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Generalized Weyl's Theorems for Perturbations of algebraically *K* - Quasi - Paranormal Operators

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Abstract

An operator $T \in B(H)$ is said to be k - quasi - paranormal operator if $\|T^{k+1}x\|^2 \leq \|T^{k+2}x\|\|Tx\|$ for every $x \in H$, k is a natural number. The study of operators satisfying Weyl's theorem, Browder's theorem, the SVEP and Bishop's property is of significant interest and is currently being done by a number of mathematicians around the world. In this paper, we prove Browder's and a - Browder's theorem for k - quasi - paranormal operators. Also we show that if T is an algebraically k - quasi - paranormal operator, then T + F satisfies generalized Weyl's theorem for every algebraic operator F which commutes with T.

Keywords: Paranormal operator, k - quasi - paranormal operator , Weyl's theorem, k - quasi - * - class A operator, Riesz idempotent, generalized a - Weyl's theorem, B - Fredholm, B – Weyl

Subject Classification [2010]: Primary 47A10 ; Secondary 47B20.

Introduction

Let B(H) and $B_0(H)$ denote the algebra of all bounded linear operators and the ideal of compact operators acting on an infinite dimensional separable Hilbert space H. An operator T is said to be p - hyponormal, for $p \in (0, 1)$, if $(T^*T)^p \geq (TT^*)^p$. An 1 - hyponormal operator is hyponormal which has been studied by many authors and it is known that hyponormal operators have many interesting properties similar to those of normal operators [26]. Furuta et al [12], have characterized class A operator as follows. An operator T belongs to class A if and only if $(T^*|T|T)^{\frac{1}{2}} \geq T^*T$.

An operator T is called (p, k) - quasihyponormal if $T^{*k}((T^*T)^p - (TT^*)^p)T^k \geq 0$ $(0 < p \leq 1, k \in \mathrm{N})$. A. Aluthge [4], B.C. Gupta [9], S.C. Arora and P. Arora [6] introduced p - hyponormal, p - quasihyponormal and k - quasihyponormal operators, respectively. In [25], the class of log - hyponormal operators is defined as follows: T is called log - hyponormal if it is invertible and satisfies $\log (T^*T)^p \geq \log (TT^*)^p$. Class of p - hyponormal operators and class of log hyponormal operators were defined as extension class of hyponormal operators, i.e., $T^*T \geq TT^*$. An operator T is called paranormal if $||T^2 x||\,||x|| \geq ||Tx||^2$ for all $x \in H$. It is also well known that there exists a hyponormal operator T such that T^2 is not hyponormal (see [15]).

An operator T is called quasi class A if $T^*|T|^2 T \geq T^*|T^2|T$. Fuji, Izumino and Nakamoto [14] introduced p - paranormal operators for $p > 0$ as a generalization of paranormal operators. Paranormal operators have been studied by many authors [5], [13] and [19].

In order to extend the class of paranormal operators and class of quasi - class A operators, Mecheri [23] introduced a new class of operators called k - quasi - paranormal operators. An operator T is called k - quasi - paranormal if $\left\|T^{k+1}x\right\|^2 \leq \left\|T^{k21}x\right\|\left\|T^k x\right\|$ for all $x \in H$ where k is a natural number. A 1 - quasi - paranormal operator is quasi paranormal. The following implication gives us relations among the classes of operators.

Hyponormal $\Rightarrow$ p – hyponormal $\Rightarrow$ class A $\Rightarrow$ paranormal
$\Rightarrow$ quasi - paranormal $\Rightarrow$ k - quasi – paranormal
Hyponormal $\Rightarrow$ class A $\Rightarrow$ quasi - class A $\Rightarrow$ quasi - paranormal
$\Rightarrow$ k - quasi - paranormal

If $T \in$ B(H), we shall write $N(T)$ and $R(T)$ for the null space and the range of T, respectively. Let $\alpha(T)$ and $\beta(T)$ be the nullity and deficiency of $T \in$ B(H) defined by

$$\alpha(T) = \dim\ (T^{-1}(0)) < \infty$$

and

$$\beta(T) = \dim\ (H/T(H)) < \infty .$$

The class of all upper semi - Fredholm operators denoted by $\Phi_+(H)$ and lower semi - Fredholm operators denoted by $\Phi_-(H)$ are defined by

$\Phi_+(H) = \{T \in B(H) : \alpha(T) < \infty \textit{ and } T(H) \text{ is closed } \}$.

and

$\Phi_-(H) = \{T \in B(H) : \beta(T) < \infty \}$

An operator $T \in$ B(H) is said to be semi - Fredholm, $T \in \Phi_\pm(H)$, if $T \in \Phi_+(H) \bigcup \Phi_-(H)$ and Fredholm, $T \in \Phi(H)$, if $T \in \Phi_+(H) \bigcap \Phi_-(H)$. If T is semi - Fredholm then the index of T is defined by

ind (T) = $\alpha(T)$ - $\beta(T)$.

The ascent (length of the null chain) of an operator $T \in$ B(H) is the smallest non negative integer p := p (T) such that $T^{-p}(0) = T^{-(p+1)}(0)$. If there is no such integer, i.e., $T^{-p}(0) \neq T^{-(p+1)}(0)$ for all p, then set $p(T) = \infty$. The descent (length of the image chain) of T is defined as the smallest non negative integer q = q(T) such that $T^q(H) = T^{(q+1)}(H)$. If there is no such integer, ie., $T^q(H) \neq T^{(q+1)}(H)$ for all q, then set $q(T) = \infty$. It is well known that if p(T) and q(T) are both finite then they are equal [17, Proposition 38.6]

It appears that the concept of Drazin invertibility plays an important role for the class of B - Fredholm operators. Let A be a unital algebra. We say that an element $x \in A$ is Drazin invertible of degree k if there exists an element $a \in A$ such that

$x^k ax = x^k$, $axa = a$ and $xa = ax$

Let $a \in A$. Then the Drazin spectrum is defined by

$\sigma_D(a) = \{\lambda \in C : a - \lambda \text{ is not Drazin invertible}\}$.

It is well known that T is Drazin invertible if and only if it has finite ascent and descent, which is also equivalent to the fact that

$T = T_1 \oplus T_2$ where T_1 is invertible and T_2 is nilpotent.

An operator T is said to have the Single Valued Extension Property at $\lambda_0 \in C$, abbreviated as T has the SVEP at λ_0, if for every neighborhood U of λ_0 the only analytic function $f : U \to X$ which satisfies the equation

$(I\lambda - T) f(\lambda) = 0,$

is the constant function $f \equiv 0$. An operator T is said to have the SVEP if T has the SVEP at every $\lambda \in C$. An operator $T \in B(H)$ is called Weyl if it is Fredholm of index zero and Browder if T is Fredholm and p(T) = q(T) < ∞ . The essential spectrum $\sigma_e(T)$, the Weyl spectrum $\sigma_w(T)$ and Browder spectrum $\sigma_b(T)$ of T are defined by

$\sigma_e(T) = \{\lambda \in C : T-\lambda \text{ is not Fredholm}\}$;
$\sigma_w(T) = \{\lambda \in C : T-\lambda \text{ is not Weyl}\}$;
$\sigma_b(T) = \{\lambda \in C : T-\lambda \text{ is not Browder}\}$

Evidently, if we let $\sigma(T)$ be the spectrum of *T* and acc $\sigma(T)$ be the set of accumulation point of $\sigma(T)$ then

$$\sigma_e(T) \subseteq \sigma_w(T) \subseteq \sigma_b(T) \subseteq \sigma_e(T) \bigcup acc\ \sigma(T)$$

Let $\pi_0(T)$ denote the set of all Riesz points of *T* (i.e., the set of isolated eigenvalues of *T* of finite algebraic multiplicity); and let $\pi_{00}(T)$ denote the set of isolated eigenvalues of *T* of finite geometric multiplicity. According to Coburn [10] Weyl's theorem holds for *T*, if

$$\sigma(T) \setminus \sigma_w(T) = \pi_{00}(T)$$

Browder's theorem, a weaker version of Weyl's theorem was introduced in [16]. Browder's theorem holds for *T* if,

$$\sigma(T) \setminus \sigma_w(T) = \pi_0(T)$$

Let *K*(*H*) denote the ideal of all compact operators on *H* and let $\sigma_a(T)$ be the approximate point spectrum of $T \in B(H)$. The essential approximate point spectrum $\sigma_{ea}(T)$ and the Browder approximate point spectrum $\sigma_{ab}(T)$ of *T* are defined by

$$\sigma_{ea}(T) = \bigcap\{\sigma_a(T+K) : K \in K(H)\}$$
$$\sigma_{ab}(T) = \bigcap\{\sigma_a(T+K) : TK = KT \quad and \quad K \in K(H)\}$$

The semigroup $\Phi_{\pm}(H) = \{T \in \Phi_{+}(H) : ind\ (T) \leq 0\}$ was introduced in [20]. It is well known that

$$\sigma_{ea}(T) = \{\lambda \in C : T-\lambda \notin \Phi_{\pm}(H)\}$$

and

$$\sigma_{ab}(T) = \sigma_{ea}(T) \bigcup \{\text{limits points of } \sigma_a(T)\} [21].$$

Evidently, $\sigma_{ea}(T) \subseteq \sigma_{ab}(T)$.

Let $\pi_{a0}(T)$ denote the set of isolated points in the approximate point spectrum of finite geometric multiplicity i.e., $\pi_{a0}(T) = \{\lambda \in C : \lambda \in iso\ \sigma_a(T)\ \ and\ \ 0 < \alpha(T-\lambda) < \infty\}$. A bounded operator $T \in B(H)$ is said to satisfy *a* - Weyl's theorem if

$$\sigma_{ea}(T) = \sigma_a(T) \setminus \pi_{a0}(T).$$

Recall that a bounded operator $T \in B(H)$ is said to satisfy *a* - Browder's theorem if

$\sigma_{ea}(T) = \sigma_{ab}(T)$.

It is well known that a - Browder's theorem $\Rightarrow$ Browder's theorem. A stronger version of Weyl's theorem , generalised Weyl's theorem, was introduced by Berkani [7]. Generalised Weyl's theorem has been studied in [7, 8].

An operator $T \in B(H)$ is called B - Fredholm, $T \in \Phi_{BF}(H)$, if there exist a natural number n, for which the induced operator $T_n : T^n(H) \to T^n(H)$ is Fredholm in usual sense, and B - Weyl, $T \in \Phi_{BW}(H)$, if $T \in \Phi_{BF}(H)$ and ind $T_n = 0$.

Let $E(T)$ is the set of all eigenvalues of T which are isolated in $\sigma(T)$. Let C denotes the set of all complex numbers. The B - Fredholm spectrum $\sigma_{BF}(T)$ and B - Weyl spectrum $\sigma_{BW}(T)$ of T are defined by

$\sigma_{BF}(T) = \{\lambda \in C : T - \lambda$ is not B - Fredholm$\}$;

and

$\sigma_{BW}(T) = \{\lambda \in C : T - \lambda$ is not B - Weyl$\}$.

A bounded operator $T \in B(H)$ is said to satisfy generalized Weyl's theorem if

$\sigma_{BW}(T) = \sigma(T) \setminus E(T)$.

An operator T is semi B - Fredholm, $T \in \Phi_{SBF}(H)$, if $T^n(H)$ is closed for some $n \in N$ and the induced operator T_n is either upper semi - Fredholm or lower semi - Fredholm [8]. For a $T \in \Phi_{SBF}(H)$, the index of T is defined by

ind (T) = ind (T_d)

where $d \in N$ is the degree of stable iteration of T [8, Definition 2.2].

Let $\Phi_{SBF_+^-}(H)$ denote the class of all upper B - Fredholm operators such that ind $(T) \leq 0$. The semi - B - essential approximate point spectrum $\sigma_{SBF_+^-}(T)$ is defined by

$\sigma_{SBF_+^-}(T) = \{\lambda \in C : T - \lambda \notin \Phi_{SBF_+^-}(H)\}$

Let $E^a(T)$ is the set of all eigenvalues of T which are isolated in $\sigma_a(T)$. Now, we consider the following sets:

$BF_+(H) = \{T \in B(H) : T$ is upper semi – B – Fredholm $\}$,

$BF_+^-(H) = \{T \in B(H) : T \in BF_+(H)$ *and* $i(T) \leq 0\}$,

$LD(H) = \{T \in B(H) : p(T) < \infty$ *and* $R(T^{p(T)+1})$ is closed $\}$.

By definition,

$\sigma_{B_{ea}}(T) = \{\lambda \in C : T - \lambda \notin BF_+^-(H)\}$

is the upper semi - B - essential approximate point spectrum and

$$\sigma_{LD}(T) = \{\lambda \in C : T - \lambda \notin LD(H)\}$$

is the left Drazin spectrum. It is well known that

$$\sigma_{B_{ea}}(T) = \sigma_{LD}(T) = \sigma_{B_{ea}}(T) \cup acc\ \sigma_a(T) \subseteq \sigma_D(T),$$

where we write acc K for the accumulation points of $K \subseteq C$. If we write iso $K = K \setminus$ acc K then we let

$$p_0^a(T) = \{\lambda \in \sigma_a(T) : T - \lambda \in LD(H)\},$$

$$\pi_0^a(T) = \{\lambda \in\ iso\ \sigma_a(T) : \lambda \in \sigma_p(T)\}.$$

A bounded operator $T \in B(H)$ is said to satisfy generalized a - Weyl's theorem if

$$\sigma_a(T) \setminus E^a(T) = \Phi_{SBF_+^-}(T).$$

It is well known that generalized a - Weyl's theorem implies a - Weyl's theorem but the converse is not true [8, Example 3.1.12].

We say that Generalized Browder's theorem holds for T if

$$\sigma(T) \setminus \sigma_{BW}(T) = \pi(T).$$

We say that Generalized a - Browder's theorem holds for T if

$$\sigma_a(T) \setminus \sigma_{B_{ea}}(T) = p_0^a(T).$$

In local spectral theory, the quasi - nilpotent part $H_0(T)$ of an operator T is defined by

$$H_0(T) = \left\{ x \in H :\ \lim_{n\to\infty} \left\| T^n x \right\|^{\frac{1}{n}} = 0 \right\}$$

and the analytic core $K(T)$ is defined as

$K(T) = \{x \in H :$ there exists a sequence $\{x_n\} \subset H$ and $\delta > 0$ for which $x = x_0,\ T(x_{n+1}) = x_n$
and $\|x_n\| \le \delta^n \|x\|$ for all $n = 1,2,3,\ldots\}$.

Let $\mathrm{P}(H)$ denotes the class of all operators for which there exists $p = p(\lambda) \in N$ for which

$$H_0(T - \lambda) = N(T - \lambda)^p \text{ for all } \lambda \in E(T).$$

Evidently, $\mathrm{P}(H) \subseteq \mathrm{P}_1(H)$. In second section, we prove Browder's and a - Browder's theorem for k - quasi - paranormal operators. In third section, we show that if T is an algebraically k - quasi - paranormal operator, then $T + F$ satisfies generalized

Weyl's theorem for every algebraic operator *F* which commutes with *T*.

Weyl's theorem for *k* - quasi - paranormal operators

Salah Mecheri [23] has introduced *k* - quasi - paranormal operators and has proved many interesting properties of it.

Lemma 2.1 : [23]

(1) Let $T \in B(H)$ be a *k* - quasi - paranormal, the range of T^k be not dense and

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$$

on $H = \overline{ran\,(T^k)} \oplus \ker(T^{*k})$. Then T_1 is paranormal, $T_3^k = 0$ and $\sigma(T) = \sigma(T_1) \cup \{0\}$.

(2) Let *M* be a closed *T* - invariant subspace of *H*. Then the restriction $T|_M$ of a *k* - quasi - paranormal operator *T* to *M* is a *k* - quasi – paranormal.

Lemma 2.2 ([23]) :

Let T $\in B(H)$ be a *k* - quasi - paranormal operator. Then *T* has Bishop's property (β), i.e., if $f_n(z)$ is analytic on *D* and $(T - z) f_n(z) \to 0$ uniformly on each compact subset of *D*. Hence *T* has the single valued extension property.

Theorem 2.3 :

Let T $\in B(H)$ be a *k* - quasi - paranormal operator. If $S \sim T$, then *S* has SVEP.

Proof:

Since *T* is *k* - quasi - paranormal operator, it follows from Lemma 2.2 that *T* has SVEP. Let *U* be any open set and $f : U \to H$ be any analytic function such that $(S - \lambda) f(\lambda) = 0$ for all $\lambda \in U$. Since $S \sim T$, there exists an injective operator *A* with dense range such that *A S* = *T A*. Thus $A(S - \lambda) = (T - \lambda)A$ for all $\lambda \in U$. Since $(S - \lambda) f(\lambda) = 0$ for all $\lambda \in U$, $A(S - \lambda) = 0 = (T - \lambda)A$ for all $\lambda \in U$. But *T* has SVEP, hence $Af(\lambda) = 0$ for all $\lambda \in U$. Since *A* is injective, $f(\lambda) = 0$ for all $\lambda \in U$. Thus *S* has SVEP.

Theorem 2.4 :

Let T $\in B(H)$ be a *k* - quasi - paranormal operator. Then *T* obeys *a* - Browder's theorem.

Proof:

It is well known that $\sigma_{ea}(T) \subseteq \sigma_{ab}(T)$. Conversely, assume that $\lambda \in \sigma_{ap}(T) \setminus \sigma_{ea}(T)$.

Then $T-\lambda I \in \Phi_+^-(H)$ and $T-\lambda I$ is not bounded below. Since T has SVEP by Lemma 2.2 and $T-\lambda I \in \Phi_+^-(H)$ [2, Theorem 2.6] implies that $T-\lambda I$ has finite ascent. Hence [22, Theorem 2.1] implies that $\lambda \in \sigma_{ap}(T) \setminus \sigma_{ab}(T)$. This implies that *a* - Browder's theorem holds for *T*.

Theorem 2.5 [2]:
Let T $\in$ *B*(*H*) have SVEP. Then $\sigma_{ea}(f(T)) = f(\sigma_{ea}(T))$ for every function $f(z)$ that is analytic in some open neighbourhood *G* of $\sigma(T)$.

Theorem 2.6 :
Let T $\in$ *B*(*H*) be a *k* - quasi - paranormal operator. Then *a* - Browder's theorem holds for *f*(*T*) for every analytic function *f* on some open neighborhood of $\sigma(T)$.\end{theorem

Proof:
Since $\sigma_{ea}(f(T)) = f(\sigma_{ea}(T))$, By applying Theorem 2.5, we get

$$\sigma_{ab}(f(T)) = f(\sigma_{ab}(T)) = f(\sigma_{ea}(T)) = \sigma_{ea}(f(T)).$$

Therefore *a* - Browder's theorem holds for *f*(*T*) .

Theorem 2.7 :
Let T $\in$ *B*(*H*) be a *k* - quasi - paranormal operator. Then *a* - Browder's theorem holds for *f*(*T*) for every analytic function *f* on some open neighborhood of $\sigma(T)$. If $S \sim T$, then *a* - Browder's theorem holds for *f*(*S*) for every analytic function *f* on some open neighborhood of $\sigma(T)$.

Proof:
Since *a* - Browder's theorem holds for *S*, it follows from Theorem 2.4 that

$$\sigma_{ab}(f(S)) = f(\sigma_{ab}(S)) = f(\sigma_{ea}(S)) = \sigma_{ea}(f(S)).$$

Hence *a* - Browder's theorem holds for *f*(*S*).

Theorem 2.8 :
Let T $\in$ *B*(*H*) be a *k* - quasi - paranormal operator. Then *T* satisfies Browder's theorem.

Proof:
It is known that operators with SVEP satisfies Browder's theorem [11]. Hence *T* satisfies Browder's theorem.

Theorem 2.9 :
Let T $\in$ *B*(*H*) be a *k* - quasi - paranormal operator. Then *f*(*T*) satisfies Browder's

theorem for each function *f* analytic in a neighbourhood of $\sigma(T)$.

Proof:
It is known that operators with SVEP satisfies Browder's theorem [11]. Hence *f*(*T*) satisfies Browder's theorem.

Generalized weyl's theorem for perturbations of algebraically *k* - quasi - paranormal operators

An operator *T* is called algebraically *k* - quasi - paranormal if there exists a non constant complex polynomial *s* such that *s*(*T*) belongs to *k* - quasi - paranormal. An operator *T* is said to be algebraic if there exists a nontrivial polynomial *h* such that *h* (*T*) = 0. From the spectral mapping theorem it easily follows that the spectrum of an algebraic operator is a finite set. It is known that generalized Weyl's theorem is not generally transmitted to perturbation of operators satisfying generalized Weyl's theorem. In [3], they proved that if *T* is paranormal and *F* is an algebraic operator commuting with *T*, then Weyl's theorem holds for *T* + *F*. We now extend this result to generalized Weyl's theorem for algebraically *k* - quasi - paranormal operators.

Corollary 3.1 : [18, Corollary 3.2]:
Let T $\in$ *B*(*H*). Suppose *T* has SVEP. Then T $\in$ generalized Weyl's theorem if and only if T $\in$ $P_1(H)$.

Lemma 3.2 : [18, Lemma 3.3]:
Suppose T $\in$ *B*(*H*) and *N* is nilpotent such that *T N* = *N T*. Then T $\in$ $P_1(H)$ if and only if *T* + *N* $\in$ $P_1(H)$.

Theorem 3.3 :
Suppose *T* is algebraically *k* - quasi - paranormal. If *F* is algebraic with *T F* = *F T*, then *T* + F $\in$ generalized Weyl's theorem.

Proof:
Since *F* is algebraic, $\sigma(F)$ is finite. Let $\sigma(F)=\{\mu_1,\mu_2,...,\mu_n\}$. Let P_i denote the spectral projection associated with *F* and the spectral set $\{\mu_i\}$. Let Y_i := R(P_i) and Z_i := N(P_i). Then $H=Y_i\oplus Z_i$ and the closed subspaces Y_i and Z_i are invariant under *T* and *F*. Moreover, $\sigma(F\,|\,Y_i)=\{\mu_i\}$. Define $F_i := F\,|\,Y_i$ and $T_i := T\,|\,Y_i$. Then clearly, the restrictions T_i and F_i commute for every *i* = 1, 2, ..., *n* and

$$\sigma(T+F)=\sigma((T+F)\,|\,Y_i)\cup\sigma((T+F)\,|\,Z_i)$$

Let *h* be a nontrivial complex polynomial such that *h*(*F*) = 0. Then $h(F_i)=h(F\,|\,Y_i)=h(F)\,|\,Y_i=0$, and from $\{0\}=\sigma(h(F_i))=h(\sigma(F_i))=h(\{\mu\})$, we obtain that $h(\mu_i)=0$. Write $h(\mu)=(\mu-\mu_i)^m g(\mu)$ with $g(\mu_i)\neq 0$. Then

$0 = h(F_i) = (F_i - \mu_i)^m g(F_i)$, where $g(F_i)$ is invertible. Hence $N_i = F_i - \mu_i$ are nilpotent for all i = 1, 2, ..., n. Observe that

$$T_i + F_i = (T_i + \mu_i) + (F_i - \mu_i) = T_i + N_i + \mu_i \quad \text{---------------} \quad (3.1)$$

Since $T_i + \mu_i$ is algebraically k - quasi - paranormal for all i = 1, 2, ..., n, $T_i + \mu_i$ has SVEP. Moreover, since N_i is nilpotent with $T_i N_i = N_i T_i$, it follows from [1, Corollary 2.12] that $T_i + N_i + \mu_i$ has SVEP, and hence $T_i + F_i$ has SVEP. From [1, Theorem 2.9] we obtain that

$$T + F = \bigoplus_{i=1}^{n} (T_i + F_i) \text{ has SVEP}$$

Now, we show that $T + F \in P_1(H)$. Since $T_i + \mu_i$ is algebraically k - quasi - paranormal, $T_i + \mu_i \in P_1(Y_i)$ by [24, Theorem 3.4]. By Lemma 3.2 and equation (3.1), $T_i + F_i \in P_1(Y_i)$ for every i = 1, 2, ..., n. Now assume that $\lambda_0 \in E(T + F)$. Fix $i \in N$ such that $1 \le i \le n$. Since the equality $T_i + N_i - \lambda_0 + \mu_i = T_i + F_i - \lambda_0$ holds, we consider two cases:

Case I: Suppose that $T_i - \lambda_0 + \mu_i$ is invertible. Since N_i is quasi - nilpotent commuting with $T_i - \lambda_0 + \mu_i$, it is clear that $T_i + F_i - \lambda_0$ is also invertible. Hence $H_0(T_i + F_i - \lambda_0) = N(T_i + F_i - \lambda_0) = \{0\}$.

Case II: Suppose that $T_i - \lambda_0 + \mu_i$ is not invertible. Then $\lambda_0 - \mu_i \in \sigma(T_i)$. We claim that $\lambda_0 \in E(T_i + F_i)$. Note that $\lambda_0 \in \sigma(T_i + \mu_i) = \sigma(T_i + F_i)$. Since $\sigma(T_i + F_i) \subseteq \sigma(T + F)$ and $\lambda_0 \in iso\,\sigma(T + F)$, $\lambda_0 \in \sigma(T_i + N_i + \mu_i)$. Therefore $\lambda_0 - \mu_i \in iso\ \sigma(T_i + N_i) = iso\ \sigma(T_i)$. Since $T_i - \lambda_0 + \mu_i$ is algebraically k - quasi - paranormal, $\lambda_0 - \mu_i \in \pi(T_i)$. Since $\pi(T_i) = E(T_i)$ by [24, Theorem 1.1] and $T_i \in$ generalized Weyl's theorem by [24, Theorem 3.5], $\lambda_0 - \mu_i \in \mu E(T_i) = \sigma(T_i) \setminus \sigma_{BW}(T_i)$. But N_i is nilpotent with $T_i N_i = N_i T_i$, hence $\sigma_D(T_i) = \sigma_D(T_i + N_i)$ and $T_i + N_i \in$ generalized Browder's theorem. Therefore we have $\sigma_{BW}(T_i + N_i) = \sigma_D(T_i + N_i)$. Hence

$$E(T_i) = \sigma(T_i) \setminus \sigma_{BW}(T_i) = \sigma(T_i + N_i) \setminus \sigma_{BW}(T_i + N_i)$$

Hence $T_i + F_i - \lambda_0$ is B - Weyl. Assume to the contrary that $T_i + F_i - \lambda_0$ is injective. Then $\beta(T_i + F_i - \lambda_0) = \alpha(T_i + F_i - \lambda_0) = 0$. Therefore $T_i + F_i - \lambda_0$ is invertible, and so $\lambda_0 \neq \sigma(T_i + F_i)$. This is a contradiction. Hence $\lambda_0 \in E(T_i + F_i)$. Since $T_i + F_i \in P_1(Y_i)$ by [24, Theorem 1.1], there exists a positive integer m_i such that $H_0(T_i + F_i - \lambda_0) = N(T_i + F_i - \lambda_0)^{m_i}$

From Cases I and II we have

$$H_0(T_i + F_i - \lambda_0) = \bigoplus_{i=1}^{n} H_0(T_i + F_i - \lambda_0)$$

$$= \bigoplus_{i=1}^{n} N(T_i + F_i - \lambda_0)^{m_i}$$

$$= N(T + F - \lambda_0)^m,$$

where m = max $\{m_1, m_{2,......,} m_n\}$. Since the last equality holds for every $\lambda_0 \in E(T+F),\ T+F \in P_1(H)$. Therefore $T+F \in$ generalized Weyl's theorem by Corollary 3.1.

It is well known that if for an operator $F \in B(H)$ there exists a natural number n for which F^n is finite - dimensional, then F is algebraic.

Corollary 3.4 :
Suppose $T \in B(H)$ is algebraically k - quasi - paranormal and F is an operator commuting with T such that F^n is a finite - dimensional operator for some $n \in N$. Then $T+F \in$ generalized Weyl's theorem.

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