Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 9, Number 3 (2013), pp. 227-231

http://www.ripublication.com/gjpam.htm

An Inequality Similar to Hardy-Hilbert's Integral Inequality

Banyat Sroysang

Department of Mathematics and Statistics, Faculty of Science and Technology, Thammasat University, Pathumthani 12121, Thailand.
E-mail: banyat@mathstat.sci.tu.ac.th

Abstract

In 2010, Sulaiman gave two new inequalities similar to Hardy-Hibert's integral inequality. In this paper, we present an other inequality similar to Hardy-Hibert's integral inequality.

Mathematics Subject Classification: 26D15.

Keywords: Hardy-Hibert's inequality, integral inequality.

1. Introduction

For any $f, g \geq 0$, if

$$0 < \int_0^\infty f^2(x)dx < \infty \text{ and } 0 < \int_0^\infty g^2(y)dy < \infty,$$

then we have the Hibert's integral inequality

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dxdy < \pi\left(\int_0^\infty f^2(x)dx \int_0^\infty g^2(y)dy\right)^{1/2}.$$

In 1925, Hardy [2] present the Hardy-Hibert's integral inequality as follows. For $f, g \geq 0$, $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, if

$$0 < \int_0^\infty f^p(x)dx < \infty \text{ and } 0 < \int_0^\infty g^q(y)dy < \infty,$$

then

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{x+y}dxdy < \frac{\pi}{\sin(\pi/p)}\left(\int_0^\infty f^p(x)dx\right)^{1/p}\left(\int_0^\infty g^q(y)dy\right)^{1/q}.$$

This inequality is important in analysis and application (see [3]).

In 2010, Sulaiman [4] gave two new inequalities similar to Hardy-Hibert's integral inequality. In this paper, we present an other inequality similar to Hardy-Hibert's integral inequality.

2. Preliminaries

The beta function is defined by

$$B(a,b) = \int_0^\infty \frac{t^{a-1}}{(1+t)^{a+b}}dt,$$

where $a,b>0$.

2.1. Proposition

[4] If $0<p<\frac{1}{2}$, then

$$\int_0^\infty \frac{t^{p-1}}{|1-t|^{2p}}dt = 2B(p,1-2p).$$

2.2. Theorem

[1] Let f be a nonneqative integrable function, $p>1$, and

$$F(x) = \int_0^x f(t)dt.$$

Then

$$\int_0^\infty \left(\frac{F(x)}{x}\right)^p dx \le \left(\frac{p}{p-1}\right)^p \int_0^\infty f^p(x)dx.$$

3. Main Results

3.1. Theorem

Let $f,g\ge 0$. Assume that $1<p<\frac{\alpha}{2}$, $1<q<\frac{\beta}{2}$, and $\frac{1}{p}+\frac{1}{q}=1$. Define

$$F(x) = \int_0^x f(t)dt \text{ and } G(x) = \int_0^x g(t)dt.$$

for all $x > 0$. Then

$$\int_0^\infty \int_0^\infty \frac{x^{\frac{1}{\alpha}+\frac{2}{\beta}-1} y^{\frac{2}{\alpha}+\frac{1}{\beta}-1} F^{\frac{1}{\alpha}+\frac{1}{p}}(x) G^{\frac{1}{\beta}+\frac{1}{q}}(y)}{\left|x^2-y^2\right|^{2+\frac{2}{\beta}}} dxdy$$

$$\leq C_{p,\alpha}^{\frac{1}{p}} C_{q,\beta}^{\frac{1}{q}} \left(\int_0^\infty f^{\frac{p}{\alpha}+1}(x)dx\right)^{\frac{1}{p}} \left(\int_0^\infty g^{\frac{q}{\beta}+1}(y)dy\right)^{\frac{1}{q}} \tag{1}$$

where

$$C_{p,\alpha} = \left(1+\tfrac{\alpha}{p}\right)^{\frac{p}{\alpha}} B\left(\tfrac{p}{\alpha}, 1-\tfrac{2p}{\alpha}\right) \text{ and } C_{q,\beta} = \left(1+\tfrac{\beta}{q}\right)^{\frac{q}{\beta}} B\left(\tfrac{q}{\beta}, 1-\tfrac{2q}{\beta}\right).$$

Proof. By the Hölder inequality, we have

$$\int_0^\infty \int_0^\infty \frac{x^{\frac{1}{\alpha}+\frac{2}{\beta}-1} y^{\frac{2}{\alpha}+\frac{1}{\beta}-1} F^{\frac{1}{\alpha}+\frac{1}{p}}(x) G^{\frac{1}{\beta}+\frac{1}{q}}(y)}{\left|x^2-y^2\right|^{\frac{2}{\alpha}+\frac{2}{\beta}}} dxdy$$

$$= \int_0^\infty \int_0^\infty \left(\frac{x^{\frac{1}{\alpha}-\frac{1}{p}} y^{\frac{2}{\alpha}-\frac{1}{p}} F^{\frac{1}{\alpha}+\frac{1}{p}}(x)}{\left|x^2-y^2\right|^{\frac{2}{\alpha}}}\right) \left(\frac{x^{\frac{2}{\beta}-\frac{1}{q}} y^{\frac{1}{\beta}-\frac{1}{q}} G^{\frac{1}{\beta}+\frac{1}{q}}(y)}{\left|y^2-x^2\right|^{\frac{2}{\beta}}}\right) dxdy$$

$$\leq \left(\int_0^\infty \int_0^\infty \frac{x^{\frac{p}{\alpha}-1} y^{\frac{2p}{\alpha}-1} F^{\frac{p}{\alpha}+1}(x)}{\left|x^2-y^2\right|^{\frac{2p}{\alpha}}} dxdy\right)^{\frac{1}{p}}$$

$$\times \left(\int_0^\infty \int_0^\infty \frac{x^{\frac{2q}{\beta}-1} y^{\frac{q}{\beta}-1} G^{\frac{q}{\beta}+1}(y)}{\left|y^2-x^2\right|^{\frac{2q}{\beta}}} dxdy\right)^{\frac{1}{q}}$$

$$= \left(\int_0^\infty \left(\frac{F(x)}{x}\right)^{\frac{p}{\alpha}+1} \int_0^\infty \frac{x^{\frac{2p}{\alpha}} y^{\frac{2p}{\alpha}-1}}{\left|x^2-y^2\right|^{\frac{2p}{\alpha}}} dydx\right)^{\frac{1}{p}}$$

$$\times \left(\int_0^\infty \left(\frac{G(y)}{y}\right)^{\frac{q}{\beta}+1} \int_0^\infty \frac{x^{\frac{2q}{\beta}-1} y^{\frac{2q}{\beta}}}{\left|y^2-x^2\right|^{\frac{2q}{\beta}}} dxdy\right)^{\frac{1}{q}}$$

$$= \left(\begin{array}{l} \int_0^\infty \left(\frac{F(x)}{x}\right)^{\frac{p}{\alpha}+1} \frac{1}{2} \int_0^\infty \frac{\left(\frac{y^2}{x^2}\right)^{\frac{p}{\alpha}-1}}{\left|1-\frac{y^2}{x^2}\right|^{\frac{2p}{\alpha}}} d\left(\frac{y^2}{x^2}\right) dxright)^{\frac{1}{p}} \\ \times \left(\int_0^\infty \left(\frac{G(y)}{y}\right)^{\frac{q}{\beta}+1} \frac{1}{2} \int_0^\infty \frac{\left(\frac{x^2}{y^2}\right)^{\frac{q}{\beta}-1}}{\left|1-\frac{x^2}{y^2}\right|^{\frac{2q}{\beta}}} d\left(\frac{x^2}{y^2}\right) dy\right)^{\frac{1}{q}}. \end{array} \right.$$

By Proposition 2.1, we have

$$\int_0^\infty\int_0^\infty \frac{x^{\frac{1}{\alpha}+\frac{2}{\beta}-1}y^{\frac{2}{\alpha}+\frac{1}{\beta}-1}F^{\frac{1}{\alpha}+\frac{1}{p}}(x)G^{\frac{1}{\beta}+\frac{1}{q}}(y)}{\left|x^2-y^2\right|^{\frac{2}{\alpha}+\frac{2}{\beta}}}dxdy$$

$$\le\left(\int_0^\infty\left(\frac{F(x)}{x}\right)^{\frac{p}{\alpha}+1}\frac{1}{2}\left[2B\left(\frac{p}{\alpha},1-\frac{2p}{\alpha}\right)\right]dx\right)^{\frac{1}{p}}$$

$$\times\left(\int_0^\infty\left(\frac{G(y)}{y}\right)^{\frac{q}{\beta}+1}\frac{1}{2}\left[2B\left(\frac{q}{\beta},1-\frac{2q}{\beta}\right)\right]dy\right)^{\frac{1}{q}}$$

$$=\left[B\left(\frac{p}{\alpha},1-\frac{2p}{\alpha}\right)\int_0^\infty\left(\frac{F(x)}{x}\right)^{\frac{p}{\alpha}+1}dx\right]^{\frac{1}{p}}$$

$$\times\left[B\left(\frac{q}{\beta},1-\frac{2q}{\beta}\right)\int_0^\infty\left(\frac{G(y)}{y}\right)^{\frac{q}{\beta}+1}dy\right]^{\frac{1}{q}}.$$

By Theorem 2.2, we have

$$\int_0^\infty\int_0^\infty \frac{x^{\frac{1}{\alpha}+\frac{2}{\beta}-1}y^{\frac{2}{\alpha}+\frac{1}{\beta}-1}F^{\frac{1}{\alpha}+\frac{1}{p}}(x)G^{\frac{1}{\beta}+\frac{1}{q}}(y)}{\left|x^2-y^2\right|^{\frac{2}{\alpha}+\frac{2}{\beta}}}dxdy$$

$$\le\left(B\left(\frac{p}{\alpha},1-\frac{2p}{\alpha}\right)\left[\left(\frac{\frac{p}{\alpha}+1}{\frac{p}{\alpha}}\right)^{\frac{p}{\alpha}+1}\int_0^\infty f^{\frac{p}{\alpha}+1}(x)dx\right]\right)^{\frac{1}{p}}$$

$$\times\left(B\left(\frac{q}{\beta},1-\frac{2q}{\beta}\right)\left[\left(\frac{\frac{q}{\beta}+1}{\frac{q}{\beta}}\right)^{\frac{q}{\beta}+1}\int_0^\infty g^{\frac{q}{\beta}+1}(y)dy\right]\right)^{\frac{1}{q}}$$

$$=C_{p,\alpha}^{\frac{1}{p}}C_{q,\beta}^{\frac{1}{q}}\left(\int_0^\infty f^{\frac{p}{\alpha}+1}(x)dx\right)^{\frac{1}{p}}\left(\int_0^\infty g^{\frac{q}{\beta}+1}(y)dy\right)^{\frac{1}{q}}.$$

where

$$C_{p,\alpha}=\left(1+\tfrac{\alpha}{p}\right)^{\frac{p}{\alpha}}B\left(\tfrac{p}{\alpha},1-\tfrac{2p}{\alpha}\right)\text{ and }C_{q,\beta}=\left(1+\tfrac{\beta}{q}\right)^{\frac{q}{\beta}}B\left(\tfrac{q}{\beta},1-\tfrac{2q}{\beta}\right).$$

This proof is completed.

References

[1] Hardy, G. H., 1920, Note on a theorem of Hilbert, Math. Z., 6, pp. 314-317.
[2] Hardy, G. H., 1925, Note on a theorem of Hilbert concerning series of positive terms, Proc. London Math. Soc., 23(2), pp. 45-46.
[3] Mitrinovic, D. S., Pecaric, J. E., and Fink, A. M., 1991, Inequalities involving functions and their integrals and derivatives, Kluwer Academic Publishers, Boston.
[4] Sulaiman, W. T., 2010, On two inequalities similar to Hardy-Hilbert's integral inequality, Appl. Math. Sci., 4(3), pp. 133-138.