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Absolutely Flatness of Modules in Exact Sequences

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Abstract

We prove that in a short exact sequence $0 \to M' \to M \to M'' \to 0$, of R- modules, M is absolutely flat if both M' and M'' are absolutely flat modules. Further if M' and M'' are finitely generated then the converse is also true. We prove that a Noetherian R-module M is absolutely flat over R if and only if, in a primary decomposition of zero submodule in M, M/N is absolutely flat for each primary submodule N occurs in the decomposition.

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Throughout this article we will consider only commutative rings with unity and all modules are unital. For standard terminology, the reference will be [1] and [3]. The concept of absolutely flat ring is extended to modules as absolutely flat modules and studied in [2]. An R module M is said to be absolutely flat module if for every R-module N, $M \otimes_R N$ is flat R-module. In this paper we study the influence of absolute flatness of a module in a short exact sequence to other modules in the sequence. We also extend the following well known result to absolutely flat modules.

Proposition 1. Let

$$0 \to E' \to E \to E'' \to 0$$

be an exact sequence of R-modules and E'' is flat. Then E is flat if and only if E' is flat.

We need the following characterization of finitely generated absolutely flat modules.

Proposition 2. Let M be a finitely generated R-module. Then M is absolutely flat over R if and only if $R_{\mathfrak{m}}$ is a field for every maximal ideal $\mathfrak{m} \in \mathrm{Supp}(M)$.

Proof. See [2, theorem 2.2]. ∎

Theorem 3. Let

$$0 \to M' \to M \to M'' \to 0$$

be an exact sequence of R-modules. Then M is absolutely flat over R if both M' and M'' are absolutely flat over R. The converse holds if both M' and M'' are finitely generated over R.

Proof. Let M' and M'' be absolutely flat R -modules. Let N be any R-module. Then we have the long exact sequence

$$\cdots \to \mathrm{Tor}_1^R(M, N) \to \mathrm{Tor}_1^R(M'', N) \to M' \otimes_R N \to M \otimes_R N \to M'' \otimes_R N \to 0.$$

Since M'' is R-flat, $\mathrm{Tor}_1^R(M'', N) = 0$. Then the sequence

$$0 \to M' \otimes_R N \to M \otimes_R N \to M'' \otimes_R N \to 0$$

is exact. Now for any R-module K, we have the long exact sequence

$$\cdots \to \mathrm{Tor}_1^R(M' \otimes_R N, K) \to \mathrm{Tor}_1^R(M \otimes_R N, K) \to \mathrm{Tor}_1^R(M'' \otimes_R N, K) \to \cdots .$$

Since M' and M'' are absolutely flat over R, we have $M' \otimes_R N$ and $M'' \otimes_R N$ as R-flat and hence $\mathrm{Tor}_1^R(M' \otimes_R N, K) = 0$ and $\mathrm{Tor}_1^R(M'' \otimes_R N, K) = 0$. So, for every R-module K, $\mathrm{Tor}_1^R(M \otimes_R N, K) = 0$ and hence $M \otimes_R N$ is flat R-module. This proves that M is a absolutely flat over R.

Conversely, we assume that M is absolutely flat R-module and M' and M'' are finitely generated R-modules. Since M is finitely generated absolutely flat over R-module, by proposition 2, $R_{\mathfrak{m}}$ is a field for every maximal ideal $\mathfrak{m} \in \mathrm{Supp}\ M$. The exactness of the sequence

$$0 \to M' \to M \to M'' \to 0$$

implies that Supp $M = $ Supp $M' \cup$ Supp M''. So $R_{\mathfrak{m}}$ is a field for every maximal ideal $\mathfrak{m} \in$ Supp M' and for every maximal ideal $\mathfrak{m} \in$ SuppM''. Then by proposition 2, both M' and M'' are absolutely flat R-modules. ∎

Theorem 4. Let

$$0 \to M' \to M \to M'' \to 0$$

be an exact sequence of R- modules with M'', absolutely flat over R. Then M is absolutely flat over R if and only if M' is absolutely flat over R.

Proof. Consider the exact sequence $0 \to M' \to M \to M'' \to 0$ with M'' as an absolutely flat R -module. If M' is absolutely flat over R, then by theorem 3, M is absolutely flat over R.

Conversely, assume M is absolutely flat over R. Let N be any R-module. We get the long exact sequence

$$\cdots \to \mathrm{Tor}_1^R(M, N) \to \mathrm{Tor}_1^R(M'', N) \to M' \otimes_R N \to M \otimes_R N \to M'' \otimes_R N \to 0.$$

Since M'' is R-flat, from the above sequence we get $0 \to M' \otimes_R N \to M \otimes_R N \to M'' \otimes_R N \to 0$ as exact sequence. Now $M \otimes_R N$ and $M'' \otimes_R N$ are R-flat. Then by proposition 1, $M' \otimes_R N$ is R-flat and this proves that M' is absolutely flat over R. ∎

Theorem 5. Let M be Noetherian R-module. Then M is absolutely flat over R if and only if in a primary decomposition of zero submodule of M, M/N is absolutely flat for each primary submodule N occurs in the decomposition.

Proof. Let

$$(0) = N_1 \bigcap N_2 \bigcap N_3 \cdots \bigcap N_n$$

be a primary decomposition of the submodule (0) in M.

Let M be absolutely flat over R. Since M is Noetherian, by theorem 3, M/N_i is absolutely flat over R for each i.

Conversely, assume that for each i, M/N_i is absolutely flat over R. Then we have the R-module monomorphism

$$M \to M/N_1 \oplus M/N_2 \oplus \cdots \oplus M/N_n.$$

Since by assumption, each M/N_i is absolutely flat, we have that $M/N_1 \oplus M/N_2 \oplus \cdots \oplus M/N_n$ is absolutely flat and finitely generated R-module. Then by theorem 3, M is is absolutely flat over R. ∎

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