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Quantum Algorithm for Vertex Coloring Problem by Numbering Method

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Abstract

A quantum algorithm for the vertex coloring problem by a numbering method and its example are reported. When n vertexes are connected m edges, and both vertexes of each edge are different colors, a number of colors that is k is decided. A computational complexity of a classical computation is k^n. The computational complexity becomes about n^2 by the quantum algorithm that uses quantum phase inversion gates, quantum inversion about mean gates and the numbering method. Therefore, a polynomial time process becomes possible.

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1. Introduction

Haroche and Wineland [1] made the very first steps towards building a quantum computer. Deutsch - Jozsa's algorithm for the rapid solution [2−4], Shor's algorithm for the factorization [3−5], Grover's algorithms for the database search [3, 6, 7] and so on are known. A quantum algorithm for the traveling salesman problem by a numbering method has recently been reported by Fujimura [8]. Its computational complexity becomes a polynomial time. The vertex coloring problem [9, 10] is examined by the numbering method this time. Therefore, its result is reported.

2. Vertex Coloring Problem

When n vertexes are connected by m edges, and both vertexes of each edge are different colors, a number of colors that is k is decided.

3. Quantum Algorithm

It is assumed that n vertexes are connected by m edges, and two vertexes have only one edge, because several edges don't change the essence of this problem. Therefore, when there is an edge between the i - th vertex and the j - th vertex, $x_{i,j}$ [$1 \le i < j \le n$. i and j are integers.] is 1, and when there isn't an edge between these vertexes, it is 0. Now, it is assumed that a number of colors is k, and a_i [$1 \le i \le n$. i is the integer.] is 0 or 1 or $\cdots$ or $k - 1$. When the number of the n times repeated permutation of 0, 1, $\cdots$, $k - 2$ and $k - 1$ is k^n, $a_1 k^{n-1} + a_2 k^{n-2} + \cdots + a_n k^0 = \sum_{i=1 \to n} (a_i k^{n-i}) = U$ is the numbering datum from 0 to $k^n - 1$ [The 0-th datum is 0, 0, $\cdots$, 0 and 0. The $(k^n - 1)$-th datum is $(k - 1)$, $(k - 1)$, $\cdots$, $(k - 1)$ and $(k - 1)$.]. This method is named the numbering method for this problem. g is the minimum integer that follows $k^n / k! \le 4^g = 2^{2g}$, because a number of combinations of an answer is $k!$ at least.

First of all, quantum registers $|a_1>$, $|a_2>$, $\cdots$, $|a_n>$, $|b_1>$, $|b_2>$, $|c>$ and $|d>$ are prepared. When α is the minimum integer that is $log_2 k$ or more, each of $|a_f>$ that f is an integer from 1 to n is consisted of α quantum bits [= qubits]. States of $|a_f>$, $|b_1>$, $|b_2>$, $|c>$ and $|d>$ are a_f, b_1, b_2, c and d, respectively.

Step 1: Each qubit of $|a_f>$, $|b_1>$, $|b_2>$, $|c>$ and $|d>$ is set $|0>$.

Step 2: The Hadamard gate $\boxed{H}$ [3, 4] acts on each qubit of $|a_f>$. It changes them for entangled states. The total states are $(2^\alpha)^n$.

Step 3: It is assumed that a quantum gate (A) changes $|b_1>$ for $|1>$ in $a_f < k$, or it changes $|b_1>$ for $|0>$ in the others of a_f. As a target state for $|b_1>$ is 1, quantum phase inversion gates (PI) and quantum inversion about mean gates (IM) [3, 6, 7] act on $|b_1>$. When β is the minimum even integer that is $(2^\alpha/k)^{1/2}$ or more, the total number that (PI) and (IM) act on $|b_1>$ is β, because they are a couple. Next, an observation gate (OB) observes $|b_1>$. These actions are repeated sequentially from $|a_1>$ to $|a_n>$. Therefore, each state of $|a_f>$ is 0, 1, $\cdots$, $k - 2$ and $k - 1$, and the total states become k^n [= W_0].

Step 4: It is assumed that a quantum gate ($B_{i,j}$) [$1 \le i < j \le n$. i and j are integers.] changes $|b_1>$ for $|b_1 + x_{i,j}>$ at $a_i \ne a_j$, or it doesn't change $|b_1>$ at $a_i = a_j$, and it changes $|b_2>$ for $|b_2 + (a_i k^{n-i} + a_j k^{n-j})/(n - 1)>$ from a_i and a_j. These actions are repeated sequentially at i and j. Therefore, $|b_1>$ becomes from $|0>$ to $|x_{1,2} + x_{1,3} + \cdots$

$+ x_{n-1, n} >$, and $|b_2 >$ becomes $|a_1 k^{n-1} + a_2 k^{n-2} + \cdots + a_n k^0 >$.

Step 5: It is assumed that a quantum gate (C) doesn't change $|c >$ at $b_1 = m$, or it changes $|c >$ for $|c + 1 + b_2 >$ in the others of b_1.

Step 6: It is assumed that a quantum gate (D_1) changes $|d >$ for $|1 >$ in $0 \le c \le (k^n/4) - k!$, or it changes $|d >$ for $|0 >$ in the others of c. As the target state for $|d >$ is 1, (PI) and (IM) act on $|d >$. The number of the data that is included in $0 \le c \le (k^n/4) - k!$ is $W_1 \approx k^n/4$. When γ_1 is the minimum even integer that is $(W_0/W_1)^{1/2} \approx (k^n/(k^n/4))^{1/2}$ or more, the total number that (PI) and (IM) act on $|d >$ is $\gamma_1 \approx 2$. Next, (OB) observes $|d >$, and the data of W_1 remain.

Similarly, (D_i) [$2 \le i \le g - 1$. i is the integer.] changes $|d >$ for $|1 >$ in $0 \le c \le (k^n/4^i) - k!$, or it changes $|d >$ for $|0 >$ in the others of c. As the target state for $|d >$ is 1, (PI) and (IM) act on $|d >$. The number of the data that is included in $0 \le c \le (k^n/4^i) - k!$ is $W_i \approx k^n/4^i$. When γ_i is the minimum even integer that is $(W_{i-1}/W_i)^{1/2} \approx ((k^n/4^{i-1})/(k^n/4^i))^{1/2}$ or more, the total number that (PI) and (IM) act on $|d >$ is $\gamma_i \approx 2$. Next, (OB) observes $|d >$, and the data of W_i remain.

(D_g) changes $|d >$ for $|1 >$ at $c = 0$, or it changes $|d >$ for $|0 >$ in the others of c. As the target state for $|d >$ is 1, (PI) and (IM) act on $|d >$. The number of the data that is included at $c = 0$ is $W_g \approx k! \approx k^n/4^g$. When γ_g is the minimum even integer that is $(W_{g-1}/W_g)^{1/2} \approx ((k^n/4^{g-1})/(k^n/4^g))^{1/2}$ or more, the total number that (PI) and (IM) act on $|d >$ is $\gamma_g \approx 2$. Next, (OB) observes $|a_f >$, $|b_1 >$, $|b_2 >$, $|c >$ and $|d >$, and one of the data W_g remains. Therefore, one example of combinations that are $b_1 = m$ is obtained.

4. Numerical Computation

It is assumed that there are $n = 4$, $x_{1,2} = x_{1,3} = x_{1,4} = x_{2,3} = x_{3,4} = 1$, $x_{2,4} = 0$, $m = 5$, $k = 3$ and $g = 2$ [$k^n/k! = 3^4/6 = 13.5 \le 4^g = 4^2 = 16$].

First of all, $|a_1 >$, $|a_2 >$, $|a_3 >$, $|a_4 >$, $|b_1 >$, $|b_2 >$, $|c >$ and $|d >$ are prepared. When α is the minimum integer that is $log_2 3 \approx 1.6 \le 2 = \alpha$, each of $|a_f >$ that f is the integer from 1 to 4 is consisted of 2 qubits. States of $|a_f >$, $|b_1 >$, $|b_2 >$, $|c >$ and $|d >$ are a_f, b_1, b_2, c and d, respectively.

Step 1: Each qubit of $|a_f >$, $|b_1 >$, $|b_2 >$, $|c >$ and $|d >$ is set $|0 >$.

Step 2: $\boxed{H}$ acts on each qubit of $|a_f >$. It changes them for entangled states. The total states are $(2^2)^4$.

Step 3: (A) changes $|b_1>$ for $|1>$ in $a_f < 3$, or it changes $|b_1>$ for $|0>$ in the others of a_f. As the target state for $|b_1>$ is 1, (PI) and (IM) act on $|b_1>$. When β is the minimum even integer that is $(2^2/3)^{1/2} \approx 1.2 \leq 2 = \beta$, the total number that (PI) and (IM) act on $|b_1>$ is $\beta \approx 2$. Next, (OB) observes $|b_1>$. These actions are repeated sequentially from $|a_1>$ to $|a_4>$. Therefore, each state of $|a_f>$ is 0, 1 and 2, and the total states become 3^4 [$= W_0$].

Step 4: ($B_{i,j}$) [$1 \leq i < j \leq 4$. i and j are integers.] changes $|b_1>$ for $|b_1 + x_{i,j}>$ at $a_i \neq a_j$, or it doesn't change $|b_1>$ at $a_i = a_j$, and it changes $|b_2>$ for $|b_2 + (a_i\, 3^{4-i} + a_j\, 3^{4-j})/3>$ from a_i and a_j. These actions are repeated sequentially at i and j. Therefore, $|b_1>$ becomes from $|0>$ to $|x_{1,2} + x_{1,3} + \cdots + x_{3,4}>$, and $|b_2>$ becomes $|a_1\, 3^3 + a_2\, 3^2 + a_3\, 3^1 + a_4\, 3^0>$.

Step 5: (C) doesn't change $|c>$ at $b_1 = 5$, or it changes $|c>$ for $|c + 1 + b_2>$ in the others of b_1.

Step 6: (D_1) changes $|d>$ for $|1>$ in $0 \leq c \leq (3^4/4) - 6$, or it changes $|d>$ for $|0>$ in the others of c. As the target state for $|d>$ is 1, (PI) and (IM) act on $|d>$. The number of the data that is included in $0 \leq c \leq (3^4/4) - 6$ is $W_1 \approx 3^4/4$. When γ_1 is the minimum even integer that is $(W_0/W_1)^{1/2} \approx (3^4/(3^4/4))^{1/2} \approx 2 \leq 2 = \gamma_1$, the total number that (PI) and (IM) act on $|d>$ is $\gamma_1 \approx 2$. Next, (OB) observes $|d>$, and the data of W_1 remain.
(D_2) changes $|d>$ for $|1>$ at $c = 0$, or it changes $|d>$ for $|0>$ in the others of c. As the target state for $|d>$ is 1, (PI) and (IM) act on $|d>$. The number of the data that is included at $c = 0$ is $W_2 \approx 6 \approx 3^4/16$. When$\gamma_2$ is the minimum even integer that is $(W_1/W_2)^{1/2} \approx ((3^4/4)/(3^4/16))^{1/2} \approx 2 \leq 2 = \gamma_2$, the total number that (PI) and (IM) act on $|d>$ is $\gamma_2 \approx 2$. Next, (OB) observes $|a_1>$, $|a_2>$, $|a_3>$, $|a_4>$, $|b_1>$, $|b_2>$, $|c>$ and $|d>$, and one of the data of W_2 remains. For example, when a_1, a_2, a_3, a_4, b_1, b_2, c and d are 0, 1, 2, 1, 5, 16, 0 and 1, respectively, it is obtained that the 0th vertex is the 0th color, the 1st and 3rd vertexes are the 1st color, and the 2nd vertex is the 2nd color. Therefore, 3! = 6 combinations are obtained from this answer at least.

5. Discussion and Summary

The computational complexity of this quantum algorithm [= S] becomes the following. In the order of the actions by the gates, the number of them is αn at $\boxed{\mathrm{H}}$, n at (A), $\beta n \approx 2n$ at (PI) and (IM), n at (OB), $n(n-1)$ at ($B_{i,j}$) [$1 \leq i < j \leq n$. i and j are integers.], 2 at (C), g at (D_i) [$1 \leq i \leq g$. i is the integer.], $\sum_{i=1\to g} \gamma_i \approx 2g$ at (PI) and (IM), and g at (OB). Therefore, S becomes $n^2 + (\alpha + 3)n + 2 + 4g$. In the example of the section 4, S is 46. The computational complexity of the classical computation

[= Z] is $k^n = 3^4 = 81$. After all, S/Z becomes about 1/2. When n is large enough, S becomes about n^2. And then S/Z is about n^2/k^n. For example, as for $n = 100$ and $k = 4$, S/Z is about $100^2/4^{100} \approx 1/10^{56}$.

Therefore, the polynomial time process becomes possible.

References

[1] Kungl. Vetenskapsakademien (The Royal Swedish Academy of Sciences), The Nobel Prize in Physics 2012, [On line], Available: http: //www. kva. se/en /pressroom /Press-releases-2012 /The-Nobel-Prize-in-Physics-2012/, 2012.

[2] Deutsch D., and Jozsa R., Rapid solution of problems by quantum computation, *Proc. Roy. Soc. Lond. A*, 439:553-558, 1992.

[3] Takeuchi S., Ryoshi Konpyuta (Quantum Computer), Kodansha, Tokyo, Japan [in Japanese], 2005.

[4] Miyano K., and Furusawa A., Ryoshi Konpyuta Nyumon (An Introduction to Quantum Computation), Nippon Hyoron sha, Tokyo, Japan [in Japanese], 2008.

[5] Shor P.W., Algorithms for quantum computation: discrete logarithms and factoring, *Proc. 35th Annu. Symp. Foundations of Computer Science*, IEEE, pp.124-134, 1994.

[6] Grover L.K., A fast quantum mechanical algorithm for database search, *Proc. 28th Annu. ACM Symp. Theory of Computing*, pp.212-219, 1996.

[7] Grover L.K., A framework for fast quantum mechanical algorithms, *Proc. 30th Annu. ACM Symp. Theory of Computing*, pp.53-62, 1998.

[8] Fujimura T., Quantum algorithm for traveling salesman problem by numbering method, *Glob. J. Pure Appl. Math.*, 9, 2013, [to be published].

[9] Weisstein E.W., Vertex coloring, [On line], Available: http: // mathworld. wolfram. com/ VertexColoring. html, 2011.

[10] Fujimura T., Quantum algorithm for vertex coloring problem by central limit theorem, *Glob. J. Pure Appl. Math.*, 7:401-405, 2011.