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# Hall and Transverse Magnetic Field Effects on Peristaltic Flow of a Maxwell Fluid through a Porous Medium

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**Abstract**

In this article, we investigate peristaltic motion induced by traveling sinusoidal wave in the porous walls of a two-dimensional channel occupied by incompressible, viscous, and electrically conducting Maxwell fluid through a porous medium. A constant transverse magnetic field is applied on the flow and the Hall effect is taken into account. The fluid entering the flow region through one plate is considered at the same rate as it is leaving through the other plate. The problem is formulated and solved analytically in terms of small wave number and for low Reynolds number. The effects of the magnetic parameter, Hall parameter, porosity, and permeability are discussed graphically. Expressions for the stream function, velocity components, and rate of volume flow are computed and discussed for various values of the pertinent parameters. This work can be considered as mathematical modeling to the case of gall bladder with stones.

**Keywords:** Peristaltic transport. Maxwell fluid. Hall effect. Porous medium. Porous boundaries.

## 1 Introduction

Peristalsis is the mechanism of fluid transport that occurs generally from a region of lower pressure to higher pressure when a progressive wave of area contraction and expansion travels along the flexible wall of a tube. In ionized gas where the density is low and/or the magnetic field is reduced due to the free spiralling of electrons and ions about the magnetic lines of force before suffering collisions, a current is induced in a direction normal to both electric and magnetic fields. This phenomenon is called Hall effect.

The steady two-dimensional laminar flow of a conducting fluid between two stationary porous disks in the presence of a transverse magnetic field was studied by

Chandrasekhara and Rudraiah [3] for uniform suction or injection Reynolds number. Heat transfer characteristics of this flow problem were studied by Sato [12].

The study of magnetohydrodynamic viscous flows with Hall currents has important engineering applications in problems of magnetohydrodynamics. In fact, the Hall effect is important when the Hall parameter, which is due to the ratio between the electron-cyclotron frequency and the electron-atom-collision frequency, is high. This happens, when the magnetic field is high or when the collision frequency is low. Various investigations examined the mechanisms of peristaltic transport. Yin and Fung [15] represented the analysis of peristaltic pumping at moderate amplitude of motion in the two dimensions greater than the critical value which accordingly would induce a backward flow (reflux) in the core region of the stream. There will be no reflux if the pressure gradient is smaller than the critical value. Biomagnetism was studied by Agrawal and Anwaruddin [2] who investigated the effects of magnetic field on blood flow. They noticed that the velocity of fluid increases with an increase in the magnetic field. Abd El Hakeem *et al*. [11] studied peristaltic motion of an incompressible Newtonian fluid with variable viscosity subjected to a constant transverse magnetic field. They noticed that as the magnetic field increases, the pressure rise increases. Elshehawey *et al*. **[**6**]** studied the peristaltic motion of Carreau fluid in a non-uniform channel. Mekheimer [8] studied the effect of a uniform magnetic field on peristaltic transport of blood in a non-uniform two-dimensional channel. Also, Elshehawey *et al*. [7] studied the axisymmetric peristaltic motion of a viscous compressible liquid through a flexible pore of changing cross section. Mekheimer [9] studied the peristaltic transport of a viscous incompressible fluid through the gap between coaxial tubes, where the outer tube is non uniform and has a sinusoidal wave traveling down its wall and the inner one is rigid uniform tube and moving with a constant velocity. Nevertheless, to the best of the author's knowledge, a limited amount of investigations exist on peristalsis of non-Newtonian fluid including the magnetic field in a transverse direction together with the Hall effect through a porous medium.

The purpose of the present work is to study the influence of the Hall parameter, the magnetic field, the porosity of walls, and the permeability of medium on peristaltic transport of a Maxwell fluid through a medium bounded by two flexible plates where a sinusoidal traveling wave is imposed. A very strong magnetic field is imposed on the flow so as to take Hall effect into consideration. Calculations are performed for various values for the indicated parameters and the results for the velocity field components along and in the cross section of the channel are given.

We formulate the problem in Sect. 2. In Sect. 3, we solve the problem. In Sect. 4, we discuss the rate of volume flow. The graphical results and discussion as well as the conclusions are presented in Sects. 5 and 6, respectively.

## 2 Formulation of the problem

Consider a two-dimensional infinite channel of uniform thickness 2d, filled with viscous, incompressible, and electrically conducting Maxwell fluid. The walls of the channel are porous and flexible on which imposed traveling sinusoidal waves of small amplitude. The fluid is subject to a transverse magnetic field of constant flux density

$B_o$ through the porous medium. The induced magnetic field is neglected since the magnetic Reynolds number is assumed to be very small. The electron-atom collision frequency is assumed to be very high, so that the Hall effect cannot be neglected [14]. The effect of Hall current gives rise to a force in z-direction, which induces a cross flow in that direction, and hence the flow becomes three-dimensional. The walls are assumed to be infinite in the x- and z- directions which means that the physical quantities do not change in these directions. We choose a rectangular coordinate system for the channel with $X$ along the center line in the direction of wave propagation and $Y$ transverse to it. The geometry of the problem is described in Figure 1. The wavelength $\lambda$ is considered to be long enough to have a flow free from inertial and curvature effects, this assumption is very common and has been used by several workers [10, 13]. The flow is inherently unsteady in the fixed frame of reference. However, the flow is shown to become steady in a wave frame with velocity $c$ along the wave. Under these assumptions, we shall carry out this investigation in a coordinate system with wave speed $c$ in the direction of wave propagation in which the boundary shape is stationary. The vertical displacements of the upper and lower walls $\eta$ and $-\eta$ are defined by $\eta(X,t) = a\cos\frac{2\pi}{\lambda}(X-ct)$. The transformation from the fixed frame to the wave frame of reference $(x, y)$ is given by

$$\begin{aligned} &x = X - ct;\ y = Y;\ u = U - c;\ v = V; \\ &w = W;\ p(x) = P(X,t);\ \eta(x) = \eta(X,t); \end{aligned} \qquad (1)$$

where $(u,v,w)$ and $(U,V,W)$ are the velocity components associated with the directions of coordinates $(x, y, z)$ and $(X,Y,Z)$, respectively. Also, p and P are pressures in the wave and fixed frames of reference. Moreover, $a$, $\lambda$ and $c$ are the amplitude, wave length and wave speed, respectively.

The fundamental equations for the incompressible Maxwell fluid in the infinite channel together with generalized Ohm's law taking Hall effect and Maxwell's equations into account are:

$$\rho[(\mathbf{V}.\nabla)\mathbf{V}] = -\nabla P + \nabla.\mathbf{S} + \mathbf{R} + \mathbf{J}x\mathbf{B} \qquad (2)$$

$$\nabla\ .\mathbf{V} = 0, \qquad (3)$$

$$\mathbf{J} = \sigma[\mathbf{E} + \mathbf{V}\,x\,\mathbf{B}] - \frac{\sigma}{en_e}\mathbf{J}\,x\,\mathbf{B}\ , \qquad (4)$$

$$\nabla x\,\mathbf{H} = \mathbf{J}\ ,\ \nabla\,x\,\mathbf{E} = 0\ ,\nabla.\mathbf{B} = 0\ . \qquad (5)$$

where ρ, t, $\upsilon$, μ, σ, e, and $n_e$ are the density of fluid, time, kinematic viscosity, dynamic viscosity, electrical conductivity, electric charge and number density of electrons, respectively. **V**, **J**, **B**, **E**, and **H** are the velocity vector, electric current density, magnetic induction vector, intensity of the electric field, and magnetic field strength, respectively. For the current density $\mathbf{J}=(J_x, J_y, J_z)$ we obtain from the relation $\nabla.\mathbf{J} = 0$ that $J_y$=constant. Hence, we consider that the channel is non-conducting and therefore $J_y$=0 at the channel. In the absence of an externally applied electric field and with negligible effects of polarization of the ionized gas, we assume that the electric field vector equals zero. i.e. **E** = 0. The basic equations governing the three–

dimensional motion of a viscous incompressible Maxwell fluid in the presence of magnetic field are (in the moving coordinates):

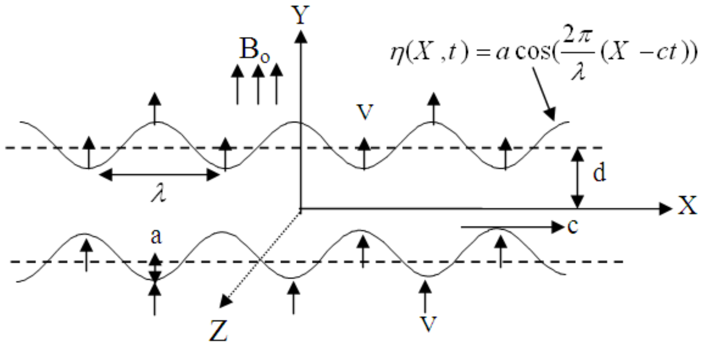


**Figure 1:** Geometry of the problem.

$$(1-c\tau\frac{\partial}{\partial x})\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=-\frac{1}{\rho}(1-c\tau\frac{\partial}{\partial x})\frac{\partial P}{\partial x}+\upsilon\nabla^2 u-\frac{\upsilon}{W_1}(u+c)$$

$$-\frac{\sigma B_0^2}{\rho(1+m^2)}(1-c\tau\frac{\partial}{\partial x})(u+mw+c)\,, \tag{6}$$

$$(1-c\tau\frac{\partial}{\partial x})\left(u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right)=-\frac{1}{\rho}(1-c\tau\frac{\partial}{\partial x})\frac{\partial P}{\partial y}+\upsilon\nabla^2 v-\frac{\upsilon}{W_1}v\,, \tag{7}$$

$$(1-c\tau\frac{\partial}{\partial x})\left(u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}\right)=\upsilon\nabla^2 w-\frac{\upsilon}{W_1}w$$

$$+\frac{\sigma B_0^2}{\rho(1+m^2)}(1-c\tau\frac{\partial}{\partial x})(mu-w+mc)\,, \tag{8}$$

and the continuity equation is

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0\,. \tag{9}$$

where x and y are Cartesian coordinates with x measured in the directions of the wave propagation and y measured in the direction normal to the mean position of the channel walls. m=$\sigma B_o / en_e$ is the Hall parameter and $W_1$ is the permeability parameter. Also, the components for the current-density as obtained from the generalized Ohm's law are $J_x = (mu - w)\sigma B_0 /(1+m^2)$ and $J_z = (u+mw)\sigma B_0 /(1+m^2)$ where the subscripts x and z denote the x- and z-components of the vectorial quantities. The horizontal displacement will be assumed zero. The fluid is entering the flow region through one plate at the same rate as it is leaving through the other plate with velocity V in the positive direction of the y-axis. The boundary conditions are

$$u(x,\pm d \pm \eta) = -c\,, \tag{10}$$

$$v(x,\pm d \pm \eta) = V \mp c\frac{\partial \eta(x)}{\partial x}\,, \tag{11}$$

$$w(x,\pm d \pm \eta) = 0\,. \tag{12}$$

The pressure p remains constant across any axial station of the channel under the assumption that the wave length is large and the curvature effects are negligible. Using the following non-dimensional quantities where c and d are the characteristics velocity and length:

$$x^* = \frac{x}{\lambda},\ y^* = \frac{y}{d},\ u^* = \frac{u}{c},\ v^* = \frac{v}{c},\ w^* = \frac{w}{c}, \psi^* = \frac{\psi}{cd}, \eta^* = \frac{\eta}{d},\ t^* = \frac{ct}{d},$$
$$p^* = \frac{d^2 p}{c\mu\lambda},\ \varepsilon = \frac{a}{d},\ \alpha = \frac{d}{\lambda}, \nabla^2 \equiv \frac{1}{d^2}\frac{\partial^2}{\partial y^2}, V^* = \frac{V}{c\alpha}, W_1^* = \frac{W_1}{d^2}, \tau^* = \frac{\tau c}{d} \tag{13}$$

where $\alpha$ and $\varepsilon$ are the wave number and the amplitude ratio, respectively.

Equation (9) allows the use of the stream function $\psi$ (x, y, t) in terms of which

$$u = \psi_y,\ v = -\alpha\psi_x. \tag{14}$$

Dropping the stars, we find that the continuity equation (9) is automatically satisfied. By the aid of Eqs. (13) and (14), Eqs. (6) - (12) become

$$R(1-\alpha\tau\frac{\partial}{\partial x})\alpha(\psi_y\psi_{xy} - \psi_x\psi_{yy}) = -(1-\alpha\tau\frac{\partial}{\partial x})p_x + \alpha^2\psi_{yxx} + \psi_{yyy}$$
$$-\frac{1}{W_1}(\psi_y + 1) - RM\beta(1-\alpha\tau\frac{\partial}{\partial x})(\psi_y + mw + 1), \tag{15}$$

$$R(1-\alpha\tau\frac{\partial}{\partial x})\alpha^3(-\psi_y\psi_{xx} + \psi_x\psi_{xy}) = -(1-\alpha\tau\frac{\partial}{\partial x})p_y$$
$$-\alpha^4\psi_{xxx} - \alpha^2\psi_{xyy} - \frac{\alpha^2}{W_1}\psi_x, \tag{16}$$

$$R(1-\alpha\tau\frac{\partial}{\partial x})\alpha(\psi_y w_x - \psi_x w_y) = \alpha^2 w_{xx} + w_{yy} - \frac{w}{W_1}$$
$$+RM\beta(1-\alpha\tau\frac{\partial}{\partial x})(m\psi_y - w + m) \tag{17}$$

where the subscripts x and y denote the partial differentiation with respect to the variables. $R = cd/\upsilon$, $M = (d\sigma/\rho c)B_0^2$ and $\beta = 1/(1+m^2)$ are Reynolds number, the magnetic parameter and the Hall parameter, respectively. It may be remarked that if $m \to \infty$, $\beta$ vanishes and hence we return to the common hydrodynamic case.

Eliminating the pressure gradient $p_x$ and $p_y$ from Eqs. (15) and (16), and using the long wave length approximation $(\alpha = 0)$ for low Reynolds number ($R \ll 1$), the governing Eqs. (15) - (17) reduce to

$$\psi_{yyyy} - (RM\beta + 1/W_1)\psi_{yy} - RM\beta m w_y = 0, \tag{18}$$

$$w_{yy} - (RM\beta + 1/W_1)w + RM\beta m\psi_y + RM\beta m = 0. \tag{19}$$

It is noticed that the relaxation time parameter disappears; hence, the problem is reduced to the case of Newtonian fluid.

The corresponding boundary conditions for the dimensionless stream function in the wave frame are

$$\psi_y = -1,\ \psi_x = -V \pm \frac{\partial \eta}{\partial x},\ w = 0,\ \text{at}\ y = \pm 1 \pm \eta. \tag{20a}$$

## 3. Method of solution

The solutions of Eqs. (18) and (19) for the stream function $\psi$ and for the velocity component $w$ subject to the boundary conditions (20a) can be obtained as follows:

By substituting with Eq. (19) into Eq. (18) yields

$w_{yyyyy} - 2RM\beta\, w_{yyy} + R^2 M^2 \beta\, w_y = 0$, (21)

which is a fifth-degree differential equation with constants coefficients whose roots are complex and are of the form 0, $\pm\sqrt{r}\, Exp[\pm i\theta/2]$, in which

$$r = \sqrt{\left(1/W_1 + RM\,\beta\right)^2 + (RM\,\beta m)^2},\qquad \theta = Arc\tan\left[R\,M\,\beta\, m / \left(1/W_1 + R\,M\,\beta\right)\right]$$

The differential equation in $w$ with its corresponding boundary conditions is satisfied by

$w(x,y) = w_1(x)\ w_2(y)$

Hence,

$$w(x,y) = w_1(x)[c_0 + e^{\sqrt{r}\cos\frac{\theta}{2}y}(c_1\cos(\sqrt{r}\sin\frac{\theta}{2}y) + c_2\sin(\sqrt{r}\sin\frac{\theta}{2}y))$$
$$+ e^{-\sqrt{r}\cos\frac{\theta}{2}y}(c_3\cos(\sqrt{r}\sin\frac{\theta}{2}y) + c_4\sin(\sqrt{r}\sin\frac{\theta}{2}y))], \tag{22}$$

where

$$w_2(y) = c_0 + e^{\sqrt{r}\cos\frac{\theta}{2}y}(c_1\cos(\sqrt{r}\sin\frac{\theta}{2}y) + c_2\sin(\sqrt{r}\sin\frac{\theta}{2}y))$$
$$+ e^{-\sqrt{r}\cos\frac{\theta}{2}y}(c_3\cos(\sqrt{r}\sin\frac{\theta}{2}y) + c_4\sin(\sqrt{r}\sin\frac{\theta}{2}y)).$$

Expanding the boundary conditions given in Eq. (20a), we obtain

$$\psi_y \pm \eta\psi_{yy} = -1,\ \psi_x \pm \eta\psi_{xy} = -V \pm \frac{\partial\eta}{\partial x},\ w \pm \eta\, w_y = 0,\ \text{at}\ y = \pm 1 \tag{20b}$$

After differentiating the expression of $w$ given by Eq. (22) and substituting in the third equation of the boundary conditions given by Eq. (20b), the five unknown constants of $w$ are reduced into three yielding

$$w(x,y) = w_1(x)[c_0 + e^{\sqrt{r}\cos\frac{\theta}{2}y}((c_0A_8 + c_2A_9 + c_4A_{10})$$
$$\cos(\sqrt{r}\sin\frac{\theta}{2}y) + c_2\sin(\sqrt{r}\sin\frac{\theta}{2}y))$$
$$+ e^{-\sqrt{r}\cos\frac{\theta}{2}y}((c_0A_5 + c_2A_6 + c_4A_7)$$

$$\cos(\sqrt{r}\sin\frac{\theta}{2}y)+c_4\sin(\sqrt{r}\sin\frac{\theta}{2}y))]. \tag{23}$$

Re-differentiating the last obtained expression of $w$ twice with respect to $y$ and then substituting in Eq. (19) gives an expression for $\psi_y$ which in turn gives a solution for $\psi$

$$\psi(x,y)=\frac{w_1(x)}{m}\{c_0(1+\frac{1}{RM\beta W_1})y$$
$$+[(\frac{1}{\sqrt{r}}+\frac{1}{RM\beta W_1\sqrt{r}}-\frac{\sqrt{r}\cos\theta}{RM\beta})(c_1\cos(\frac{\theta}{2}-\sqrt{r}\sin\frac{\theta}{2}y)$$
$$-c_2\sin(\frac{\theta}{2}-\sqrt{r}\sin\frac{\theta}{2}y))-\frac{\sqrt{r}\sin\theta}{RM\beta}(c_1\sin(\frac{\theta}{2}-\sqrt{r}\sin\frac{\theta}{2}y)$$
$$+c_2\cos(\frac{\theta}{2}-\sqrt{r}\sin\frac{\theta}{2}y))]e^{\sqrt{r}\cos\frac{\theta}{2}y}+[(\frac{1}{\sqrt{r}}+\frac{1}{RM\beta W_1\sqrt{r}}-\frac{\sqrt{r}\cos\theta}{RM\beta})$$
$$(-c_3\cos(\frac{\theta}{2}+\sqrt{r}\sin\frac{\theta}{2}y)-c_4\cos(\frac{\theta}{2}-\sqrt{r}\sin\frac{\theta}{2}y))+\frac{\sqrt{r}\sin\theta}{RM\beta}$$
$$(c_3\sin(\frac{\theta}{2}+\sqrt{r}\sin\frac{\theta}{2}y)-c_4\cos(\frac{\theta}{2}+\sqrt{r}\sin\frac{\theta}{2}y))]e^{-\sqrt{r}\cos\frac{\theta}{2}y}\}-y, \tag{24}$$

where

$$c_1=c_0A_8+c_2A_9+c_4A_{10},\ c_3=c_0A_5+c_2A_6+c_4A_7,$$

in which

$$A_1=[(1+\eta\sqrt{r}\cos\frac{\theta}{2})\sin(\sqrt{r}\sin\frac{\theta}{2})+\eta\sqrt{r}\sin\frac{\theta}{2}\cos(\sqrt{r}\sin\frac{\theta}{2})]e^{\sqrt{r}\cos\frac{\theta}{2}},$$
$$A_2=[(1-\eta\sqrt{r}\cos\frac{\theta}{2})\cos(\sqrt{r}\sin\frac{\theta}{2})-\eta\sqrt{r}\sin\frac{\theta}{2}\sin(\sqrt{r}\sin\frac{\theta}{2})]e^{-\sqrt{r}\cos\frac{\theta}{2}},$$
$$A_3=[(1-\eta\sqrt{r}\cos\frac{\theta}{2})\sin(\sqrt{r}\sin\frac{\theta}{2})+\eta\sqrt{r}\sin\frac{\theta}{2}\cos(\sqrt{r}\sin\frac{\theta}{2})]e^{-\sqrt{r}\cos\frac{\theta}{2}},$$
$$A_4=[(1+\eta\sqrt{r}\cos\frac{\theta}{2})\cos(\sqrt{r}\sin\frac{\theta}{2})-\eta\sqrt{r}\sin\frac{\theta}{2}\sin(\sqrt{r}\sin\frac{\theta}{2})]e^{\sqrt{r}\cos\frac{\theta}{2}},$$
$$A_5=-\frac{1}{A_2+A_4},A_6=-\frac{A_1A_2+A_3A_4}{A_2^2-A_4^2},A_7=-\frac{A_1A_4+A_2A_3}{A_2^2-A_4^2}, \tag{25}$$
$$A_8=-\frac{1}{A_4}(1+A_2A_5),A_9=-\frac{1}{A_4}(A_1+A_2A_6),A_{10}=-\frac{1}{A_4}(A_3+A_2A_7).$$

where the constant of integration is chosen to be 0 throughout this study since it has no influence on the velocity components given by Eq. (14).

Using the expression for the stream function of Eq. (24), we substitute in the first equation of Eq. (20b), and hence we reduce the undetermined constants from three to one;

$$c_1 = c_0 N_1,\ c_2 = c_0 N_2,\ c_3 = c_0 N_3,\ c_4 = c_0 N_4,$$

where

$$N_1 = A_8 + A_9 N_2 + A_{10} N_4,\ N_2 = A_{23} + A_{24} N_4,$$
$$N_3 = A_5 + A_6 N_2 + A_7 N_4,\ N_4 = \frac{A_{25} - A_{23}}{A_{24} - A_{26}}, \tag{26}$$

in which

$$A_{11} = (K_1 \cos(\frac{\theta}{2} - \gamma_2) - K_2 \sin(\frac{\theta}{2} - \gamma_2))e^{\gamma_1}, A_{12} = (K_1 \sin(\frac{\theta}{2} - \gamma_2) + K_2 \cos(\frac{\theta}{2} - \gamma_2))e^{\gamma_1},$$
$$A_{13} = (K_1 \cos(\frac{\theta}{2} + \gamma_2) - K_2 \sin(\frac{\theta}{2} + \gamma_2))e^{-\gamma_1}, A_{14} = (K_1 \sin(\frac{\theta}{2} + \gamma_2) + K_2 \cos(\frac{\theta}{2} + \gamma_2))e^{-\gamma_1},$$
$$A_{15} = (K_3 \cos\gamma_2 + K_4 \sin\gamma_2)e^{\gamma_1}, A_{16} = (K_3 \sin\gamma_2 - K_4 \cos\gamma_2)e^{\gamma_1}, \tag{27}$$
$$A_{17} = (K_3 \cos\gamma_2 - K_4 \sin\gamma_2)e^{-\gamma_1}, A_{18} = (K_3 \sin\gamma_2 + K_4 \cos\gamma_2)e^{-\gamma_1},$$
$$A_{19} = (K_5 \cos\gamma_2 - K_6 \sin\gamma_2)e^{\gamma_1}, A_{20} = (K_5 \sin\gamma_2 + K_6 \cos\gamma_2)e^{\gamma_1},$$
$$A_{21} = (K_5 \cos\gamma_2 + K_6 \sin\gamma_2)e^{-\gamma_1}, A_{22} = (K_5 \sin\gamma_2 - K_6 \cos\gamma_2)e^{-\gamma_1},$$

$$A_{23} = -\frac{(1 + A_8 A_{15} + A_5 A_{17}) + \eta(A_8 A_{19} - A_5 A_{21})}{(A_9 A_{15} + A_6 A_{17} + A_{16}) + \eta(A_9 A_{19} - A_6 A_{21} + A_{20})},$$
$$A_{24} = -\frac{(A_{10} A_{15} + A_7 A_{17} + A_{18}) + \eta(A_{10} A_{19} - A_7 A_{21} - A_{22})}{(A_9 A_{15} + A_6 A_{17} + A_{16}) + \eta(A_9 A_{19} - A_6 A_{21} + A_{20})},$$
$$A_{25} = -\frac{(1 + A_8 A_{17} + A_5 A_{15}) - \eta(A_8 A_{21} - A_5 A_{19})}{(A_9 A_{17} + A_6 A_{15} - A_{18}) - \eta(A_9 A_{21} - A_6 A_{19} - A_{22})}, \tag{28}$$
$$A_{26} = -\frac{(A_{10} A_{17} + A_7 A_{15} - A_{16}) - \eta(A_{10} A_{21} - A_7 A_{19} + A_{20})}{(A_9 A_{17} + A_6 A_{15} - A_{18}) - \eta(A_9 A_{21} - A_6 A_{19} - A_{22})}.$$

in which

$$K_1 = \frac{1}{\sqrt{r}} + \frac{1}{RM\beta W_1\sqrt{r}} - \frac{\sqrt{r}\cos\theta}{RM\beta},\ K_2 = \frac{\sqrt{r}\sin\theta}{RM\beta}, K_3 = \sqrt{r}\,K_1,\ K_4 = \sqrt{r}\,K_2$$
$$K_5 = \gamma_1(1 + \frac{1}{RM\beta W_1}) - \frac{\sqrt{r^3}\cos\frac{3\theta}{2}}{RM\beta},\ K_6 = \gamma_2(1 + \frac{1}{RM\beta W_1}) - \frac{\sqrt{r^3}\sin\frac{3\theta}{2}}{RM\beta},$$
$$\gamma_1 = \sqrt{r}\cos\frac{\theta}{2},\ \gamma_2 = \sqrt{r}\sin\frac{\theta}{2}\ .$$

Therefore, the expressions for $w$ and $\psi$ given by Eqs. (23) and (24) can be restated as

$$w(x,y)=c_0 w_1(x)[1+(N_1\cos\gamma_2+N_2\sin\gamma_2)e^{\gamma_1}$$

$$+(N_3\cos\gamma_2+N_4\sin\gamma_2)e^{-\gamma_1}], \tag{29}$$

$$\psi(x,y)=c_0\,w_1(x)[(1+\frac{1}{RM\beta W_1})y$$

$$+(K_1(N_1\cos(\frac{\theta}{2}-\gamma_2\mathrm{y})-\mathrm{N}_2\sin(\frac{\theta}{2}-\gamma_2\mathrm{y}))$$

$$-K_2(N_1\sin(\frac{\theta}{2}-\gamma_2\mathrm{y})+\mathrm{N}_2\cos(\frac{\theta}{2}-\gamma_2\mathrm{y})))\mathrm{e}^{\gamma_1\mathrm{y}}$$

$$+(K_1(-N_3\cos(\frac{\theta}{2}+\gamma_2\mathrm{y})-\mathrm{N}_4\sin(\frac{\theta}{2}+\gamma_2\mathrm{y}))$$

$$+K_2(N_3\sin(\frac{\theta}{2}+\gamma_2\mathrm{y})-\mathrm{N}_4\cos(\frac{\theta}{2}+\gamma_2\mathrm{y})))\mathrm{e}^{-\gamma_1\mathrm{y}}]-\mathrm{y}. \tag{30}$$

Now, we use the long wave length assumption to approximate the functions of $\eta$ into $b_i+\eta\, b_j$ by neglecting the terms of $O(\varepsilon)^2$ and higher (which express the induced non-linear terms), and keeping only the linear terms (which express the applied and induced external forces). According to this approximation, the As and Ns given by Eqs. (25), (26), (27), and (28) take the form stated in the appendix.

Next, by substituting in the second equation of Eq. (20b), we get

$$\left.\begin{aligned}(b_{119}+b_{120}\eta)w_1{}'(x)+(b_{121}+b_{122}\eta)\eta' w_1(x)&=\frac{m}{c_0}(-V+\eta'),\\ (b_{123}+b_{124}\eta)w_1{}'(x)+(b_{125}+b_{126}\eta)\eta' w_1(x)&=-\frac{m}{c_0}(V+\eta'),\end{aligned}\right\} \tag{31}$$

where the dash denotes partial differential with respect to x.

By taking $\eta(x)=\varepsilon\cos 2\pi x$, Eq. (31) can be solved for $w_1(x)$ giving

$$c_0\,w_1(x)=-\frac{m}{M_7}\frac{V+\eta'}{\eta'}, \tag{32}$$

where $Ms$ are given in the appendix.

Now, the expressions for $w$ and $\psi$ given by Eqs. (29) and (30) can be put in the following forms

$$w(x,y)=-\frac{m}{M_7}\left(\frac{V}{\eta'}+1\right)[1+(N_1\cos\gamma_2+N_2\sin\gamma_2)e^{\gamma_1 y}$$

$$+(N_3\cos\gamma_2+N_4\sin\gamma_2)e^{-\gamma_1 y}] \tag{33}$$

$$\psi(x,y)=-\frac{1}{M_7}\left(\frac{V}{\eta'}+1\right)[(1+\frac{1}{RM\beta W_1})y$$

$$+(K_1(N_1\cos(\frac{\theta}{2}-\gamma_2\mathrm{y})-\mathrm{N}_2\sin(\frac{\theta}{2}-\gamma_2\mathrm{y}))$$

$$-K_2(N_1\sin(\frac{\theta}{2}-\gamma_2 y)+N_2\cos(\frac{\theta}{2}-\gamma_2 y)))e^{\gamma_1 y}$$
$$+(K_1(-N_3\cos(\frac{\theta}{2}+\gamma_2 y)-N_4\sin(\frac{\theta}{2}+\gamma_2 y))$$
$$+K_2(N_3\sin(\frac{\theta}{2}+\gamma_2 y)-N_4\cos(\frac{\theta}{2}+\gamma_2 y)))e^{-\gamma_1 y}]-y. \quad (34)$$

## 4. Rate of volume flow

In this section, we shall discuss theoretically the volume flow rate as follows:

The instantaneous volume flow rate in the fixed frame of reference is given by

$$Q=\int_0^{\eta}U(X,Y,t)dY, \quad (35)$$

where $\eta$ is a function of $X$ and $t$. Also, the rate of volume flow in the wave frame is defined

$$F=\int_0^{\eta}u(x,y)dy, \quad (36)$$

where $\eta$ is a function of $x$ alone.

Substituting Eq. (1) into Eq. (35) and making use of Eq. (36), we find that the two rates of volume flow are related through

$$Q=F+c\eta. \quad (37)$$

The time mean flow over a period $T(=\lambda/c)$ at a fixed position $X$ is defined as

$$\overline{Q}=\frac{1}{T}\int_0^{T}Q\ dt. \quad (38)$$

We assume that the peristaltic wave propagating on the walls by a sinusoidal wave in the fixed frame has the form

$$\eta(x,t)=a\cos[\frac{2\pi}{\lambda}(X-ct)], \quad (39)$$

which on nondimensionalization in the wave frame gives

$$\eta(x)=\varepsilon\cos 2\pi x. \quad (40)$$

where $a$ is the amplitude of the peristaltic wave. Substituting Eq. (37) into Eq. (38) and integrating, we get

$$Q=q=-\frac{1}{M_7}\left(\frac{V}{\eta'}+1\right)[H_3-H_1+\eta(H_4-H_2-1)] \quad (41)$$

where $Q(=\overline{Q}/dc)$ and $q(=F/dc)$ are the dimensionless time mean flows in the fixed and wave frames, respectively. Such that

$$H_1=(b_{73}-b_{79})\left(K_1\cos\frac{\theta}{2}-K_2 sin\frac{\theta}{2}\right)-(b_{67}+b_{63})\left(K_1\sin\frac{\theta}{2}+K_2\cos\frac{\theta}{2}\right),$$
$$H_2=(b_{74}-b_{80})\left(K_1\cos\frac{\theta}{2}-K_2 sin\frac{\theta}{2}\right)-(b_{68}+b_{64})\left(K_1\sin\frac{\theta}{2}+K_2\cos\frac{\theta}{2}\right),$$

$$H_3 = K_1\left(b_{73}\cos\frac{\theta}{2} - b_{67}\sin\frac{\theta}{2} - b_{79}\cos\frac{\theta}{2} - b_{63}\sin\frac{\theta}{2}\right)$$
$$-K_2\left(b_{73}\sin\frac{\theta}{2} + b_{67}\cos\frac{\theta}{2} - b_{79}\sin\frac{\theta}{2} + b_{63}\cos\frac{\theta}{2}\right),$$
$$H_4 = (1+\frac{1}{RM\beta W_1}) + K_1\left(b_{74}\cos\frac{\theta}{2} + b_{73}\gamma_2\sin\frac{\theta}{2} + b_{67}\gamma_2\cos\frac{\theta}{2}\right.$$
$$-b_{68}\sin\frac{\theta}{2} - b_{80}\cos\frac{\theta}{2}$$
$$\left.+b_{79}\gamma_2\sin\frac{\theta}{2} - b_{64}\sin\frac{\theta}{2} - b_{63}\gamma_2\cos\frac{\theta}{2}\right) + K_2\left(b_{80}\sin\frac{\theta}{2} + b_{79}\gamma_2\cos\frac{\theta}{2} - b_{64}\cos\frac{\theta}{2}\right.$$
$$\left.+b_{63}\gamma_2\sin\frac{\theta}{2} - b_{74}\sin\frac{\theta}{2} + b_{73}\gamma_2\cos\frac{\theta}{2} - b_{68}\cos\frac{\theta}{2} - b_{67}\gamma_2\sin\frac{\theta}{2}\right)$$
$$+K_1\gamma_1(b_{73}\cos\frac{\theta}{2}$$
$$-b_{67}\sin\frac{\theta}{2} + b_{79}\cos\frac{\theta}{2} + b_{63}\sin\frac{\theta}{2})$$
$$-K_2\gamma_1\left(b_{73}\sin\frac{\theta}{2} + b_{67}\cos\frac{\theta}{2} + b_{79}\sin\frac{\theta}{2} - b_{63}\cos\frac{\theta}{2}\right).$$

## 5. Graphical results and discussion

In order to have an estimate of the quantitative effects of various parameters involved in the results of the present analysis, the graphs of the analytical results obtained for the stream function $\psi(x,y)$, the fluid velocities $w(x,y)$, $v(x,y)$, $u(x,y)$, and the volume flow rate $\hat{Q}$ are introduced for different value of the magnetic parameter M, the Hall parameter m, the porosity of the boundaries V, and the permeability parameter $W_1$.

Figure 3 illustrates the behavior of the non-dimensional fluid velocity $w(x,y)$ with changing in parameters M, m, V, and $W_1$. Figures 3a and 3b depict that the magnitude of $w$ decreases with an increase in M and m, respectively. Conversely, Figs. 3c and 3d reveal that the magnitude of $w$ is enhanced by the increase in both V and $W_1$, respectively. That is, the magnitude of $w$ parabolically increases by increasing the values of V and $W_1$. It is noted that $w$ has its maximum value at the center of the channel (y=0). Figure 4 depicts the behavior of the velocity distribution $v$ with changing the parameters under consideration. Figures 4a, 4b, and 4d show that the magnitude of $v$ decreases steadily in the channel by increasing the values of M, m, and $W_1$, respectively. On the other hand, Figure 4c shows that the magnitude of values of $v$ increases with increasing the values of V. It is noted that the value of $v$ vanishes on y=0.

The variations in the behavior of the velocity distribution $u$ for various values of the indicated parameters are illustrated in Figure 5. Figures 5a, 5b, and 5d depict that the magnitude of $u$ parabolically decreases with increasing the values of M, m, and $W_1$ throughout the channel. However, in Figure 5c, it is shown that the magnitude of $u$ parabolically increases across the channel with increasing the values of V. It is shown that $u$ has its maximum value at the centerline of the channel (y=0).

Figure 6 depict the behavior of the rate of volume flow $\hat{Q}$ which is plotted against x within the interval $0 \le x \le 0.5$ for different values of M, m, V and $W_1$. Apparently, M, V and $W_1$ have increasing effects on the magnitude of $\hat{Q}$ as shown in Figs. 6a, 6c, and 6d, while m has a decreasing effect on the magnitude of $\hat{Q}$ as shown in Figure 6b. It is noted that $\hat{Q}$ has its maximum value on x=0.

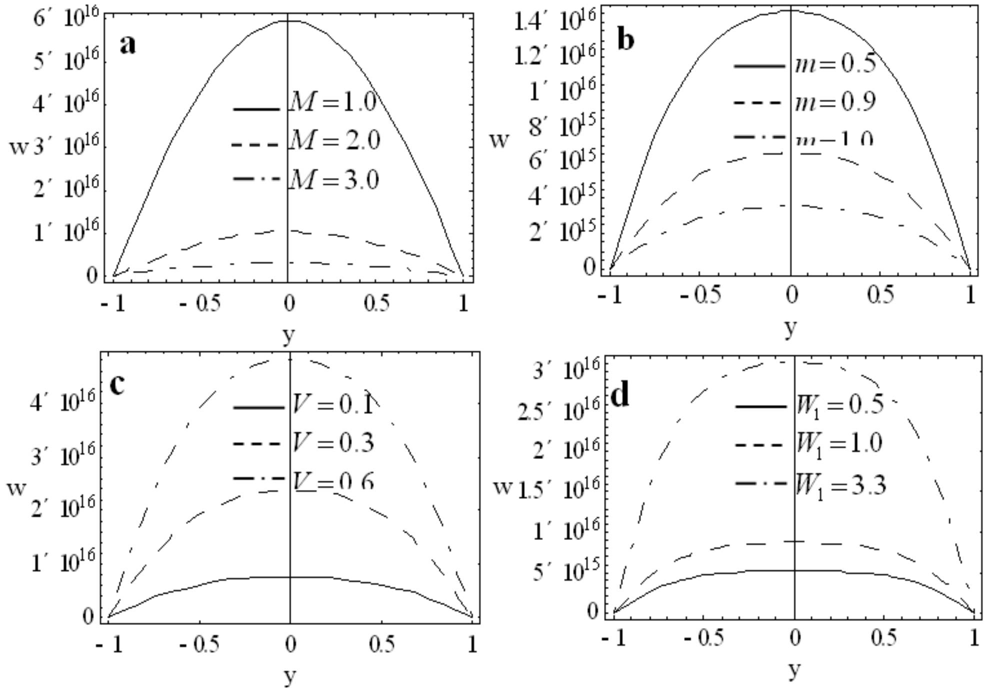


**Figure 3:** The variation of $w$ with y for three different values of the magnetic parameter M (panel **a**), the Hall parameter m (panel **b**), the porosity parameter V (panel **c**), and the permeability parameter $W_1$ (panel **d**). The other chosen parameters are V=0.05, $W_1$=1, m=5, x=0.25, R=10, $\varepsilon = 0.001$, and $\alpha = 0.5$ (panel **a**); V=0.05, $W_1$=1, M=2, x=0.25, R=10, $\varepsilon = 0.001$, and $\alpha = 0.5$ (panel **b**); $W_1$=1, M=2, m=1, x=0.25, R=10, $\varepsilon = 0.001$, and $\alpha = 0.5$ (panel **c**); and V=0.05, M=4, m=1, x=0.25, R=10, $\varepsilon = 0.001$, and $\alpha = 0.5$ (panel **d**)

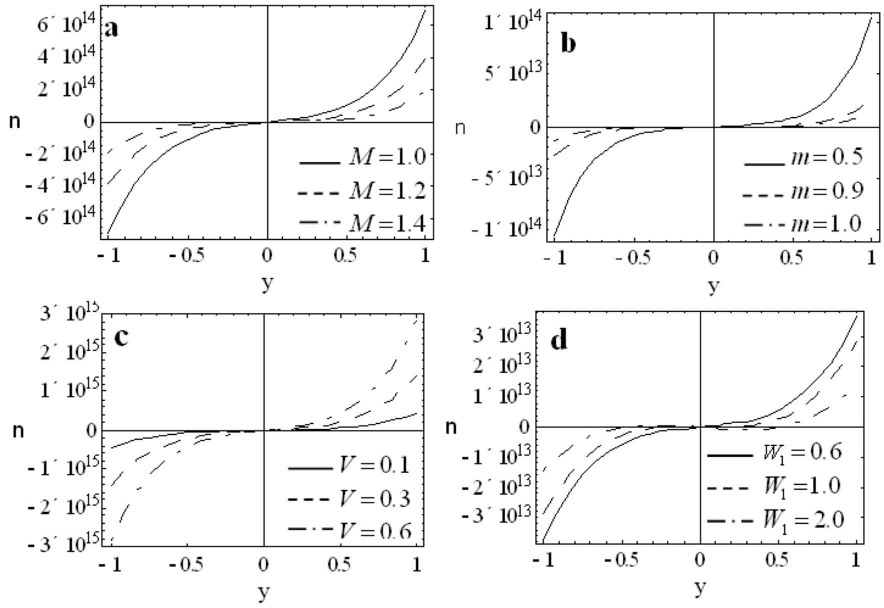


**Figure 4** The variation of $v$ with y for three different values of the magnetic parameter M (panel **a**), the Hall parameter m (panel **b**), the porosity parameter V (panel **c**), and the permeability parameter $W_1$ (panel **d**). The other chosen parameters are V=0.15, $W_1$=1, m=1, x=0.25, R=10, $\varepsilon = 0.001$, and $\alpha = 0.5$ (panel **a**); V=0.05, $W_1$=1, M=2, x=0.25, R=10, $\varepsilon = 0.001$, and $\alpha = 0.5$ (panel **b**); $W_1$=1, M=1, m=1, x=0.25, R=10, $\varepsilon = 0.001$, and $\alpha = 0.5$ (panel **c**); and V=0.05, M=2, m=5, x=0.25, R=10, $\varepsilon = 0.001$, and $\alpha = 0.5$ (panel **d**)

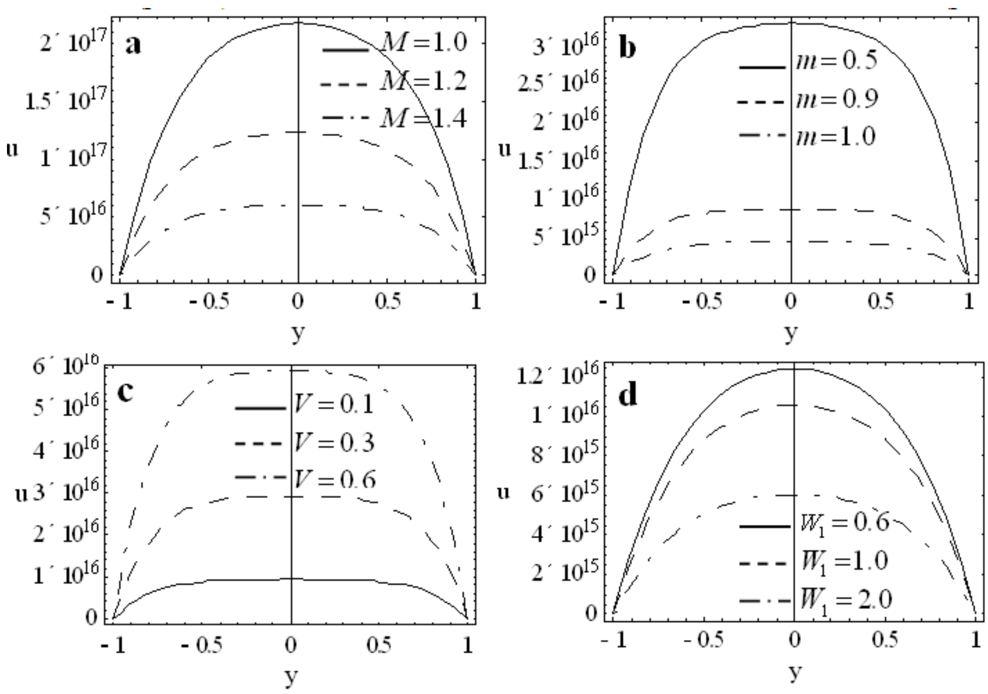


**Figure 5** The variation of $u$ with y for three different values of the magnetic parameter M (panel **a**), the Hall parameter m (panel **b**), the porosity parameter V

(panel **c**), and the permeability parameter $W_1$ (panel **d**). The other chosen parameters are V=0.15, $W_1$=1, m=1, x=0.25, R=10, $\varepsilon = 0.001$, and $\alpha = 0.5$ (panel **a**); V=0.05, $W_1$=1, M=2, x=0.25, R=10, $\varepsilon = 0.001$, and $\alpha = 0.5$ (panel **b**); $W_1$=1, M=2, m=1, x=0.25, R=10, $\varepsilon = 0.001$, and $\alpha = 0.5$ (panel **c**); and V=0.05, M=2, m=5 , x=0.25, R=10, $\varepsilon = 0.001$, and $\alpha = 0.5$ (panel **d**)

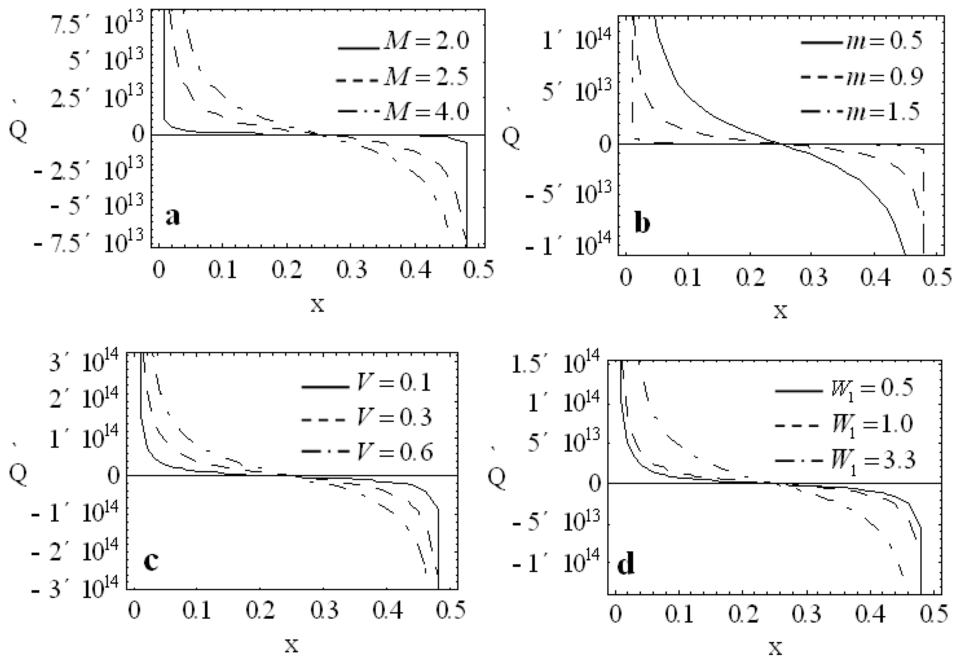


**Figure 6** The variation of $\hat{Q}$ with y for three different values of the magnetic parameter M (panel **a**), the Hall parameter m (panel **b**), the porosity parameter V (panel **c**), and the permeability parameter $W_1$ (panel **d**). The other chosen parameters are V=0.05, $W_1$=1, m=1.5, R=10, $\varepsilon = 0.001$, and $\alpha = 0.5$ (panel **a**); V=0.05, $W_1$=1, M=2, R=10, $\varepsilon = 0.001$, and $\alpha = 0.5$ (panel **b**); $W_1$=1, M=2, m=1, R=10, $\varepsilon = 0.001$, and $\alpha = 0.5$ (panel **c**); and V=0.05, M=4, m=1, R=10, $\varepsilon = 0.001$, and $\alpha = 0.5$ (panel **d**)

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## Appendix

The functions of $\eta$ and the constants appearing in Eqs. (25), (26), (28), (29) and (30) are given by:

$$A_1 = b_1 + \eta\, b_2,\ A_2 = b_3 + \eta\, b_4,\ A_3 = b_5 + \eta\, b_6,\ A_4 = b_7 + \eta\, b_8$$

$$A_5 = b_9 + \eta\, b_{10},\ A_6 = b_{11} + \eta\, b_{12},\ A_7 = b_{13} + \eta\, b_{14},\ A_8 = b_{15} + \eta\, b_{16},$$

$$A_9 = b_{17} + \eta\, b_{18},\ A_{10} = b_{19} + \eta\, b_{20},\ A_{23} = b_{31} + \eta b_{32},\ A_{24} = b_{38} + \eta\, b_{39},$$

$$A_{25} = b_{50} + \eta\, b_{51},\ A_{26} = b_{57} + \eta\, b_{58},\ N_1 = b_{73} + \eta\, b_{74}\ N_2 = b_{67} + \eta\, b_{68},$$

$N_3 = b_{79} + \eta\, b_{80}$, $N_4 = b_{63} + \eta\, b_{64}$, where $A_{1j}\ (j=\overline{1,9})$ and $A_{2j}\ (j=\overline{0,2})$ are constants, in which $b_1 = \sin(\gamma_2)e^{\gamma_1}$, $b_2 = \sqrt{r}\sin(\frac{\theta}{2}+\gamma_2)e^{\gamma_1}$, $b_3 = \cos(\gamma_2)e^{-\gamma_1}$,

$$b_4 = -\sqrt{r}\cos(\frac{\theta}{2}-\gamma_2)e^{-\gamma_1},\ b_5 = \sin(\gamma_2)e^{-\gamma_1},\ b_6 = \sqrt{r}\sin(\frac{\theta}{2}-\gamma_2)e^{-\gamma_1},$$

$$b_7 = \cos(\gamma_2)e^{\gamma_1},\ b_8 = \sqrt{r}\cos(\frac{\theta}{2}+\gamma_2)e^{\gamma_1},\ b_9 = -1/b_3 + b_7,$$

$$b_{10} = b_4 + b_8/(b_3+b_7)^2,\ b_{11} = -b_1 b_3 + b_5 b_7/b_3^{\,2} - b_7^{\,2},$$

$$b_{12} = \frac{(b_1 b_3 + b_5 b_7)(2b_3 b_4 - 2b_7 b_8)}{(b_3^{\,2} - b_7^{\,2})^2} - \frac{(b_1 b_4 + b_2 b_3 + b_5 b_8 + b_6 b_7)}{(b_3^{\,2} - b_7^{\,2})},$$

$$b_{13} = -b_1 b_7 + b_3 b_5/b_3^{\,2} - b_7^{\,2},$$

$$b_{14} = \frac{(b_1 b_7 + b_3 b_5)(2b_3 b_4 - 2b_7 b_8)}{(b_3^{\,2} - b_7^{\,2})^2} - \frac{(b_1 b_8 + b_2 b_7 + b_3 b_6 + b_4 b_5)}{(b_3^{\,2} - b_7^{\,2})},$$

$$b_{15} = -(1/b_7)(1 + b_3 b_9),\ b_{16} = -(1/b_7)(b_3 b_{10} + b_4 b_9 - b_8/b_7 - b_8 b_3 b_9/b_7),$$

$$b_{17} = -(1/b_7)(b_1 + b_3 b_{11}),$$

$$b_{18} = -(1/b_7)(b_2 + b_3 b_{12} + b_4 b_{11} - b_1 b_8/b_7 - b_3 b_8 b_{11}/b_7),$$

$$b_{19} = -(1/b_7)(b_5 + b_3 b_{13}),$$

$$b_{20} = -(1/b_7)(b_6 + b_3 b_{14} + b_4 b_{13} - b_5 b_8/b_7 - b_3 b_8 b_{13}/b_7),\ b_{25} = b_{22} + b_{23}$$

$$b_{21} = 1 + 1 \,|\, RM\ \beta W_1 + A_{15} b_{15} + A_{17} b_9,\ b_{22} = A_{15} b_{16} + A_{17} b_{10},\ b_{23} = b_{15} A_{19} - A_{21} b_9$$

$$b_{24} = b_{16} A_{19} - A_{21} b_{10},\ b_{26} = A_{15} b_{17} + A_{17} b_{11} + A_{16},\ b_{27} = A_{15} b_{18} + A_{17} b_{12},$$

$$b_{28} = A_{19} b_{17} - A_{21} b_{11} + A_{20},\ b_{29} = A_{19} b_{18} - A_{21} b_{12},\ b_{30} = b_{27} + b_{28},\ b_{31} = -b_{21}/b_{26}$$

$$,\quad b_{32} = -(1/b_{26})(b_{25} - b_{21} b_{30}/b_{26}),\ b_{33} = A_{15} b_{19} + A_{17} b_{13} + A_{18},$$

$$b_{34} = A_{15} b_{20} + A_{17} b_{14},\quad b_{35} = A_{19} b_{19} - A_{21} b_{13} - A_{22},\ b_{36} = A_{19} b_{20} - A_{21} b_{14},$$

$$b_{37} = b_{34} + b_{35},\ b_{38} = -b_{33}/b_{26},\quad b_{39} = -(1/b_{26})(b_{37} - b_{30} b_{33}/b_{26}),$$

$$b_{40} = 1 + 1 \,|\, RM\ \beta W_1 + A_{17} b_{15} + A_{15} b_9,\ b_{41} = A_{17} b_{16} + A_{15} b_{10},$$

$$b_{42} = A_{21} b_{15} - A_{19} b_9,\ b_{43} = A_{21} b_{16} - A_{19} b_{10},\ b_{44} = b_{41} - b_{42},$$

$$b_{45} = A_{17}b_{17} + A_{15}b_{11} - A_{18}\,, b_{46} = A_{17}b_{18} + A_{15}b_{12}\,, b_{47} = A_{21}b_{17} - A_{19}b_{11} - A_{22}\,,$$
$$b_{48} = A_{21}b_{18} - A_{19}b_{12}\,, b_{49} = b_{46} - b_{47}\,, b_{50} = -b_{40}/b_{45}\,,$$
$$b_{51} = -(1/b_{45})(b_{44} - b_{40}b_{49}/b_{45})\,, M_4 = -V b_{121} + V b_{125}\,,$$
$$b_{52} = A_{17}b_{19} + A_{15}b_{13} - A_{16}\,, b_{53} = A_{17}b_{20} + A_{15}b_{14}\,, b_{54} = A_{21}b_{19} - A_{19}b_{13} + A_{20}\,,$$
$$b_{55} = A_{21}b_{20} - A_{19}b_{14}\,, b_{56} = b_{53} - b_{54}\,, b_{57} = -b_{52}/b_{45}\,,$$
$$b_{58} = -(1/b_{45})(b_{56} - b_{49}b_{52}/b_{45})\,, b_{59} = b_{38} - b_{57}\,, b_{60} = b_{39} - b_{58}\,,$$
$$b_{61} = b_{50} - b_{31}\,, b_{62} = b_{51} - b_{32}\,, b_{63} = b_{61} | b_{59}\,, b_{64} = (1/b_{59})(b_{62} - b_{60}b_{61} | b_{59})\,,$$
$$b_{65} = b_{38}b_{63}\,, b_{66} = b_{39}b_{63} + b_{38}b_{64}\,, b_{67} = b_{31} + b_{65}\,, b_{68} = b_{32} + b_{66}\,, b_{69} = b_{17}b_{67}\,,$$
$$b_{70} = b_{18}b_{67} + b_{17}b_{68}\,, b_{71} = b_{19}b_{63}\,, b_{72} = b_{19}b_{64} + b_{20}b_{63}\,, b_{73} = b_{15} + b_{69} + b_{71}\,,$$
$$b_{74} = b_{16} + b_{70} + b_{72}\,, b_{75} = b_{11}b_{67}\,, b_{76} = b_{12}b_{67} + b_{11}b_{68}\,, b_{77} = b_{13}b_{63}\,,$$
$$b_{78} = b_{14}b_{63} + b_{13}b_{64}\,, b_{79} = b_9 + b_{75} + b_{77}\,, M_7 = -b_{123}M_4 | M_1 + b_{125}\,,$$
$$b_{119} = 1 + 1 | RM\,\beta W_1 + b_{73}A_{11} - b_{67}A_{12} - b_{79}A_{13} - b_{63}A_{14}\,,$$
$$M_1 = -V b_{119} + V b_{123}\,, M_6 = -b_{119}M_4 | M_1 + b_{121}\,,$$
$$b_{120} = 1 + 1 | RM\,\beta W_1 + b_{73}A_{15} + b_{67}A_{16} + b_{79}A_{17}$$
$$+ b_{63}A_{18} + b_{68}A_{12} - b_{80}A_{13} - b_{14}A_{14}\,, M_2 = -V b_{120} + V b_{124}\,,$$
$$b_{121} = b_{74}A_{11} - b_{68}A_{12} - b_{80}A_{13} - b_{64}A_{14}\,,$$
$$b_{122} = b_{74}A_{15} + b_{68}A_{16} + b_{80}A_{17} + b_{64}A_{18}$$
$$b_{123} = -(1 + 1 | RM\,\beta W_1) + b_{73}A_{13} - b_{67}A_{14} - b_{79}A_{11} - b_{63}A_{12}\,,$$
$$M_3 = -b_{119} + b_{123}\,, b_{124} = -(1 + 1 | RM\,\beta W_1) - b_{73}A_{17} + b_{67}A_{18}$$
$$- b_{79}A_{15} + b_{63}A_{16} + b_{74}A_{13} - b_{68}A_{14} - b_{80}A_{11} - b_{64}A_{12}\,,$$
$$b_{125} = b_{74}A_{13} - b_{68}A_{14} - b_{80}A_{11} - b_{64}A_{12}\,,$$
$$b_{126} = -b_{74}A_{17} + b_{68}A_{18} - b_{80}A_{15} + b_{64}A_{16}\,,$$