Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 9, Number 3 (2013), pp. 255-264

http://www.ripublication.com/gjpam.htm

Oscillation Results for Odd Order Difference Equations

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Abstract

In this paper, we provide sufficient condition for the oscillation of every solution of the difference equation

$$\Delta^r x_n + p_n x_{n-k} = 0, \; n = 0, 1, 2, \dots ,$$

where $k \in \mathbb{N}$, $\{p_n\}$ is the sequence of real terms, $\lim_{n\to\infty} p_n = p > 0$ and $r > 1$ is an odd positive integer; and also provide sufficient conditions for the oscillation of every solution of the difference equation

$$\Delta^r x_n + \sum_{i=1}^{m} p_{in} x_{n-k_i} = 0, \; n = 0, 1, 2, \dots ,$$

where $k_i \in \{0, 1, 2, \dots\}$, $p_{in} \geq 0$, $\lim_{n\to\infty} p_{in} = p_i \geq 0$ for $i = 1, 2, \dots , m$ and $r > 1$ is an odd positive integer. Here, Δ is the forward difference operator defined by $\Delta x_n = x_{n+1} - x_n$.

AMS subject classification: 39A10.
Keywords: Difference equation, difference inequality, oscillation, positive solution, negative solution.

1. Introduction

In the recent years the oscillatory behavior of difference equations has been investigated by some authors (see, for instance, [1–13]). In particular, Erbe and Zhang [6] have introduced a sufficient condition for the oscillation of all solutions of the following difference equations:

$$x_{n+1} - x_n + p_n x_{n-k} = 0, \quad n = 0, 1, 2, \ldots, \tag{1.1}$$

where $k \in \mathbb{N}$ and $\{p_n\}$ is the sequence of real terms, and

$$x_{n+1} - x_n + \sum_{i=1}^{m} p_{in} x_{n-k_i} = 0, \quad n = 0, 1, 2, \ldots, \tag{1.2}$$

where $k_i \in \mathbb{N}$ and $p_{in} \geq 0$ for $i = 1, 2, \ldots, m$.

By a solution of equation (1.1) we mean a sequence $\{x_n\}$ which is defined for $n \geq -k$ and which satisfies equation (1.1) for $n \geq 0$. We recall that a solution $\{x_n\}$ of equation (1.1) is said to be oscillatory if the terms x_n of the sequence $\{x_n\}$ are neither eventually positive nor eventually negative. Otherwise, the solution is called nonoscillatory.

The aim of the present paper is to provide sufficient condition for the oscillation of every solution of difference equation

$$\Delta^r x_n + p_n x_{n-k} = 0, \; n = 0, 1, 2, \ldots, \tag{1.3}$$

where $k \in \mathbb{N}$, $\{p_n\}$ is the sequence of real terms, $\lim_{n\to\infty} p_n = p \geq 0$ and also we obtain sufficient conditions for the oscillation of every solution of difference equation

$$\Delta^r x_n + \sum_{i=1}^{m} p_{in} x_{n-k_i} = 0, \quad n = 0, 1, 2, \ldots, \tag{1.4}$$

where $k_i \in \mathbb{N}$, $p_{in} \geq 0$, $\lim_{n\to\infty} p_{in} = p_i \geq 0$ for $i = 1, 2, \ldots, m$ and Δ^r is the r^{th} order forward difference operator defined by

$$\Delta^r x_n = \sum_{i=0}^{r} (-1)^{r-i} \binom{r}{i} x_{n+i}, \; r \geq 1.$$

In [4], Agarwal, Thandapani and Wong, in [5] Agarwal and Grace, in [7] Grzegorczyk and Werbowski have investigated the oscillatory behavior of the solutions of equation (1.3). Furthermore, all mentioned papers concern equation (1.3) under the hypothesis

$$\sum_{n=0}^{\infty} p_n = \infty.$$

which has played a significant role in the study of the oscillation of equation (1.3). Recently, in [13] Zhou given some results for oscillation of equation (1.4). In this paper, we obtain the oscillatory behavior of the solutions of equations (1.3) and (1.4) in case of the lim p_n exists.

We shall need the following lemmas which are given in [1].

Lemma 1.1. Let x_n be defined for $n \geq n_0$ and $x_n > 0$ with $\Delta^r x_n$ of constant sign for $n \geq n_0$ and not identically zero. Then, there exists an integer j, $0 \leq j \leq r$ with $(r + j)$ odd for $\Delta^r x_n \leq 0$ and $(r + j)$ even for $\Delta^r x_n \geq 0$ such that

(i) $j \leq r - 1$ implies $(-1)^{j+i} \Delta^i x_n > 0$ for all $n \geq n_0$, $\quad j \leq i \leq r - 1$

(ii) $j \geq 1$ implies $\Delta^i x_n > 0$ for all large $n \geq n_0$, $1 \leq i \leq j - 1$.

Specially, if $\Delta^r x_n \leq 0$ for $n \geq n_0$, and $\{x_n\}$ is bounded, then

$$(-1)^{i+1} \Delta^{r-i} x_n \geq 0, \text{ for all large } n \geq n_0,\ i = 1, \dots, r - 1,$$

and

$$\lim_{n\to\infty} \Delta^i x_n = 0,\ 1 \leq i \leq r - 1.$$

Lemma 1.2. Let x_n be defined for $n \geq n_0$, and $x_n > 0$ with $\Delta^r x_n \leq 0$ for $n \geq n_0$ and not identically zero. Then, there exists a large integer $n_1 \geq n_0$ such that

$$x_n \geq \frac{1}{(r-1)!}(n - n_1)^{r-1} \Delta^{r-1} x_{2^{r-j-1}n}, \quad n \geq n_1$$

where j is defined as in Lemma 1.1. Further, if x_n is increasing, then

$$x_n \geq \frac{1}{(r-1)!}\left(\frac{n}{2^{r-1}}\right)^{r-1} \Delta^{r-1} x_n, \quad n \geq 2^{r-1} n_1.$$

2. Sufficient Condition for the Oscillation of Eq. (1.3)

In this section, we provide a sufficient condition for the oscillation of every solution of equation (1.3). In 1989, Erbe and Zhang [6] have proved the following result.

Theorem A. Assume that

$$\liminf_{n\to\infty} p_n = p > \frac{k^k}{(k+1)^{k+1}},\ k \in \mathbb{N}. \tag{C}$$

Then every solution of equation (1.1) oscillates.

We remark that, later condition (C) was improved, by Ladas et al. [9], to

$$\liminf_{n\to\infty} \sum_{i=n-k}^{n-1} p_i > \left(\frac{k}{k+1}\right)^{k+1}. \tag{C*}$$

It is clear that if $\lim_{n\to\infty} p_n$ exists, then condition C and C^* are equivalent.

We first need the following lemma.

Lemma 2.1. Let $k \in \mathbb{N}$ and r is an odd positive integer. If

$$\liminf_{n\to\infty} \sum_{i=n-k}^{n-1} c_i > \left(\frac{k}{k+1}\right)^{k+1} \quad \text{and} \quad \lim_{n\to\infty} p_n = p > r^r \frac{k^k}{(k+r)^{k+r}} \tag{2.5}$$

where $c_n = p_n \dfrac{1}{(r-1)!}\left(\dfrac{n-k}{2^{r-1}}\right)^{r-1}$, then the following holds:

(i) the difference inequality

$$\Delta^r x_n + p_n x_{n-k} \le 0 \tag{2.6}$$

has no eventually positive solution,

(ii) the difference inequality

$$\Delta^r x_n + p_n x_{n-k} \ge 0 \tag{2.7}$$

has no eventually negative solution.

Proof. (i) The proof is by contradiction, assume that inequality (2.2) has an eventually positive solution. Then, there exists a number $n_0 \ge 0$ such that $x_n > 0$ for all $n \ge n_0$. Also by (2.1) there is a number $n_1 \ge 0$ such that $p_n > 0$ for all $n \ge n_1$. Let $N = \max\{n_0 + k, n_1\}$ and by using (2.1) and (2.2), we have

$$\Delta^r x_n \le -p_n x_{n-k} \le 0$$

for all $n \ge N$. Then by Lemma 1.1 we have $\Delta^j x_n,\ j = 0, 1, \dots, r$ are eventually of one sign and $\Delta^{r-1} x_n > 0$. Now, there are two possibilities to consider: (a) $\Delta x_n \ge 0$ for $n \ge N$, and (b) $\Delta x_n \le 0$ for $n \ge N$.

Case (a). Suppose $\Delta x_n \ge 0$ for $n \ge N$. Then, it is clear that $\{x_n\}$ nondecreasing and applying Lemma 1.2, there exists an integer $n_2 \ge N$ such that

$$x_n \ge \frac{1}{(r-1)!}\left(\frac{n}{2^{r-1}}\right)^{r-1} \Delta^{r-1} x_n\,, \quad n \ge 2^{r-1} n_2 \tag{2.8}$$

Using (2.4) in (2.2), we have

$$\Delta^r x_n + p_n \frac{1}{(r-1)!}\left(\frac{n-k}{2^{r-1}}\right)^{r-1} \Delta^{r-1} x_{n-k} \le 0, \;\; n \ge 2^{r-1} n_2$$

if we choose $\Delta^{r-1}x_n = z_n$, then we have the following

$$\Delta z_n + p_n \frac{1}{(r-1)!}\left(\frac{n-k}{2^{r-1}}\right)^{r-1} z_{n-k} \le 0, \ n \ge n_2. \tag{2.9}$$

Thus, in view of (2.1) and condition (C^*), (2.5) has no eventually positive solution, which is a contradiction.

Case (b). Suppose $\Delta x_n \le 0$ for $n \ge N$. This implies that $\{x_n\}$ is nonincreasing for $n \ge N$. Now dividing inequality (2.2) by x_n we have

$$\frac{\Delta^r x_n}{x_n} + p_n \frac{x_{n-k}}{x_n} \le 0$$

for all $n \ge N$. This yields, for $n \ge N$, that

$$\frac{x_{n+r}}{x_n} - r\frac{x_{n+r-1}}{x_n} + \cdots + r\frac{x_{n+1}}{x_n} - 1 + p_n\left\{\frac{x_{n-k}}{x_{n-k+1}}\frac{x_{n-k+1}}{x_{n-k+2}}\cdots\frac{x_{n-1}}{x_n}\right\} \le 0 \tag{2.10}$$

Let $z_n = \dfrac{x_n}{x_{n+1}}$. Then $z_n \ge 1$ for $n \ge N$. By (2.6) we get

$$\frac{1}{z_{n+r-1}\cdots z_n} - r\frac{1}{z_{n+r-2}\cdots z_n} + \cdots + r\frac{1}{z_n} - 1 \le -p_n\{z_{n-k}\cdots z_{n-1}\}. \tag{2.11}$$

Since $\lim_{n\to\infty} x_n$ exists, either $\lim_{n\to\infty} z_n = q \in [1,\infty)$ or $\lim_{n\to\infty} z_n = \infty$. We claim that $\lim_{n\to\infty} z_n = q \in [1,\infty)$, otherwise taking limit as $n \to \infty$ on both sides of (2.7) we have

$$\lim_{n\to\infty}\left[\frac{1}{z_{n+r-1}\cdots z_n} + \cdots + r\frac{1}{z_n} - 1\right] \le \lim_{n\to\infty}\left[-p_n\{z_{n-k}\cdots z_{n-1}\}\right] \to -\infty$$

contradiction. Therefore we have $\lim_{n\to\infty} z_n = q \ge 1$. So, taking limit as $n \to \infty$ on both sides of (2.7) we get

$$(q-1)^r \ge pq^{k+r}$$

So, we conclude that

$$p \le (q-1)^r q^{-(k+r)}. \tag{2.12}$$

Consider the function f defined by $f(q) = (q-1)^r q^{-(k+r)}$. Then observe that $f'\left(\frac{k+r}{k}\right) = 0$ and $f''\left(\frac{k+r}{k}\right) < 0$. Therefore, by (2.8) we obtain

$$p \le f\left(\frac{k+r}{k}\right) = r^r \frac{k^k}{(k+r)^{k+r}},$$

which contradicts (2.1).

(ii) It is easily shown that, under condition (2.1), inequality (2.3) has no eventually negative solution by using a method similar to that of (i). ■

By using Lemma 2.1 one can deduce the following main result immediately.

Theorem 2.2. (Main Theorem) Let $k \in \mathbb{N}$ and r is an odd positive integer. If condition (2.1) holds, then every solution of the difference equation (1.3) oscillates.

Proof. Combining (i) and (ii) in Lemma 2.1 we conclude that under condition (2.1) every solution of (1.3) oscillates. ■

Corollary 2.3. Let $k \in \mathbb{N}$ and r is an odd positive integer. If

$$\lim_{n\to\infty} p_n = p > r^r \frac{k^k}{(k+r)^{k+r}},$$

then every bounded solution of the difference equation (1.3) oscillates.

Proof. Let $\{x_n\}$ be a bounded eventually positive solution of (1.3). Then, by Lemma 1.1, we have $\Delta x_n \leq 0$. Therefore the rest of the proof is similar to that of Theorem 2.2. ■

Remark 2.4. When $p_n = p \in \mathbb{R}$ for $n = 0, 1, 2, \ldots$ and $k > 0$ or $k < -r$ in equation (1.3), then every solution of equation (1.3) oscillates if and only if the following condition hold;

$$p > r^r \frac{k^k}{(k+r)^{k+r}},$$

which is given in [11].

3. Sufficient Conditions for the Oscillation of Eq. (1.4)

In this section we extend the results in Section 2 to equation (1.4). We remark that throughout this paper we will use the convention that $0^0 = 1$. Erbe and Zhang [6] have proved the following result.

Theorem B. Let $k_i \in \mathbb{N}$, $p_{in} \geq 0$ and $\liminf_{n\to\infty} p_{in} = p_i$ for $i = 1, 2, \ldots, m$. Assume that

$$\sum_{i=1}^{m} p_i \frac{(k_i+1)^{k_i+1}}{{k_i}^{k_i}} > 1.$$

Then every solution of equation (1.4) oscillates.

Lemma 3.1. Let $k_i \in \mathbb{N}, r$ is an odd positive integer and $\lim_{n\to\infty} c_{in} = c_i$, $\lim_{n\to\infty} p_{in} = p_i$ for $i = 1, 2, \ldots, m$. If $c_{in}, p_{in} \geq 0$ and

$$\frac{1}{(r-1)!}\left(\frac{1}{2^{r-1}}\right)^{r-1} \liminf_{n\to\infty} \sum_{i=1}^{m} \frac{(k_i+1)^{k_i+1}}{{k_i}^{k_i}} \sum_{s=n-k_i}^{n-1} c_s > 1 \quad \text{and} \quad \sum_{i=1}^{m} p_i \frac{(k_i+r)^{k_i+r}}{{k_i}^{k_i}} > r^r, \tag{3.13}$$

where $c_{in} = p_{in}(n-k_i)^{r-1}$ for $i = 1, 2, \ldots, m$, then the following holds:

(i) the difference inequality

$$\Delta^r x_n + \sum_{i=1}^{m} p_{in} x_{n-k_i} \leq 0 \tag{3.14}$$

has no eventually positive solution,

(ii) the difference inequality

$$\Delta^r x_n + \sum_{i=1}^{m} p_{in} x_{n-k_i} \geq 0 \tag{3.15}$$

has no eventually negative solution.

Proof. (i) Assume that $\{x_n\}$ is an eventually positive solution of (3.2). So, there is a number $N_1 > 0$ such that $x_n > 0$ for all $n \geq N_1$. As in the proof of Lemma 2.1, we have (a) $\Delta x_n \geq 0$ for $n \geq N_1$, (b) $\Delta x_n \leq 0$ for $n \geq N_1$. If $\Delta x_n \geq 0$, the proof is similar to the proof of Lemma 2.1. If $\Delta x_n \leq 0$ for $n \geq N_1$, then $\{x_n\}$ is nonincreasing. Let $z_n = \dfrac{x_n}{x_{n+1}}$, then $z_n \geq 1$ for $n \geq N_1$ and $\lim_{n\to\infty} z_n = q \in [1, \infty)$. Now, dividing the inequality (3.2) by x_n we have

$$\frac{1}{z_{n+r-1}\cdots z_n} + \cdots + r\frac{1}{z_n} - 1 \leq -\sum_{i=1}^{m} p_{in}\, z_{n-k_i} \cdots z_{n-1} \tag{3.16}$$

for all $n \geq N$, where $N = \max\{N_1, N_1 + k_1, \ldots, N_1 + k_m\}$. Taking limit as $n \to \infty$ on both sides of (3.4) we can write

$$\sum_{i=1}^{m} p_i q^{k_i+r} \leq (q-1)^r,$$

which implies that $q \neq 1$ and that

$$\sum_{i=1}^{m} p_i \frac{q^{k_i+r}}{(q-1)^r} \leq 1. \tag{3.17}$$

Now consider the function f defined by $f(q) = \dfrac{q^{k_i+r}}{(q-1)^r}$. Then, observe that $f'\left(\dfrac{k_i+r}{k_i}\right) = 0$ and $f''\left(\dfrac{k_i+r}{k_i}\right) > 0$. It follows that

$$\begin{aligned}\sum_{i=1}^{m} p_i \frac{1}{r^r} \frac{(k_i+r)^{k_i+r}}{{k_i}^{k_i}} &= \sum_{i=1}^{m} p_i f\left(\frac{k_i+r}{k_i}\right) \\ &\leq \sum_{i=1}^{m} p_i \frac{q^{k_i+r}}{(q-1)^r}.\end{aligned}$$

Hence by (3.5)

$$\sum_{i=1}^{m} p_i \frac{1}{r^r} \frac{(k_i+r)^{k_i+r}}{{k_i}^{k_i}} \leq 1, \tag{3.18}$$

which contradicts (3.1). ■

(ii) By using a method similar to that of (i) the fact that (3.3) has no eventually negative solution under condition (3.1) is clear.

Theorem 3.2. Let $k_i \in \mathbb{N}, r$ is an odd positive integer and $\lim_{n\to\infty} c_{in} = c_i$, $\lim_{n\to\infty} p_{in} = p_i$ for $i = 1, 2, \ldots, m$. If $c_{in}, p_{in} \geq 0$. If condition (3.1) holds, then every solution of the difference equation (1.4) oscillates.

Proof. Combining (i) and (ii) in Lemma 3.1 we conclude that under condition (3.1) every solution of (1.4) oscillates. ■

If $\{x_n\}$ is bounded, then in view of Lemma 3.1, we have the following corollary.

Corollary 3.3. Let $k_i \in \mathbb{N}$, r is an odd positive integer and $\lim_{n\to\infty} p_{in} = p_i$ for $i = 1, 2, \ldots, m$. If $p_{in} \geq 0$ and

$$\sum_{i=1}^{m} p_i \frac{(k_i+r)^{k_i+r}}{{k_i}^{k_i}} > r^r,$$

then every bounded solution of the difference equation (1.4) oscillates.

Remark 3.4. If $r = 1$ in equation (1.4) and $p_{in} = p_i \in \mathbb{R}, k_i \in \mathbb{Z}$ for $(i = 1, 2, \ldots, m)$. If

$$\sum_{i=1}^{m} p_i \frac{(k_i+1)^{k_i+1}}{{k_i}^{k_i}} > 1,$$

then every solution of equation (1.4) oscillates, which is obtained by Ladas in [10].

Theorem 3.5. Let $k_i \in \mathbb{N}$, r is an odd positive integer and $\lim_{n\to\infty} p_{in} = p_i$ for $i = 1, 2, \dots, m$. If $p_{in} \geq 0$ and

$$m\left(\prod_{i=1}^{m} p_i\right)^{1/m} > r^r \frac{k^k}{(k+r)^{k+r}}, \tag{3.19}$$

where $k = \frac{1}{m}\sum_{i=1}^{m} k_i$, then every bounded solution of (1.4) oscillates.

Proof. Assume that $\{x_n\}$ is an eventually positive solution of equation (1.4). It is clear that, by (3.7) and Lemma 3.1, we have $\Delta x_n \leq 0$. Therefore, by using (3.5) and (3.6), and also applying the arithmetic-geometric mean inequality, we conclude that

$$\begin{aligned} 1 &\geq \sum_{i=1}^{m} p_i \frac{q^{k_i+r}}{(q-1)^r} \\ &\geq m\left(\prod_{i=1}^{m} p_i \frac{q^{k_i+r}}{(q-1)^r}\right)^{1/m} \\ &= m\frac{q^{k+r}}{(q-1)^r}\left(\prod_{i=1}^{m} p_i\right)^{1/m} \\ &\geq m\frac{1}{r^r}\frac{(k+r)^{k+r}}{k^k}\left(\prod_{i=1}^{m} p_i\right)^{1/m}, \end{aligned}$$

which contradicts (3.7). By a similar way, one can obtain that equation (1.4) has no eventually negative solution. ■

Remark 3.6. If $r = 1$ in equation (1.4) and $\liminf_{n\to\infty} p_{in} = p_i$, $k_i \in \mathbb{N}$ for $i = 1, 2, \dots, m$, $p_{in} \geq 0$, and

$$m\left(\prod_{i=1}^{m} p_i\right)^{1/m} > \frac{k^k}{(k+1)^{k+1}}$$

where $k = \frac{1}{m}\sum_{i=1}^{m} k_i$, then every solution of equation (1.4) oscillates, which are obtained by Erbe and Zhang in [6]. Also $p_{in} = p_i \in \mathbb{R}$ for $(i = 1, 2, \dots, m)$, $n = 0, 1, 2, \dots$, $k \in \mathbb{Z}$ and

$$m\left(\prod_{i=1}^{m} |p_i|\right)^{1/m}\left|\frac{(k+1)^{k+1}}{k^k}\right| > 1,$$

where $k = \frac{1}{m} \sum_{i=1}^{m} k_i$, then every solution of equation (1.4) oscillates, which is obtained by Ladas in [10].

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