Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 9, Number 4 (2013), pp. 335-340

http://www.ripublication.com/gjpam.htm

Relative Topological Entropy of Continuous Functions on Compact Topological Spaces

Mohamad Hosein Asadian and Abolfazl Ebrahimzadeh

Department of Mathematics, Islamic Azad University,
Bandar Abbas Branch, Bandar Abbas, Iran.
Department of Mathematics, Islamic Azad University,
Zahedan Branch, Zahedan, Iran.

Abstract

In this paper, we introduce a new type of topological entropy of a continuous function on compact space with open covers, and investigate several of its properties. We also express the relationship between this new concept of topological entropy and the classic topological entropy [10].

M.S.C. 2010: 54E45; 37B40; 54F65.

Keywords: Compact space; Topological entropy; Open cover.

1 Introduction and preliminiaries

An important application of covering properties of topological spaces is topological entropy. In 1965, the concept of topological entropy of continuous functions for a compact dynamical system on general topological spaces is introduced by Adler in [1]; subsequently, Liu generalized this in [2] for an arbitrary dynamical system. Bowen [3], CÃ¡novas and RodrÃ-guez [4], Goodwyn [5, 6], Kwietniak and Oprocha [7], and Thomas [8] have studied the concept of topological entropy. The concept of topological entropy due to Adler et al. [1] is generalized in [9] by Bowen. In this paper a generalization of entropy of a continuous function on compact space with open covers is introduced, and several of its properties are investigated.

Let X be a compacttopological space and ξ, γ be two open covers of X. Their join is the open cover $\xi \vee \gamma = \{B \cap C: B \in \xi, C \in \gamma\}$.

Definition 1.1. Let ξ, γ be two open covers of X. γ is a refinement of ξ, denoted by $\xi \prec \gamma$, if every member of γ is a subset of a member of ξ.

Hence we have $\xi \prec \xi \vee \gamma$ for any open covers ξ and γ. If γ is a subcover of ξ, then $\xi \prec \gamma$.

Definition 1.2. If γ is an open cover of X. The entropy of γ is defined in [10] by $H(\gamma)=\log N(\gamma)$, where $N(\gamma)$ is the number of sets in finite subcover of ξ with smallest cardinality.

Proposition 1.3. Let ξ, γ be two open covers of X. If $\xi \prec \gamma$ then $H(\xi) \leq H(\gamma)$.

Proposition 1.4. If $T:X \longrightarrow X$ is a continuous function then $H(T^{-1}\gamma) \leq H(\gamma)$. If T is also surjective then $H(T^{-1}\gamma)=H(\gamma)$.

Definition 1.5. If $T:X \longrightarrow X$ is continuous and ξ is an open cover of X. The entropy of T relative to ξ is defined in [10] by, $h(T,\xi) = \lim_{n\to\infty} \frac{1}{n} H\left(\vee_{i=0}^{n-1} T^{-i}\xi\right)$.

Definition 1.6. If $T:X \longrightarrow X$ is continuous. The topological entropy of T is defined by $h(T) = \sup_\xi h(T,\xi)$, where the supremum is taken over all open covers of X.

2 Relative entropy of an open cover of X

Definition 2.1. Let ξ, γ be two open covers of X with $N(\xi) \leq N(\gamma)$. The relative entropy of γ with respect to ξ is denoted by $H_\xi(\gamma)$ and is defined by $H_\xi(\gamma) = \log \frac{N(\gamma)}{N(\xi)}$.

Proposition 2.2 Let ξ, η, γ be open covers of X and $T:X \longrightarrow X$ be continuous and serjective. Then the following hold.

i. $H_\xi(\gamma) \geq 0$;

ii. $H_\xi(\gamma)=H(\gamma)-H(\xi)$;

iii. $H_{\eta^0}(\xi)=H(\xi)$, where $\eta^0 = \{X\}$;

iv. $\xi \prec \gamma$ implies that $H_\eta(\xi) \leq H_\eta(\gamma)$;

v. $\xi \prec \eta$ implies that $H_\xi(\gamma) \geq H_\eta(\gamma)$;

vi. $H_{T^{-1}\eta}(\xi)=H_\eta(\xi)$.

Proof.

i. Since $N(\xi) \leq N(\gamma)$, $H_\eta(\xi) \geq 0$.

ii. $H_\xi(\gamma) = \log \frac{N(\gamma)}{N(\xi)} = \log N(\gamma) - \log N(\xi) = H(\gamma) - H(\xi)$.

iii. $H(\eta^0)=0$. This together with ii) proves iii).

iv. By proposition 1.3, $\xi \prec \gamma$ implies that $H(\xi) \leq H(\gamma)$. So by ii) we have, $H_\gamma(\xi)=H(\xi)-H(\eta) \leq H(\gamma)-H(\eta)=H_\eta(\gamma)$.

v. By proposition 1.3, $\xi \prec \eta$ implies that $H(\xi) \leq H(\eta)$. So by ii) we have, $H_\xi(\gamma)=H(\gamma)-H(\xi) \geq H(\gamma)-H(\eta)=H_\eta(\gamma)$.

vi. Since $T:X \longrightarrow X$ is continuous and surjective, by proposition 1.4 we have, $H(T^{-1}\eta)=H(\eta)$. Therefore by ii), $H_{T^{-1}\eta}(\xi)=H(\xi)-H(T^{-1}\eta)=H(\xi)-H(\eta)=H_\eta(\xi)$.

Corollary 2.3. Let ξ, η, γ be open covers of X. Then $H_{\eta\xi}(\gamma) \leq \min\{H_\eta(\gamma), H_\xi(\gamma)\}$.

Proof. Since $\eta \prec \eta \vee \xi$ and $\xi \prec \eta \vee \xi$, it follows from ii).

Proposition 2.4. Let ξ, η, γ be open covers of X. Then $H_\eta(\gamma)=H_\eta(\xi)+H_\xi(\gamma)$.

Proof. By proposition 2.2 ii), we have,
$H_\eta(\gamma)=H(\gamma)-H(\eta)=H(\xi)-H(\eta)+H(\gamma)-H(\xi)=H_\eta(\xi)+H_\xi(\gamma)$.

Definition 2.5. Let ξ, γ be open covers of X with $N(\gamma) \leq N(\xi \vee \gamma)$. We define the conditional entropy by, $H(\xi|\gamma) = \log\frac{N(\xi\vee\gamma)}{N(\gamma)}$.

Proposition 2.6. Let ξ, γ, δ be open covers of X. Then

i. $H(\xi|\gamma) \geq 0$;

ii. $H(\xi|\gamma)=H(\xi\vee\gamma)-H(\gamma)$;

iii. $H(\xi\vee\gamma|\delta)=H(\xi|\delta)+H(\gamma|\xi\vee\delta)$;

iv. $H(\xi\vee\gamma)=H(\xi)+H(\gamma|\xi)$;

v. $\xi \prec \gamma$ implies that $H(\xi|\delta) \leq H(\gamma|\delta)$.

Proof.

i. By ussing the definition it is proved.

ii. $H(\xi|\gamma) = \log\frac{N(\xi\vee\gamma)}{N(\gamma)} = \log N(\xi \vee \gamma) - \log N(\gamma) = H(\xi \vee \gamma) - H(\gamma)$.

iii. $H(\xi\vee\gamma|\delta)= \log\frac{N(\xi\vee\gamma\vee\delta)}{N(\delta)} = \log(\frac{N(\xi\vee\delta)}{N(\delta)}.\frac{N(\gamma\vee\xi\vee\delta)}{N(\xi\vee\delta)}) = H(\xi|\delta)+H(\gamma|\xi\vee\delta)$;

iv. $H(\xi \vee \gamma) = \log N(\xi \vee \gamma) = \log\left(N(\xi).\frac{N(\gamma\vee\xi)}{N(\xi)}\right) = H(\xi) + H(\gamma|\xi)$;

v. By proposition 1.3, $\xi\vee\delta \prec \gamma\vee\delta$ implies that $H(\xi\vee\delta) \leq H(\gamma\vee\delta)$. So by ii) we have,
$H(\xi|\delta)=H(\xi\vee\delta)-H(\delta) \leq H(\gamma\vee\delta)-H(\delta)=H(\gamma|\delta)$.

Definition 2.7. Let ξ, γ and η be open covers of X with $N(\xi\vee\gamma) \geq N(\eta)N(\gamma)$. We define $H_\eta(\xi|\gamma) = \log\frac{N(\xi\vee\gamma)}{N(\eta)N(\gamma)}$.

Proposition 2.8. If ξ, γ and η be open covers of X. Then

i. $H_\eta(\xi|\gamma) \geq 0$;
ii. $H_\eta(\xi|\gamma) = H(\xi|\gamma) - H(\eta)$;
iii. $H_{\eta^0}(\xi|\gamma) = H(\xi|\gamma)$, where $\eta^0 = \{X\}$;
iv. $H_\eta(\xi \vee \gamma|\delta) = H(\xi|\delta) + H_\eta(\gamma|\xi \vee \delta)$;
v. $H_\eta(\xi \vee \gamma) = H(\xi) + H_\eta(\gamma|\xi)$;
vi. $\xi \prec \gamma$ implies that $H_\eta(\xi|\delta) \leq H_\eta(\gamma|\delta)$;
vii. If T:X⟶X is continuous and surjective, then $H_{T^{-1}\eta}(\xi|\gamma) = H_\eta(\xi|\gamma)$;

Proof.
i. By definition it is proved.
ii. $H_\eta(\xi|\gamma) = \log\frac{N(\xi\vee\gamma)}{N(\eta)N(\gamma)} = \log\frac{N(\xi\vee\gamma)}{N(\gamma)} - \log N(\eta) = H(\xi|\gamma) - H(\eta)$;
iii. Since $H(\eta^0)=0$, by ii) it is proved.
iv. $H_\eta(\xi \vee \gamma|\delta) = \log\frac{N(\xi\vee\gamma\vee\delta)}{N(\eta)N(\delta)} = \log\frac{N(\xi\vee\delta)}{N(\delta)} + \log\frac{N(\gamma\vee\xi\vee\delta)}{N(\eta)N(\xi\vee\delta)} = H(\xi|\delta) + H_\eta(\gamma|\xi \vee \delta)$v) $H_\eta(\xi \vee \gamma) = \log\frac{N(\xi\vee\gamma)}{N(\eta)} = \log N(\xi) + \log\frac{N(\gamma\vee\xi)}{N(\eta)N(\xi)} = H(\xi) + H_\eta(\gamma|\xi)$;
v. By ii) and proposition 2.6 v), it is proved.
vi. Since T:X⟶X is continuous and surjective, by proposition 1.4, we have $H(T^{-1}\eta)=H(\eta)$. Therefore by ii),
$H_{T^{-1}\eta}(\xi|\gamma) = H(\xi|\gamma) - H(T^{-1}\eta) = H(\xi|\gamma) - H(\eta) = H_\eta(\xi|\gamma)$.

3 Relative entropy of a continuous function

Proposition 3.1. Let T:X⟶X be a continuous function. If η, ξ, are open covers of X. Then $\lim_{n\to\infty}\frac{1}{n}H_\eta(\vee_{i=0}^{n-1} T^{-i}\xi)$ exists.

Proof. Let $a_n = H_\eta\left(\vee_{i=0}^{n-1} T^{-i}\xi\right) \geq 0$.Then

$$H_\eta\left(\vee_{i=0}^{n+p-1} T^{-i}\xi\right) \leq H_\eta\left(\vee_{i=0}^{n-1} T^{-i}\xi\right) + H_\eta\left(\vee_{i=n}^{n+p-1} T^{-i}\xi\right) = a_n + a_p.$$

So $a_{n+p} \leq a_n + a_p$, ∀n, p. Hence $\lim_{n\to\infty}\frac{a_n}{n}$ exists.

Definition 3.2. Let T:X⟶X be a be a continuous function. If η, ξ are open covers of X. we define generalized entropy of T relative to ξ by $h_\eta(T,\xi) = \lim_{n\to\infty}\frac{1}{n}H_\eta\left(\vee_{i=0}^{n-1} T^{-i}\xi\right)$.

Proposition 3.3. Let T:X⟶X be a continuous function. If η, ξ are open covers of X. Then the following hold.
i. $\eta \prec \xi$ implies that $h_\eta(T,\gamma) \geq h_\xi(T,\gamma)$;
ii. $h_{T^{-1}\eta}(T,\xi) = h_\eta(T,\xi)$;

iii. $h_{\eta^0}(T,\xi) = h_\eta(T,\xi)$, where $\eta^0 = \{X\}$;

iv. $h_\eta(T,\xi) \le h(T,\xi)$.

Proof.

i. $\eta \prec \xi$ implies that $\vee_{i=0}^{n-1} T^{-i}\eta \prec \vee_{i=0}^{n-1} T^{-i}\xi$, $n \ge 1$. So by Proposition 2.2(v) it is proved.

ii. By definition 3.2 and by proposition 2.3, are implied.

iii. Because $\eta^0 \prec \eta$, by i) and iii) we have, $h_\eta(T,\xi) \le h_{\eta^0}(T,\xi) = h(T,\xi)$.

Definition 3.4. Let T:X⟶X be a continuous function and η be an open cover of X. The generalized entropy of T is defined as $h_\eta(T) = \sup_\xi h_\eta(T,\xi)$ where the supremum is taken over all open covers of X.

Corollary 3.5. Let T:X⟶X be a continuous function and ξ, η be open covers of X. Then

i. $\eta \prec \xi$ implies that $h_\eta(T) \ge h_\xi(T)$;

ii. $h_{T^{-1}\eta}(T) = h_\eta(T)$;

iii. $h_{\eta^0}(T) = h(T)$, where $\eta^0 = \{X\}$;

iv. $h_\eta(T) \le h(T)$.

Proof. By definition 3.4 and Proposition 3.3, it is proved. Also see [10].

Corollary 3.6. If T:X⟶X is a continuous function. Then $h(T) = \sup_\eta h_\eta(T)$.

Proof. By corollary 3.5(iv) we have $h_\eta(T) \le h(T)$. So, $\sup_\eta h_\eta(T) \le h(T)$. On the other hand by corollary 3.5 iii), $h(T) = h_{\eta^0}(T) \le \sup_\eta h_\eta(T)$. Hence $h(T) = \sup_\eta h_\eta(T)$.

Proposition 3.7. Let T:X⟶X be a continuous function. Then

i. $h_\eta(id) = 0$;

ii. For $k \ge 1$, $h_\eta(T^k) = k h_\eta(T)$.

Proof.

i. By definition we have, $\vee_{i=0}^{n-1} T^{-i}\xi = \xi$, for any n∈N. Hence,

$$h_\eta(id,\xi) = \lim_{n\to\infty} \frac{1}{n} H_\eta\left(\vee_{i=0}^{n-1} T^{-i}\xi\right) = 0.$$

ii. Let ξ be an arbitrary countable partition. We have,

$$h_\eta\left(T^k, \vee_{i=0}^{n-1} T^{-i}\xi\right) = \lim_{n\to\infty} \frac{1}{n} H_\eta\left(\vee_{j=0}^{n-1} \left(T^k\right)^{-j}\left(\vee_{i=0}^{k-1} T^{-i}\xi\right)\right)$$

$$= \lim_{n\to\infty} \frac{1}{n} H_\eta\left(\vee_{j=0}^{n-1} \vee_{j=0}^{k-1} T^{-(kj+i)}\right)$$
$$= \lim_{n\to\infty} \frac{1}{n} H_\eta\left(\vee_{i=0}^{nk-1} T^{-i}\xi\right)$$
$$= \lim_{n\to\infty} \frac{nk}{n}\frac{1}{nk} H_\eta\left(\vee_{i=0}^{nk-1} T^{-i}\xi\right)$$
$$= kh_\eta(T,\xi)$$

So, $kh_\eta(T) = k\sup_\xi h_\eta(T,\xi) = \sup_\xi h_\eta\left(T^k, \vee_{i=0}^{n-1} T^{-i}\xi\right) \le \sup_\xi h_\eta\left(T^k,\xi\right) = h_\eta\left(T^k\right)$. Since, $\xi \prec \vee_{i=0}^{n-1} T^{-i}\xi$ we have, $h_\eta\left(T^k,\xi\right) \le h_\eta\left(T^k, \vee_{i=0}^{n-1} T^{-i}\xi\right) = kh_\eta(T,\xi)$.

4 Conclusion

This paper has defined a generalization of topological entropy of a continuous function on compact space with open covers. We expressed the relationship between this new concept of entropy and the classic topological entropy [10].

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