Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 9, Number 2 (2013), pp. 215-220
© Research India Publications
http://www.ripublication.com/gjpam.htm

# On Concircularly $\phi$-recurrent Sasakian Manifolds

**Amit Prakash[1] and Sunil Kumar Srivastava[2]**

*[1]Department of Mathematics,*
*National Institute of Technology, Kurukshetra, (Haryana), India*
*E-mail: apoct0185@rediffmail.com*

*[2]Department of Science & Humanities,*
*Columbia Institute of Engg. & Technology , Raipur, (C. G.), INDIA.*
*E-mail: sunilk537@gmail.com*

**Abstract**

The object of the present paper is to study concircularly $\phi$-recurrent Sasakian manifolds.

**AMS Mathematics Subject Classification (2000): 53C05, 53C20, 53C25.**

**Keywords:** and phrases: Concircularly $\phi$-symmetric manifold, Concircularly $\phi$-recurrent manifold, Einstein manifold.

## 1. Introduction

A transformation of an n-dimensional Riemannian manifold M, which transform every geodesic circle of M in to a geodesic circle, is called a concircular transformation. A concircular transformation is always a conformal transformation. Here geodesic circle means a curve in M whose first curvature is constant and second curvature is identically zero. Thus the geometry of concircular transformations, that is, the concircular geometry, is a generalization of inversive geometry in the sence that the change of metric is more general than that induced by a circle preserving diffeomorphism. An interesting invariant of a concircular transformation is the Concircular curavture tensor.

The notion of local symmetry of a Riemannian manifold has been studied by many authors. As a weaker version of local symmetry, in (1977), Takahashi [7] introduced the notion of locally $\phi$-symmetric Sasakian manifold and obtained their several interesting results .Later in (2003), De, Shaikh and Biswas [9] studied $\phi$-recurrent Sasakian manifold and obtained their some interesting results. In this paper we study a concircularly $\phi$-recurrent Sasakian manifold which generalizes the notion of locally concircularly $\phi$-symmetric Sasakian manifold and obtained some interesting results. Again it is proved that a concircularly $\phi$-recurrent Sasakian manifold is an Eiinstein manifold and a concircularly $\phi$-recurrent Sasakian manifold having a non- zero constant sectional curvature is locally concircularly $\phi$-symmetric manifold.

## 2. Preliminaries

Let $M^{2n+1}(\phi,\xi,\eta,g)$ be a Sasakian manifold with the structure $(\phi,\xi,\eta,g)$.Then the following relation hold [2]:

(2.1) $\phi^2X = -X + \eta(X)\xi$,

(2.2) (a) $\eta(\xi) = 1$, (b) $g(X,\xi) = \eta(X)$ , (c) $\eta(\phi X) = 0$,

(2.3) $g(\phi X,\phi Y) = g(X,Y) - \eta(X)\eta(Y)$,

(2.4) $(D_X\phi)(Y) = g(X,Y)\xi - \eta(Y)X$,

(2.5) $D_X\xi = -\phi X$,

(2.6) $(D_X\eta)(Y) = g(X,\phi Y)$,

(2.7) $\eta(R(X,Y)Z) = g(Y,Z)\eta(X) - g(X,Z)\eta(Y)$,

(2.8) $R(X,Y)\xi = \eta(Y)X - \eta(X)Y$,

(2.9) $S(\phi X,\phi Y) = S(X,Y) - 2n\,\eta(X)\eta(Y)$,

(2.10) $S(X,\xi) = 2n\,\eta(X)$,

for all vector fields X, Y, Z, where $D$ denotes the operator of covariant differentiation with respect to $g$, $\phi$ is skew – symmetric tensor field of type (1, 1 ), S is the Ricci tensor of type (0,2) and R is the Riemannian curvature tensor of the manifold.

**Definition (2.1).** A Sasakian manifold is said to be a locally $\phi$ −symmetric manifold if [10]

(2.11) $\phi^2\big((D_WR)(X,Y)Z\big) = 0$,

for all vector fields X,Y,Z,W orthogonal to $\xi$.

**Definition (2.2).** A Sasakian manifold is said to be a locally concircularly $\phi$ −symmetric manifold if

(2.12) $\phi^2\big((D_WC)(X,Y)Z\big) = 0$,

for all vector fields X,Y,Z,W orthogonal to $\xi$.

**Definition (2.3).** A Sasakian manifold is said to be concircularly $\phi-$ recurrent Sasakian manifold if there exists a non-zero 1-form A such that

(2.13) $\phi^2\big((D_WC)(X,Y)Z\big) = A(W)C(X,Y)Z$,

for arbitrary vector fields X,Y,Z,W where C is a Concircular curvature tensor given by

(2.14) $C(X,Y)Z = R(X,Y)Z - \frac{r}{2n(2n+1)}\,[g(Y,Z)X - g(X,Z)Y]$,

where R is the curvature tensor and r is the scalar curvature. If the 1-form A vanishes, then the manifold reduces to a locally concircularly $\phi$ −symmetric manifold

## 3. Concircularly $\boldsymbol{\phi}-$ recurrent Sasakian manifold

Let us consider a concircularly $\phi-$ recurrent Sasakian manifold. Then by virtue of (2.1) and (2.13),we get

(3.1) $-(D_W C)(X,Y)Z + \eta\big((D_W C)(X,Y)Z\big)\xi = A(W)C(X,Y)Z,$

from which it follows that

(3.2) $-g\big((D_W C)(X,Y)Z,U\big) + \eta\big((D_W C)(X,Y)Z\big)\,\eta(U) = A(W)g(C(X,Y)Z,U).$

Let $\{e_i\}, i = 1,2, \ldots \ldots \ldots, 2n+1$ *be an or*thonormal basis of the tangent space at any point of the manifold. Then putting X = U = $e_i$ in (3.2) and taking summation over $i$, $1 \le i \le 2n+1$,we get

(3.3) $(D_W S)(Y,Z) = \frac{dr(W)}{(2n+1)}\, g(Y,Z) - \frac{dr(W)}{2n\,(2n+1)}[\, g(Y,Z) - \eta(Y)\eta(Z)]$
$-A(W)\left[S(Y,Z) - \frac{r}{2n+1} g(Y,Z)\right].$

Replacing $Z$ by $\xi$ in (3.3) and using (2.5) and (2.10), we get

(3.4) $(D_W S)(Y,\xi) = \frac{dr(W)}{(2n+1)}\, \eta(Y) - A(W)[2n - \frac{r}{2n+1}\,]\eta(Y).$

Now we have $(D_W S)(Y,\xi) = D_W S(Y,\xi) - S(D_W Y,\xi) - S(Y, D_W \xi)$

Using (2.5) and (2.10) in the above relation, it follows that

(3.5) $(D_W S)(Y,\xi) = 2n\, g(W,\phi Y) + S(Y,\phi W).$

In view of (3.4) and (3.5),we get

(3.6) $S(Y,\phi W) = -2n\, g(W,\phi Y) + \frac{dr(W)}{(2n+1)}\, \eta(Y) - A(W)[2n - \frac{r}{2n+1}]\eta(Y).$

Replacing Y by $\phi Y$ in (3.6) and using (2.2),(2.3) and (2.9),we get
$S(Y,W) = 2n\, g(Y,W)$, for all Y,W.
Hence, we can state the following theorem:

**Theorem (3.1).** A Concircularly $\phi-$ recurrent Sasakian manifold $(M^{2n+1}, g)$ is an Einstein manifold.
Now from (3.1) ,we have

(3.7) $(D_W C)(X,Y)Z = \eta\big((D_W C)(X,Y)Z\big)\xi - A(W)C(X,Y)Z.$

This implies

(3.8) $(D_W R)(X,Y)Z = \eta(\,(D_W R)(X,Y)Z)\xi - A(W)R(X,Y)Z$
$-\frac{dr(W)}{2n(2n+1)}\,[g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi]$
$+\frac{dr(W)}{2n(2n+1)}\,[g(Y,Z)X - g(X,Z)Y] + \frac{r}{2n(2n+1)}\,A(W)[g(Y,Z)X - g(X,Z)Y].$

From (3.8) and the Bianchi`s identity, we get

(3.9) $A(W)\eta(R(X,Y)Z) + A(X)\eta(R(Y,W)Z) + A(Y)\eta(R(W,X)Z)$

$$= \frac{r}{2n(2n+1)} A(W)[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)]$$

$$+ \frac{r}{2n(2n+1)} A(X)[g(Z,W)\eta(Y) - g(Y,Z)\eta(W)]$$

$$+ \frac{r}{2n(2n+1)} A(Y)\begin{bmatrix} g(X,Z)\eta(W) \\ -g(W,Z)\eta(X) \end{bmatrix}.$$

Putting $Y = Z = e_i$ in (3.9) and taking summation over $i$, $1 \leq i \leq 2n+1$, we get

(3.10) $A(W)\eta(X) = A(X)\eta(W)$,

for all vector fields X, W. Replacing X by $\xi$ in (3.10) , we get

(3.11) $A(W) = \eta(W)\eta(\rho)$,

for any vector field W, where $A(\xi) = g(\xi,\rho) = \eta(\rho)$, $\rho$ being the vector field associated to the 1-form A i.e., $A(X) = g(X,\rho)$.
From (3.10) and (3.11), we can state the following theorem:

**Theorem (3.2).** In a Concircularly $\phi -$ recurrent Sasakian manifold $(M^{2n+1}, g)$ $n \geq 1$, the characteristic vector field $\xi$ and the vector field $\rho$ associated to the 1-form A are co-directional and the 1-form A is given by (3.11).
From (2.14) it follows that

(3.12) $(D_W C)(X,Y)\xi = (D_W R)(X,Y)\xi - \frac{dr(W)}{2n(2n+1)} [\eta(Y)X - \eta(X)Y]$.

In view of (2.5) and (2.8) it can be easily seen that in a Sasakian manifold the following relation holds:

(3.13) $(D_W R)(X,Y)\xi = g(W,\phi Y)X - g(W,\phi X)Y + R(X,Y)\phi W$.

Using (3.13) in (3.12), we get

(3.14) $(D_W C)(X,Y)\xi = g(W,\phi Y)X - g(W,\phi X)Y + R(X,Y)\phi W$

$$- \frac{dr(W)}{2n(2n+1)} [\eta(Y)X - \eta(X)Y].$$

By virtue of (2.7) it follows from (3.14) that

(3.15) $\eta((D_W C)(X,Y)\xi) = 0$.

Again from Tanno [6], we have

(3.16) $R(X,Y)\phi W = g(\phi X,W)Y - g(Y,W)\phi X - g(\phi Y,W)X + g(X,W)\phi Y$
$+ \phi R(X,Y)W$,

for any X, Y, Z∈ $T_p(M)$.

Using (3.16) in (3.14), we get

(3.17) $(D_W C)(X,Y)\xi = g(W,\phi Y)X - g(W,\phi X)Y + g(\phi X,W)Y - g(Y,W)\phi X$
$-g(\phi Y,W)X + g(X,W)\phi Y + \phi R(X,Y)W - \frac{dr(W)}{2n(2n+1)}[\eta(Y)X - \eta(X)Y].$

We now suppose that Sasakian manifold $(M^{2n+1},g)\ n>1$, is concircularly $\phi -$ recurrent. Then from (3.7) and (3.17), it follows that

(3.18) $(D_W C)(X,Y)Z = g(W,\phi Y)g(X,Z) - g(W,\phi X)g(Y,Z)$
$+ g(\phi X,W)g(Y,Z) - g(Y,W)g(\phi X,Z) - g(\phi Y,W)g(X,Z)$
$+g(\phi Y,Z)g(X,W) + g(\phi R(X,Y)W,Z) - \frac{dr(W)}{2n(2n+1)}\begin{bmatrix}\eta(Y)g(X,Z)\\ -\eta(X)g(Y,Z)\end{bmatrix}$
$-A(W)C(X,Y)Z.$

Next, we suppose that in a concircularly $\phi -$ recurrent Sasakian manifold, the sectional curvature of a plane $\pi \subset T_p$ (M) is defined by

(3.19) $K_p(\pi) = g(R(X,Y)Y,X)$, is a non zero constant k, where {X, Y} is any orthonormal basis of $\pi$.

Then we have

(3.20) $g\big((D_Z R)(X,Y)Y,X\big) = 0.$

Again from (2.14),we get

(3.21) $(D_Z C)(X,Y)Y = (D_Z R)(X,Y)Y - \frac{dr(Z)}{2n(2n+1)}[g(Y,Y)X - g(X,Y)Y].$

In view of (3.20) it follows from (3.21) that

(3.22) $g((D_Z C)(X,Y)Y,X) = 0.$

By virtue of (3.22) and (3.1), we have

(3.23) $g\big((D_Z C)(X,Y)Y,\xi\big)\eta(X) = A(Z)g(C(X,Y)Y,X).$

Since in a concircularly $\phi -$ recurrent Sasakian manifold, the relation (3.18) hold good, using (3.18) in (3.22), we get

(3.24) $\eta(X)[g(Z,\phi Y)g(X,Y) - g(Z,\phi X)g(Y,Y) + g(\phi X,Z)g(Y,Y)$
$-g(Y,Z)g(\phi X,Y) - g(\phi Y,Z)g(X,Y) + g(X,Z)g(\phi Y,Y)$
$+ g(Y,Z)g(\phi X,Y) - g(X,Z)g(\phi Y,Y) - \frac{dr(Z)}{2n(2n+1)}\{\eta(Y)g(X,Y) - \eta(X)g(Y,Y)\}$
$-A(Z)\left\{1 - \frac{r}{2n(2n+1)}\right\}\{g(Y,Y)\eta(X) - g(X,Y)\eta(Y)\}]$
$= A(Z)\left[k - \frac{r}{2n(2n+1)}\{g(Y,Y)g(X,X) - g(X,Y)g(X,Y)\}\right].$

Putting $Y = Z = \xi$ in (3.24) and simplifying, we get
(3.25) $\eta(\rho)$=0.

Hence by (3.11), we get
(3.26) A(W)=0.

Again using (3.26) in (3.1), we get

$$(\phi^2(D_W C)(X, Y)Z) = 0.$$

Hence, we can state the following theorem:

**Theorem (3.3).** If a concircularly $\phi-$ recurrent Sasakian manifold $(M^{2n+1}, g)$ $(n > 1)$, has a non –zero constant sectional curvature, then it reduces to a locally concircularly $\phi-$ symmetric manifold.

## References


[1] C. Özgür and U.C. De, On the quasi-conformal curvature tensor of a Kenmotsu manifold, Mathematica Pannonica, 17(2), (2006) ,221-228.

[2] D. E.Blair, Contact manifolds in Riemannian geometry, Lecture Notes in Math. No. 509. Springer (1976).

[3] J. B.Jun, U.C. De and G. Pathak, On Kenmotsu manifolds, J. Korean Math Soc., 42, (2005), 435-445.

[4] K. Kenmotsu. A class of almost contact Riemannian manifolds, Tohoku Math. J., 24,(1972), 93-103.

[5] K. Yano, Concircular geometry, I, Proc. Imp. Acad., Tokyo, 16, (1940), 195-200.

[6] S. Tanno, Isometric Immersions of Sasakian manifold in spheres, Kodai Math. Sem. Rep.21, (1969), 448-458.

[7] T. Takahashi, Sasakian $\phi-$symmetric spaces, Tohoku Math. J ., 29(1977), 91-113.

[8] U.C. De and G. Pathak, On 3-Dimensional Kenmotsu Manifolds, Indian J Pure Appl. Math. 35(2) (2004), 159-165.

[9] U.C. De, A. A. Shaikh, and S. Biswas, On $\phi-$Recurrent Sasakian manifolds, NoviSad J. Math, 33(2), (2003), 43-48.

[10] U. C. De, A. Yildiz and A.F. Yaliniz On $\phi-$Recurrent Kenmotsu manifolds,Turk J Math , 33, (2009), 17-25.