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Powers of Prestarlike Functions and Trinomials

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Abstract

Applying the convolution or Hadamard product properties to starlike and prestarlike functions of certain order, we investigate the inclusion properties of powers of prestarlike functions and prestarlike trinomials.

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1. Introduction

Let H be the set of functions that are holomorphic or analytic in the open unit disk $\mathbb{D} := \{z : |z| < 1\}$. Let H_0 be the subset of H consisting of functions f so that $f(0) = f'(0) - 1 = 0$. The class of functions $f \in H_0$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

that are univalent in the open unit disk $\mathbb{D}$ is denoted by $\ddot{U}$.

A functions $f \in H_0$ is called starlike of order $\alpha : \alpha \leq 1$, denoted by $S(\alpha)$, if and only if

$$\operatorname{Re} \frac{zf'(z)}{f(z)} \geq \alpha; z \in \mathbb{D}.$$

We note that the set $S(\alpha)$ for $0 \leq \alpha \leq 1$ is a subset of $\ddot{U}$.

The convolution or Hadamard product of two power series $p(z) = \sum_{n=0}^{\infty} b_n z^n$ analytic in $|z| < r_1$ and $q(z) = \sum_{n=0}^{\infty} c_n z^n$ analytic in $|z| < r_2$ is the power series

$$(p * q)(z) = p(z) * q(z) = \sum_{n=0}^{\infty} b_n c_n z^n$$

which is analytic in $|z| < r_1 r_2$.

Ruscheweyh [1] defined the class $R(\alpha)$ of prestarlike functions of order $\alpha; \alpha \leq 1$. A functions $f \in H_0$ is said to be prestarlike of order $\alpha; \alpha \leq 1$ if and only if

$$f(z) * \frac{z}{(1-z)^{2(1-\alpha)}} \in S(\alpha); \alpha < 1, z \in \mathbb{D},$$

or

$$Re \frac{f(z)}{z} > \frac{1}{2}; \alpha = 1, z \in \mathbb{D}.$$

Note that the set $R(\alpha)$ for $0 \leq \alpha \leq 1$ is a subset of $\ddot{U}$.

The classes of functions prestarlike of order one-half and functions prestarlike of order zero are the two most favored cases since $R(1/2) \equiv S(1/2)$ and $R(0) \equiv K(0)$, where $K(0)$ is the class of functions $f \in \ddot{U}$ that are convex in $\mathbb{D}$. For functions $f \in \ddot{U}$ we have $f \in K(0)$ if and only if $zf' \in S(0)$.

For functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in R(\alpha); 0 \leq \alpha \leq 1$, Silverman and Silvia [3] (see also [4], Theorem 3, page 472) proved that

$$|a_n| \leq \frac{1 + 2(1-\alpha)|a_2|}{3 - 2\alpha}; n \geq 3.$$

Taking the partial sums of the functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in R(\alpha); \alpha \leq 1, |z| < 1$, Ruscheweyh [2] proved that

$$z + \sum_{i=2}^{n} a_i z^i \in R(\alpha); \alpha \leq 1, |z| < \frac{1}{2(2-\alpha)}.$$

There are interesting correlations between the powers of prestarlike functions which can shed light on the powers of prestarlike polynomials. In the next section we use the properties of convolution and prestarlike functions to obtain such results which lead to an interesting prestarlike trinomial. The case for quadrinomials and higher degree polynomials is open.

2. Main Results

Theorem 2.1. Let $p(z) = 1 + \sum_{n=1}^{\infty} b_n z^n$ be analytic in $\mathbb{D}$. Then $zp(z) \in R(1/2)$ if and only if

$$g(z) = \sum_{n=0}^{\infty} \frac{b_n}{C_{mn}(m)} z^{mn+1} \in R\left(1 - \frac{1}{2}m\right)$$

where $z/(1-z)^m = \sum_{k=0}^{\infty} C_k(m) z^{k+1}$ and $b_0 = 1, m \in \mathbb{N} = \{1, 2, 3, \ldots\}$.

Proof. First assume that $zp(z) \in R(1/2)$. Then, by definition, we have

$$\operatorname{Re} \frac{z(zp(z))'}{zp(z)} \geq \frac{1}{2}; z \in \mathbb{D}.$$

This is equivalent to

$$\operatorname{Re} \frac{1 + 3b_1 z + 5b_2 z^2 + 7b_3 z^3 + \cdots}{1 + b_1 z + b_2 z^2 + b_3 z^3 + \cdots} \geq 0; z \in \mathbb{D}$$

or

$$\operatorname{Re} \frac{1 + 3b_1 z^m + 5b_2 z^{2m} + 7b_3 z^{3m} + \cdots}{1 + b_1 z^m + b_2 z^{2m} + b_3 z^{3m} + \cdots} \geq 0; z \in \mathbb{D}, m \in \mathbb{N}.$$

A simple manipulation leads to

$$\operatorname{Re} \frac{1 + (m+1)b_1 z^m + (2m+1)b_2 z^{2m} + (3m+1)b_3 z^{3m} + \cdots}{1 + b_1 z^m + b_2 z^{2m} + b_3 z^{3m} + \cdots} \geq \left(1 - \frac{1}{2}m\right).$$

This latter inequality is equivalent to the fact that

$$\operatorname{Re} \frac{z(zp(z^m))'}{zp(z^m)} \geq \left(1 - \frac{1}{2}m\right); m \in \mathbb{N}.$$

Therefore, by the definition of starlike functions,

$$zp(z^m) \in S\left(1 - \frac{1}{2}m\right); z \in \mathbb{D}, m \in \mathbb{N}.$$

Now rewriting $zp(z^m)$ as

$$zp(z^m) = \frac{z}{(1-z)^m} * \sum_{n=0}^{\infty} \frac{b_n}{C_{mn}(m)} z^{mn+1} = \frac{z}{(1-z)^m} * g(z)$$

and applying the definition of prestarlike functions, we conclude that

$$g(z) \in R\left(1 - \frac{1}{2}m\right); m \in \mathbb{N}.$$

The "only if" part of the proof of the theorem follows easily upon noting that if $g(z) \in R\left(1 - \frac{1}{2}m\right)$ then, by an argument similar to that used in the first part of the proof, $zp(z^m) \in S\left(1 - \frac{1}{2}m\right)$ and consequently $zp(z) \in R(1/2)$. ∎

As an application of the above theorem, we obtain the following prestarlike polynomial example.

Example 2.2. For arbitrary complex numbers α and β consider the trinomial $p(z) = 1 + \alpha z^n + \beta z^{2n}$ where $z \in \mathbb{D}$ and $n \in \mathbb{N}$. Then $zp(z) \in R(1/2)$ if and only if

$$z + \frac{\alpha}{C_{mn}(m)} z^{mn+1} + \frac{\beta}{C_{2mn}(m)} z^{2mn+1} \in R\left(1 - \frac{1}{2}m\right); m \in \mathbb{N}.$$

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