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Curvature Tensors on Three-Dimensional Trans-Sasakian Manifolds

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Abstract

The object of this paper is to study the geometry of three dimensional trans-Sasakian manifold when it is conformally semi-symmetric, concircularly semi-symmetric.

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1. Introduction

In 1985, J.A.Oubina [9]introduced a new class of almost contact manifold.An almost contact metric structure on a manifold M is called a trans-Sasakian structure if the product manifold M × R belongs to the class W_4.It is Known that [8] trans-Sasakian structures of type (0, 0), (0, β) and (α, 0) are cosymplectic, $\beta-$Kenmotsu and $\alpha-$Sasakian respectively. Many geometers like[2, 3, 6, 9], have studied this manifold and obtained many interesting results.

The notion of semi-symmetric manifold is defined by R(X, Y).R = 0and studied by author[11].

The conditions R(X, Y).P = 0, R(X, Y).C = 0 and R(X, Y).$\tilde{C}$ = 0 are called projectively semi-symmetric, conformal semi-symmetric and concircularly semi-symmetric respectively, where R(X, Y) is considered as derivation of tensor algebra at each point of the manifold.

The Local structure of trans-Sasakian manifolds of dimension n ≥ 5 has been completely characterized by J.C.Marrero[10].He proved that a trans-Sasakian manifold of dimension n ≥ 5 is either cosymplectic or $\beta -$Kenmotsu or $\alpha -$Sasakian manifold. But when n > 3 trans-Sasakian manifold does not exist. In this paper we consider the three dimensional trans-Sasakian manifold under the condition ϕ(gradα) = gradβ satisfying the properties R(X, Y).C = 0, R(X, Y)$\tilde{C}$ = 0 and shown that such a manifold is either Einstein or $\eta -$Einstein

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2. Preliminaries

Let M be connected almost contact metric manifold with an almost contact metric structure (ϕ,ξ,η,g),, that is ϕ is an (1, 1) tensor field, ξ is a vector field, η is an 1−form, and g is compatible Riemannian metric such that[4]

(2.1) $$\phi\xi = 0,\ \ \eta(\phi X) = 0,\ \ \phi^2 X = -X + \eta(X)\xi,$$

(2.2) $$g(\phi X, Y) = -g(X, \phi Y),\ \ \eta(\xi) = 1,\ \ g(X,\xi) = \eta(X),$$

(2.3) $$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y)$$

for all vector fields X, Y on M.The fundamental 2-form φ of the manifold is defined by

φ(X, Y) = g(X, ϕ Y)

for X, Y on M An almost contact metric structure (ϕ,ξ,η,g) on a connected manifold M is called trans-Sasakian structure[9] if (M × R, J, G) belongs to the class W_4 of the Hermitian manifolds, where J is the almost complex structure on M × R defined by

$$J(Z, f\frac{d}{dt}) = (\phi Z - f\xi, \eta(Z)\frac{d}{dt})$$

for any vector field Z on M and smooth function f on M × R and G is product metric on M × R.This may be stated by the condition[5]

(2.4) $$(\nabla_X \phi)(Y) = \alpha\{g(X,Y)\xi - \eta(Y)X\} + \beta\{g(\phi X, Y)\xi - \eta(Y)\phi X\},$$

where α, β are smooth functions on M and such a structure is said to be the trans-Sasakian structure of type (α, β). From (2.4) it follows that

(2.5) $$\nabla_X \xi = -\alpha\phi X + \beta\{X - \eta(X)\xi\},$$

(2.6) $$(\nabla_X \eta)(Y) = -\alpha g(\phi X, \phi Y).$$

Trans-Sasakian manifolds have been studied by authors [6] and they obtained the following results

(2.7) $$R(X,Y)\xi = (\alpha^2 - \beta^2)[\eta(Y)X - \eta(X)Y] - (X\alpha)\phi Y - (X\beta)\phi^2(Y) + 2\alpha\beta[\eta(Y)\phi X - \eta(X)\phi Y] + (Y\alpha)\phi X - (Y\beta)\phi^2(X)$$

(2.8) $$S(X,\xi) = [2(\alpha^2 - \beta^2) - (\xi\beta)]\eta(X) - (\phi X)\alpha - (X\beta)$$

(2.9) $$R(\xi, X)\xi = (\alpha^2 - \beta^2 - \xi\beta)[\eta(X)\xi - X],$$

(2.10) $S(\xi,\xi) = 2(\alpha^2 - \beta^2 - \xi\beta),$

(2.11) $(\xi\alpha) + 2\alpha\beta = 0$

(2.12) $Q\xi = [2(\alpha^2 - \beta^2) - (\xi\beta)]\xi + \phi(grad\alpha) - (grad\beta),$

where R is the curvature tensor of type(1, 3) of the manifold and Q is the symmetric endomorphism of the tangent space at each point of the manifold corresponding to the Ricci tensor S, that is, g(QX, Y) = S(X, Y) for any vector fields X, Y on M.
when $\phi(grad\alpha) = (grad\beta)$ (2.8) and(2.12) reduce to

(2.13) $S(X,\xi) = 2(\alpha^2 - \beta^2)\eta(X),$

(2.14) $Q\xi = 2(\alpha^2 - \beta^2)\xi$,

3. Conformal flat 3-dimensional trans-Sasakian manifold

The Conformal curvature C is defined as

$$C(X, Y)Z = R(X, Y)Z - [S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX - g(X, Z)QY] + \frac{r}{2}[g(Y, Z)X - g(X, Z)Y] \quad (3.1)$$

where R is the curvature tensor and S is the Ricci tensor.

Suppose that C = 0 then from(3.1)we have

$$R(X, Y)Z = S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX - g(X, Z)QY - \frac{r}{2}[g(Y, Z)X - g(X, Z)Y] \quad (3.2)$$

Putting z = ξ and taking inner product with W in (3.2) and using (2.7) and (2.13) we obtain

$$\begin{aligned} -(X\alpha)g(\phi Y,W)+(X\beta)g(Y,W)(-X\beta)\eta(Y)\eta(W)+2\alpha\beta[\eta(Y)g(\phi X,W)-\eta(X)g(\phi Y,W)] \\ +(Y\alpha)g(\phi X,W)-(Y\beta)[g(X,W)-\eta(X)\eta(W)] \\ =(\alpha^2-\beta^2)[\eta(Y)g(X,W)-\eta(X)g(Y,W)\}+\eta(Y)g(QX,W) \\ -\eta(X)g(QY,W)-\frac{r}{2}[g(X,W)\eta(Y)-g(Y,W)\eta(X)] \end{aligned} \quad (3.3)$$

Taking X = ξ in (3.3) and then using (2.1), (2.2) and (2.13) we obtain

(3.4) $S(Y,W) = [3(\alpha^2 - \beta^2) - \frac{r}{2}]\eta(Y)\eta(W) - [\alpha^2 - \beta^2 + \frac{r}{2}]g(Y,W)$

Hence (3.4) leads to the following theorem

Theorem 3.1. A conformal flat 3-dimensional trans-Sasakian manifold is an $\eta-$ Einstein manifold.

4. Three dimensional trans-Sasakian manifold satisfying R(X, Y).C = 0

In this section we consider a 3-dimensional trans-Sasakian manifold satisfying the condition

(4.1) R(X, Y).C = 0

Using (2.2), (2.7) in (3.1), we get

$$\eta(C(X,Y)Z) = (\alpha^2 - \beta^2 + \frac{r}{2})[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)] \tag{4.2}$$
$$-[S(Y,Z)\eta(X) - S(X,Z)\eta(Y) + g(Y,Z)S(X,\xi) - g(X,Z)S(Y,\xi)$$

for Z = ξ, we get from (4.2)

(4.3) η (C(X, Y)ξ) = 0

Again putting X = ξ in (4.2) we obtain

$$\eta(C(\xi,Y)Z) = (\frac{r}{2} - (\alpha^2 - \beta^2))g(Y,Z) + (3(\alpha^2 - \beta^2) - \frac{r}{2})\eta(Y)\eta(Z) - S(Y,Z) \tag{4.4}$$

In virtue of (4.1) we get

R(X, Y)C(U, V)Z − C(R(X, Y)U, V)Z
(4.5) −C(U, R(X, Y)V)Z − C(U, V)R(X, Y)Z = 0

Therefore,

g[R(ξ , Y)C(U, V)Z, ξ] − g[C(R(ξ , Y)U, V)Z, ξ]
−g[C(U, R(ξ , Y)V)Z, ξ] − g[C(U, V)R(ξ , Y)Z, ξ] = 0

From this, it follows that

− C(U, V, Z, U) + η (Y)η (C(U, V)Z) − η (U)η (C(Y, V)Z)
+g(Y, U)η (C(ξ , V)Z) −η (V)η (C(U, Y)Z) + g(Y, V)η (C(U, ξ)Z)
(4.6) −η (Z)η (C(U, V)Y) + g(Y, Z) η (C(U, V)ξ) = 0

where −C(U, V, Z, Y) = g(C(U, V)Z, Y)

Putting Y = U in (4.6) and with the help of (4.2) and(4.3) we get

C(U, V, Z, U) + g(U, U)η (C(ξ , V)Z)
g(U, V)η (C(U, ξ)Z) −η (Z)η (C(U, V)U) = 0

If $e_1, e_{2,} e_3$ are local orthonormal basis of the tangent space at any point then the sum for $1 \le i \le 3$ of the relation (4.7) for U = e_i yields

$$S(V,Z) = [\frac{r}{2} - (\alpha^2 - \beta^2)]g(V,Z) + [3(\alpha^2 - \beta^2) - \frac{r}{2}]\eta(Z)\eta(V) \tag{4.8}$$

Hence (4.8) leads to the following theorem:

Theorem 4.2. A conformal semi-symmetric 3-dimensional trans-Sasakian manifold is an η −Einstein manifold.

Now contracting (4.8) we get r = 0 by making use of this and (4.8), (4.2), (4.3) into the equation (4.6) we get

(4.9) $$-C(U, V, Z, Y) = 0$$

From (4.9) it follows that

(4.10) $$C(U, V)Z = 0$$

Therefore, 3-dimensional trans-Sasakian manifold is conformally flat.Then it is trivially conformally semi-symmetric. So we have the following theorem:

Theorem 4.3. 3-dimensional trans-Sasakian manifold is conformally flat if and only if it is conformally semi-symmetric.

5. Three dimensional trans-Sasakian manifold satisfying $R(X,Y).\tilde{C} = 0$

The concircular curvature tensor $\tilde{C}$ is defined as

(5.1) $$\tilde{C}(X, Y)Z = R(X, Y)Z - \frac{r}{6}[g(Y, Z)X - g(X, Z)Y]$$

where R is the curvature tensor and r is the scalar curvature

Hence, in view of (2.2) and (2.7), we get

(5.2) $$\eta(\tilde{C}(X,Y)Z) = (\alpha^2 - \beta^2 - \frac{r}{6})[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)]$$

for Z = ξ , we get from (5.2)

(5.3) $$\eta(\tilde{C}(X,Y)\xi) = 0$$

Again putting X = ξ in(5.2) we obtain

(5.4) $$\eta(\tilde{C}(\xi,Y)Z) = (\alpha^2 - \beta^2 - \frac{r}{6})[g(Y,Z) - \eta(Z)\eta(Y)]$$

In virtue of (5.1) we get

(5.5) $$R(X, Y)\tilde{C}(U, V)Z - \tilde{C}(R(X, Y)U, V)Z - \tilde{C}(U, R(X, Y)V)Z - \tilde{C}(U, V)R(X, Y)Z = 0$$

Therefore,

$$g[R(\xi, Y)\tilde{C}(U, V)Z, \xi] - g[\tilde{C}(R(\xi, Y)U, V)Z, \xi]$$

$-g[\tilde{C}(U, R(\xi, Y)V)Z, \xi] - g[\tilde{C}(U, V)R(\xi, Y)Z, \xi] = 0$

From this, it follows that

$-\tilde{C}(U, V, Z, U) + \eta(Y)\eta(\tilde{C}(U, V)Z) - \eta(U)\eta(\tilde{C}(Y, V)Z)$
$+g(Y, U)\eta(\tilde{C}(\xi, V)Z) - \eta(V)\eta(\tilde{C}(U, Y)Z) + g(Y, V)\eta(\tilde{C}(U, \xi)Z)$
$-\eta(Z)\eta(\tilde{C}(U, V)Y) + g(Y, Z)\eta(\tilde{C}(U, V)\xi) = 0$

where $\tilde{C}(U, V, Z, Y) = g(\tilde{C}(U, V)Z, Y)$

Putting Y = U in (5.6) and with the help of (5.2) and (5.3) we get

(5.7) $$-\sum_{i=1}^{i=3}\tilde{C}(U,V,Z,U) + g(U,U)\eta(\tilde{C}(\xi,V)Z)$$
$$g(U,U)\eta(\tilde{C}(U,\xi)Z) - \eta(Z)\eta(\tilde{C}(U,V)U) = 0$$

If $e_1, e_{2,} e_3$ are local orthonormal basis of the tangent space at any point then the sum for $1 \le i \le 3$ of the relation (5.7) for U = e_i yields

(5.8) $S(V, Z) = [2(\alpha^2 - \beta^2)]g(V, Z)$

Taking Z = ξ in (5.8) we get

(5.9) $r = 6(\alpha^2 - \beta^2)$

Now using (5.8), (5.9)(5.2)and(5.3) into the equation (5.6) we get

$-\tilde{C}(U, V, Z, Y) = 0$

From (5.10) it follows that

(5.11) $\tilde{C}(U, V)Z = 0$

Therefore, 3-dimensional trans-Sasakian manifold is concircularly flat.Hence we can state the next theorem.

Theorem 5.4. If in a 3-dimensional trans-Sasakian manifold the relation R(X, Y).$\tilde{C}$ = 0 holds then the manifold is concircularly flat.

As we know, in general a concircularly flat Riemannian manifold is Einstein and so, in particular, a concircularly flat 3-dimensional trans-Sasakian manifold is Einstein. Hence we can state the next theorem..

Theorem 5.5. A 3-dimensional trans-Sasakian manifold satisfying R(X, Y).$\tilde{C}$ = 0is an Einstein manifold of constant curvature $6(\alpha^2 - \beta^2)$

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