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More on a Reverse Hilbert-Type Integral Inequality

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Abstract

In 2011, a new reverse Hilbert-type integral inequality was presented. In this paper, we present a generalization of the integral inequality.

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1. Introduction

In 1934, Hardy, Littlewood and Polya [1] presented an integral inequality as follows. For $f, g \geq 0$, $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, if

$$0 < \int_0^\infty f^p(x)dx < \infty \text{ and } 0 < \int_0^\infty g^q(y)dy < \infty,$$

then

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{\max\{x, y\}} dxdy < pq\left(\int_0^\infty f^p(x)dx\right)^{\frac{1}{p}}\left(\int_0^\infty g^q(y)dy\right)^{\frac{1}{q}}.$$

In 2007, Yang [3] presented an reverse integral inequality as follows. For $f, g \geq 0$, $0 < p < 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\alpha > 0$, $2 - p < \lambda < 2 - q$, if

$$0<\int_0^\infty \frac{x^{p(1-\alpha)}}{x^{1+\alpha(\lambda-2)}} f^p(x)dx<\infty \text{ and } 0<\int_0^\infty \frac{x^{q(1-\alpha)}}{x^{1+\alpha(\lambda-2)}} g^q(y)dy<\infty,$$

then

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{\left(x^\alpha+y^\alpha\right)^\lambda}dxdy > B\left(\frac{p+\lambda-2}{\lambda p},\frac{q+\lambda-2}{\lambda q}\right)$$
$$\times\left(\int_0^\infty \frac{x^{p(1-\alpha)}}{x^{1+\alpha(\lambda-2)}} f^p(x)dx\right)^{\frac{1}{p}}\left(\int_0^\infty \frac{x^{q(1-\alpha)}}{x^{1+\alpha(\lambda-2)}} g^q(y)dy\right)^{\frac{1}{q}}$$

where B is the beta function.

In 2011, Bing He [2] presented a new reverse Hilbert-type integral inequality as follows. For $f,g\geq 0$, $0<p<1$, $\frac{1}{p}+\frac{1}{q}=1$, $\lambda+\mu>0$, if

$$0<\int_0^\infty x^{p\left(1+\frac{\lambda-\mu}{2}\right)-1} f^p(x)dx<\infty \text{ and } 0<\int_0^\infty y^{q\left(1+\frac{\lambda-\mu}{2}\right)-1} g^q(y)dy<\infty,$$

then

$$\int_0^\infty\int_0^\infty \frac{\left(\min\{x,y\}\right)^\lambda}{\left(\max\{x,y\}\right)^\mu} f(x)g(y)dxdy$$
$$\geq \frac{4}{\lambda+\mu}\left(\int_0^\infty x^{p\left(1+\frac{\lambda-\mu}{2}\right)-1} f^p(x)dx\right)^{\frac{1}{p}}\left(\int_0^\infty y^{q\left(1+\frac{\lambda-\mu}{2}\right)-1} g^q(y)dy\right)^{\frac{1}{q}}.$$

In this paper, we present a generalization of this integral inequality.

2. Main Results

2.1. Lemma

Assume that $0<p<1$, $\frac{1}{p}+\frac{1}{q}=1$, $0\leq\alpha<\lambda+\mu$, and $u>0$. Then

$$\int_0^\infty \frac{\left(\min\{u,v\}\right)^\lambda}{\left(\max\{u,v\}\right)^\mu}\cdot\frac{u^{-\frac{\lambda-\mu}{2}-\frac{\alpha}{2}}}{v^{1+\frac{\lambda-\mu}{2}-\frac{\alpha}{2}}}dv=\frac{2}{\lambda+\mu+\alpha}+\frac{2}{\lambda+\mu-\alpha}.$$

Proof. Denote $t=\frac{v}{u}$. Then

$$\int_0^{\infty} \frac{\left(\min\{u,v\}\right)^{\lambda}}{\left(\max\{u,v\}\right)^{\mu}} \cdot \frac{u^{-\frac{\lambda-\mu}{2}-\frac{\alpha}{2}}}{v^{1+\frac{\lambda-\mu}{2}-\frac{\alpha}{2}}} dv = \int_0^{\infty} \frac{\left(\min\{1,t\}\right)^{\lambda}}{\left(\max\{1,t\}\right)^{\mu}} \cdot t^{-1-\frac{\lambda-\mu}{2}+\frac{\alpha}{2}} dt$$

$$= \int_0^{1} \frac{\left(\min\{1,t\}\right)^{\lambda}}{\left(\max\{1,t\}\right)^{\mu}} \cdot t^{-1-\frac{\lambda-\mu}{2}+\frac{\alpha}{2}} dt$$

$$+ \int_1^{\infty} \frac{\left(\min\{1,t\}\right)^{\lambda}}{\left(\max\{1,t\}\right)^{\mu}} \cdot t^{-1-\frac{\lambda-\mu}{2}+\frac{\alpha}{2}} dt$$

$$= \int_0^{1} \frac{t^{\lambda}}{1} \cdot t^{-1-\frac{\lambda-\mu}{2}+\frac{\alpha}{2}} dt$$

$$+ \int_1^{\infty} \frac{1}{t^{\mu}} t^{-1-\frac{\lambda-\mu}{2}+\frac{\alpha}{2}} dt$$

$$= \frac{2}{\lambda+\mu+\alpha} + \frac{2}{\lambda+\mu-\alpha}.$$

This proof is completed. □

2.2. Theorem

Let $f, g \geq 0$. Assume that $0 < p < 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $0 \leq \alpha < \lambda + \mu$,

$$0 < \int_0^{\infty} x^{p\left(1+\frac{\lambda-\mu}{2}-\frac{\alpha}{2}\right)-1+\alpha} f^{p}(x) dx < \infty,$$

and

$$0 < \int_0^{\infty} y^{q\left(1+\frac{\lambda-\mu}{2}-\frac{\alpha}{2}\right)-1+\alpha} g^{q}(y) dy < \infty.$$

Then

$$\int_0^{\infty}\int_0^{\infty} \frac{\left(\min\{x,y\}\right)^{\lambda}}{\left(\max\{x,y\}\right)^{\mu}} f(x)g(y) dx dy$$

$$\geq \left(\frac{2}{\lambda+\mu+\alpha} + \frac{2}{\lambda+\mu-\alpha}\right)\left(\int_0^{\infty} x^{p\left(1+\frac{\lambda-\mu}{2}-\frac{\alpha}{2}\right)-1+\alpha} f^{p}(x) dx\right)^{\frac{1}{p}}$$

$$\times \left(\int_0^{\infty} y^{q\left(1+\frac{\lambda-\mu}{2}-\frac{\alpha}{2}\right)-1+\alpha} g^{q}(y) dy\right)^{\frac{1}{q}}.$$

Proof. By the reverse Hölder inequality, and by Lemma 2.1, we obtain that

$$
\begin{aligned}
&\int_0^\infty \int_0^\infty \frac{\left(\min\{x,y\}\right)^{\lambda}}{\left(\max\{x,y\}\right)^{\mu}} f(x)g(y)dxdy \\
&= \int_0^\infty \int_0^\infty \left(\frac{\left(\min\{x,y\}\right)^{\frac{\lambda}{p}}}{\left(\max\{x,y\}\right)^{\frac{\mu}{p}}} \cdot \frac{x^{\left(1+\frac{\lambda-\mu}{2}-\frac{\alpha}{2}\right)/q}}{y^{\left(1+\frac{\lambda-\mu}{2}-\frac{\alpha}{2}\right)/p}} f(x) \right) \\
&\quad \times \left(\frac{\left(\min\{x,y\}\right)^{\frac{\lambda}{q}}}{\left(\max\{x,y\}\right)^{\frac{\mu}{q}}} \cdot \frac{y^{\left(1+\frac{\lambda-\mu}{2}-\frac{\alpha}{2}\right)/p}}{x^{\left(1+\frac{\lambda-\mu}{2}-\frac{\alpha}{2}\right)/q}} g(y) \right) dxdy \\
&\geq \left(\int_0^\infty \int_0^\infty \frac{\left(\min\{x,y\}\right)^{\lambda}}{\left(\max\{x,y\}\right)^{\mu}} \cdot \frac{x^{\left(1+\frac{\lambda-\mu}{2}-\frac{\alpha}{2}\right)(p-1)}}{y^{1+\frac{\lambda-\mu}{2}-\frac{\alpha}{2}}} f^{p}(x)dxdy \right)^{\frac{1}{p}} \\
&\quad \times \left(\int_0^\infty \int_0^\infty \frac{\left(\min\{x,y\}\right)^{\lambda}}{\left(\max\{x,y\}\right)^{\mu}} \cdot \frac{y^{\left(1+\frac{\lambda-\mu}{2}-\frac{\alpha}{2}\right)(q-1)}}{x^{1+\frac{\lambda-\mu}{2}-\frac{\alpha}{2}}} g^{q}(x)dxdy \right)^{\frac{1}{q}} \\
&= \left(\int_0^\infty x^{p\left(1+\frac{\lambda-\mu}{2}-\frac{\alpha}{2}\right)-1+\alpha} f^{p}(x) \left[\int_0^\infty \frac{\left(\min\{x,y\}\right)^{\lambda}}{\left(\max\{x,y\}\right)^{\mu}} \cdot \frac{x^{-\frac{\lambda-\mu}{2}-\frac{\alpha}{2}}}{y^{1+\frac{\lambda-\mu}{2}-\frac{\alpha}{2}}} dy \right] dx \right)^{\frac{1}{p}} \\
&\quad \times \left(\int_0^\infty y^{p\left(1+\frac{\lambda-\mu}{2}-\frac{\alpha}{2}\right)-1+\alpha} g^{q}(y) \left[\int_0^\infty \frac{\left(\min\{x,y\}\right)^{\lambda}}{\left(\max\{x,y\}\right)^{\mu}} \cdot \frac{y^{-\frac{lambda-\mu}{2}-\frac{\alpha}{2}}}{x^{1+\frac{\lambda-\mu}{2}-\frac{\alpha}{2}}} dx \right] dy \right)^{\frac{1}{q}} \\
&= \left(\int_0^\infty x^{p\left(1+\frac{\lambda-\mu}{2}-\frac{\alpha}{2}\right)-1+\alpha} f^{p}(x) \left[\frac{2}{\lambda+\mu+\alpha} + \frac{2}{\lambda+\mu-\alpha} \right] dx \right)^{\frac{1}{p}} \\
&\quad \times \left(\int_0^\infty y^{p\left(1+\frac{\lambda-\mu}{2}-\frac{\alpha}{2}\right)-1+\alpha} g^{q}(y) \left[\frac{2}{\lambda+\mu+\alpha} + \frac{2}{\lambda+\mu-\alpha} \right] dy \right)^{\frac{1}{q}} \\
&= \left(\frac{2}{\lambda+\mu+\alpha} + \frac{2}{\lambda+\mu-\alpha} \right) \left(\int_0^\infty x^{p\left(1+\frac{\lambda-\mu}{2}-\frac{\alpha}{2}\right)-1+\alpha} f^{p}(x)dx \right)^{\frac{1}{p}} \\
&\quad \times \left(\int_0^\infty y^{q\left(1+\frac{\lambda-\mu}{2}-\frac{\alpha}{2}\right)-1+\alpha} g^{q}(y)dy \right)^{\frac{1}{q}}.
\end{aligned}
$$

This proof is completed.

2.3. Corollary

[2] Let $f, g \geq 0$. Assume that $0 < p < 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\lambda + \mu > 0$, and

$$0 < \int_0^\infty x^{p\left(1+\frac{\lambda-\mu}{2}\right)-1} f^p(x)dx < \infty \text{ and } 0 < \int_0^\infty y^{q\left(1+\frac{\lambda-\mu}{2}\right)-1} g^q(y)dy < \infty .$$

Then

$$\int_0^\infty \int_0^\infty \frac{\left(\min\{x, y\}\right)^\lambda}{\left(\max\{x, y\}\right)^\mu} f(x)g(y)dxdy$$
$$\geq \frac{4}{\lambda+\mu}\left(\int_0^\infty x^{p\left(1+\frac{\lambda-\mu}{2}\right)-1} f^p(x)dx\right)^{\frac{1}{p}}\left(\int_0^\infty y^{q\left(1+\frac{\lambda-\mu}{2}\right)-1} g^q(y)dy\right)^{\frac{1}{q}}.$$

Proof. This follows fr0m Theorem 2.2 where $\alpha = 0$. □

References

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