This implies that $\sum_{k=1}^{\infty}\lambda_k[d(\bar{x}_k,\bar{0})]^2$ diverges and so $\bar{x}\notin G_\lambda^i$.

This contradiction proves that $\bar{x}\in\Gamma^i$

Hence $G_\lambda^i \subseteq \Gamma^i$

Remark. G_λ^i is a Banach space with norm

$$\|\bar{x}\|_{G_\lambda^i} = \{\sum_{k=1}^{\infty}\lambda_k[d(\bar{x}_k,\bar{0})]^2\}^{1/2}$$

Theorem 2.3. If G_λ^i and G_μ^i are two entire sequences of interval numbers, then $G_\lambda^i = G_\mu^i$ if and only if $k_1 \le \frac{\lambda_k}{\mu_k} \le k_2$, where k_1 *and* k_2 are constants.

Proof: The sufficiency of the condition $k_1 \le \frac{\lambda_k}{\mu_k} \le k_2$ (2.3)

If $\lambda_k \le k_2\mu_k$ then $\lambda_k d(\bar{x}_k,\bar{0})]^2 \le k_2\mu_k d(\bar{x}_k,\bar{0})]^2$

If $(\bar{x}_k)\in G_\mu^i$, $\sum_{k=1}^{\infty}\mu_k d(\bar{x}_k,\bar{0})^2 < \infty$

Therefore $\sum_{k=1}^{\infty}\lambda_k d(\bar{x}_k,\bar{0})^2 \le \sum_{k=1}^{\infty}k_2\mu_k d(\bar{x}_k,\bar{0})^2 < \infty$

This implies that $(\bar{x}_k)\in G_\lambda^i$

Hence $G_\mu^i \subset G_\lambda^i$ (2.4)

If $k_1\mu_k \le \lambda_k$ then $k_1\mu_k d(\bar{x}_k,\bar{0})]^2 \le \lambda_k d(\bar{x}_k,\bar{0})]^2$

If $(\bar{x}_k)\in G_\lambda^i$, $\sum_{k=1}^{\infty}\lambda_k d(\bar{x}_k,\bar{0})^2 < \infty$

Therefore $\sum_{k=1}^{\infty}k_1\mu_k d(\bar{x}_k,\bar{0})^2 \le \sum_{k=1}^{\infty}\lambda_k d(\bar{x}_k,\bar{0})^2 < \infty$

This implies that $(\bar{x}_k)\in G_\mu^i$

Hence $G_\lambda^i \subset G_\mu^i$ (2.5)

From (2.4) and (2.5), $G_\lambda^i = G_\mu^i$

To prove the necessity of the condition, let us suppose that the condition is not satisfied. First consider the right hand side inequality of (2.3). Let $\frac{\lambda_k}{\mu_k}\to\infty$ *as* $k\to\infty$

.

Then it has a subsequence $\frac{\lambda_{k_n}}{\mu_{k_n}} \to \infty \ as \ k_n \to \infty$ in such a manner that $\frac{\lambda_{k_n}}{\mu_{k_n}} > n$ for the values n=1, 2, ... and $k_1 < k_2 < .....$

Now we shall define a metric as follows

$$d(\bar{x}_k, \bar{0})^2 = \begin{cases} \frac{1}{n^2 \mu_k} when \ k = k_n \\ 0 \ when \ k \neq k_n \end{cases}$$

Then $\sum_{k=1}^{\infty} \mu_k d(\bar{x}_k, \bar{0})^2 = \sum_{n=1}^{\infty} \mu_{k_n} d(\bar{x}_{k_n}, \bar{0})^2$

$$= \sum_{n=1}^{\infty} \frac{\mu_{k_n}}{n^2 \mu_{k_n}} = \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$$

Therefore $(\bar{x}_k) \in G_\mu^i$ (2.6)

But $\sum_{k=1}^{\infty} \lambda_k d(\bar{x}_k, \bar{0})^2 = \sum_{n=1}^{\infty} \lambda_{k_n} d(\bar{x}_{k_n}, \bar{0})^2$

$$= \sum_{n=1}^{\infty} \frac{\lambda_{k_n}}{n^2 \mu_{k_n}} > \sum_{n=1}^{\infty} \frac{n}{n^2} = \infty$$

Thus $\sum_{k=1}^{\infty} \lambda_k d(\bar{x}_k, \bar{0})^2 > \infty$

Therefore $(\bar{x}_k) \notin G_\lambda^i$ (2.7)

From (2.6) and (2.7) contradict (2.4)

Similarly, if the left hand side inequality of (2.3) is not satisfied, then we can contradict (2.5) by constructing a sequence of the above type.

Hence the condition $k_1 \leq \frac{\lambda_k}{\mu_k} \leq k_2$ is necessary and sufficient in order that $G_\lambda^i = G_\mu^i$

Theorem 2.4. If $(\bar{x}_k) \in G_\lambda^i$, then the sequence of partial sums of the series for $(\bar{x}_k)$ converges to $\bar{x} \ in \ G_\lambda^i$

Proof: Let $(\bar{x}^{[p]}) = (\bar{x}_1, \bar{x}_2, ..., \bar{x}_p)$

We have to prove $(\bar{x}^{[p]}) \to \bar{x} \ as \ p \to \infty$

It is enough to prove $\left\| (\bar{x}^{[p]}) - \bar{x} \right\| \to 0 \ as \ p \to \infty$ in G_λ^i

Consider $\left\| (\bar{x}^{[p]}) - \bar{x} \right\|^2 = \sum_{k=p+1}^{\infty} \lambda_k d(\bar{x}_k, \bar{0})^2$

Since $\sum_{k=1}^{\infty}\lambda_k d(\bar{x}_k,\bar{0})^2$ is convergent, $\sum_{k=p+1}^{\infty}\lambda_k d(\bar{x}_k,\bar{0})^2 \to 0\ \ as\ \ p\to\infty$ in G_λ^i

Therefore $(\bar{x}^{[p]}) \to \bar{x}\ \ as\ \ p\to\infty$ in G_λ^i

Theorem 2.5. G_λ^i is BK space.

Proof: For each $(\bar{x}_k)\in G_\lambda^i$,

$$\lambda_k d(\bar{x}_k,\bar{0})^2 \le \sum_{k=1}^{\infty}\lambda_k d(\bar{x}_k,\bar{0})^2 = \|\bar{x}\|^2_{G_\lambda^i}$$

This implies that $d(\bar{x}_k,\bar{0})^2 \le \frac{1}{\lambda_k}\|\bar{x}\|^2_{G_\lambda^i}$

$d(\bar{x}_k,\bar{0}) \le \frac{1}{\lambda_k^{1/2}}\|\bar{x}\|_{G_\lambda^i} \le M\|\bar{x}\|_{G_\lambda^i}$ where $M=\sup\{\frac{1}{\lambda_k^{1/2}}\}$ is independent of $\bar{x}$.

Hence G_λ^i is BK space.

Theorem 2.6. G_λ^i is an AK space.

Proof: For each $(\bar{x}_k)\in G_\lambda^i$, $\|(\bar{x}^{[n]})-\bar{x}\| \to 0\ \ as\ \ n\to\infty$ from theorem 2.4.

Hence G_λ^i has AK.

Theorem 2.7. G_λ^i has AB property.

Proof: It is enough to show that G_λ^i has monotone norm. Indeed for n<m and for every $(\bar{x}_k)\in G_\lambda^i$, we have $\|(\bar{x}^{[n]})\|^2 = \sum_{k=1}^{n}\lambda_k d(\bar{x}_k,\bar{0})^2 < \sum_{k=1}^{m}\lambda_k d(\bar{x}_k,\bar{0})^2 = \|(\bar{x}^{[m]})\|^2$

$$\|(\bar{x}^{[n]})\| < \|(\bar{x}^{[m]})\|$$

Also $\{\|(\bar{x}^{[n]})\|,\ n=1,2,\ldots\}$ is a monotonically increasing sequence of interval numbers bounded above by $\|\bar{x}\|_{G_\lambda^i}$

Hence $\|\bar{x}\|_{G_\lambda^i} = \lim_{n\to\infty}\|(\bar{x}^{[n]})\| = \sup_n\{\|(\bar{x}^{[n]})\|, n=1,2,\ldots\}$

Thus G_λ^i has monotone norm.

Theorem 2.8. The space G_λ^i is solid.

Proof: Let $(\bar{x}_k)$ and $(\bar{y}_k)$ be two sequences such that

$(\bar{x}_k) \in G_\lambda^i$ and $d(\bar{y}_k, \bar{0}) \le d(\bar{x}_k, \bar{0})$ for all $k \in N$

Since $(\bar{x}_k) \in G_\lambda^i$, we have $\sum_{k=1}^{\infty} \lambda_k d(\bar{x}_k, \bar{0})^2 < \infty$

Also we have $\lambda_k d(\bar{y}_k, \bar{0})^2 \le \lambda_k d(\bar{x}_k, \bar{0})^2$

$$\sum_{k=1}^{\infty} \lambda_k d(\bar{y}_k, \bar{0})^2 \le \sum_{k=1}^{\infty} \lambda_k d(\bar{x}_k, \bar{0})^2 < \infty$$

So $(\bar{y}_k) \in G_\lambda^i$. Therefore G_λ^i is solid.

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