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Domatic Subdivision Stable Graphs

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Abstract

A domatic partition of a graph G = (V, E) is a partition of V into disjoint sets $V_1, V_2, ..., V_K$ such that each V_i is a dominating set for G. A subdivision of a graph G is a graph resulting from the subdivision of edges in G. In this paper we define domatic subdivision stable graph and initiate a study on them.
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Keywords: dominating set, domatic partition, subdivision.

1. Introduction

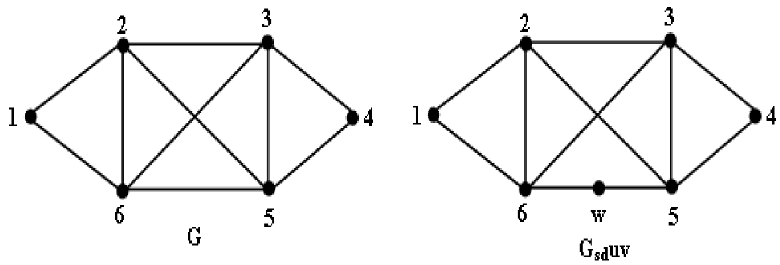
We consider only simple connected undirected graphs G = (V, E). P_n denotes the path of length n. The open neighborhood of vertex v ∈ V (G) is defined by N (v) = { u ∈ V (G) | (u v) ∈ E (G) } while its closed neighborhood is the set N [v] = N (v) ∪ { v }. The private neighborhood of v ∈ D is defined by pn [v, D] = N (v) – N (D – { v }). We indicate that u is adjacent to v by writing u ⊥ v. A set of vertices D in a graph G = (V, E) is a dominating set if every vertex of V – D is adjacent to some vertex of D. If D has the smallest possible cardinality of any dominating set of G, then D is called a minimum dominating set — abbreviated MDS. The cardinality of any MDS for G is called the domination number of G and it is denoted by γ (G). A vertex in V – D is k – dominated if it is dominated by at least k – vertices in D.

2. Materials and Methods

A domatic partition of a graph G = (V, E) is a partition of V into disjoint sets V_1, V_2, ...,V_K such that each V_i is a dominating set for G. The domatic number is the maximum number of such disjoint sets and it is denoted by d (G).
A subdivision of a graph G is a graph resulting from the subdivision of edges in G.

The subdivision of some edge e with endpoints { u, v } yields a graph containing one new vertex w, and with an edge set replacing e by two new edges, { u, w } and { w, v }. We shall denote the graph obtained by subdividing any edge uv of a graph G, by G_{sd} uv. Let w be a vertex introduced by subdividing uv. We shall denote this by G_{sd} uv = w.
In this paper we define a new graph called domatic subdivision stable graph.

Domatic Subdivision Stable Graphs
A graph G is said to be domatic subdivision stable(dss), if d (G) = d (G_{sd} uv), for all u, v $\in$ V (G), u $\perp$ v.



The graph in the above figure is dss. Here d (G) = | { 2, 5 }, { 3, 6 }, { 1, 4 } | = 3 and d (G_{sd} 65) = | { 2, 5 }, { 3, 6 }, { 1, 4, w } | = 3. This is true for all u, v $\in$ V (G), u $\perp$ v.

3. Results and Discussion
In this section we obtained a necessary and sufficient condition for a graph to be dss. We also provide condition under which a dss graph is DSS.

3.1 Necessary and sufficient condition
Theorem 1 *Let G be any graph. Then d (G_{sd} uv) $\leq$ 3.*
Proof
Claim Let G be any graph. Let P_3: uwv be a subgraph of G, where N(w) = { u, v }. Then d(G) $\leq$ 3.

Proof Let P_3: uwv be a subgraph of any graph G. Let N (w) = { u, v }. To dominate w either u or v or w should be included, that is any dominating set for G should contain atleast one of u, v, w. If G has a domatic partition d (G) = | { V_1 }, { V_2 }, { V_3 } | such that u $\in$ V_1, w $\in$ V_2, v $\in$ V_3, then d (G) = 3. In all cases where any two elements of the set { u, w, v } belong to the same dominating set d (G) < 3, which implies d (G) $\leq$ 3.
In G_{sd} uv $\langle$ uwv $\rangle$ is P_3, which implies d (G_{sd} uv) $\leq$ 3, by the above claim.

Corollary If G is a graph such that d (G) $\geq$ 4, then G is not dss for every u, v $\in$ V (G), u $\perp$ v.

Remark By corollary of theorem [1], we conclude that, it is sufficient to discuss dss for graphs, where $d (G) \leq 3$.

Theorem 2 *When $d (G) = 3$, $d (G_{sd}\, uv) = 3$ if and only if there is a partition $Z =\{ V_1, V_2, V_3 \}$ for G such that*

i. *V_1 and V_2 are dominating sets for G such that $u \in V_1$, $v \in V_2$.*
ii. *v is 2 – dominated with respect to V_1.*
iii. *u is 2 – dominated with respect to V_2.*
iv. *V_3 dominates atleast $G – \{ u \} – \{ v \}$.*

Proof Let $d (G) = d (G_{sd}\, uv) = 3$. By theorem [1], we know that there is a partition for G_{sd} uv, say $d (G_{sd}\, uv) = | \{ V_1 \}, \{ V_2 \}, \{ V_3 \} |$ such that $u \in V_1$, $v \in V_2$ and $w \in V_3$, where $w = G_{sd}$ uv. Since $w \in pn[u, V_1]$ and $w \in pn[v, V_2]$ in G_{sd} uv, v is 2 – dominated with respect to V_1 in G and u is 2 – dominated with respect to V_2 in G. Also $V_3 – \{ w \}$ dominates atleast $G – \{ u \} – \{ v \}$. $X = \{ V_1, V_2, V_3 – \{ w \}\}$ is a partition for G such that,

i. V_1 and V_2 are dominating sets for G such that $u \in V_1$, $v \in V_2$.
ii. v is 2 – dominated with respect to V_1.
iii. u is 2 – dominated with respect to V_2.
iv. $V_3 – \{ w \}$dominates atleast $G – \{ u \} – \{ v \}$.

Conversely assume that there is a partition $Z = \{ V_1, V_2, V_3 \}$ that satisfies the conditions of the theorem, then

a) V_1 is a dominating set for G_{sd} uv, since v is 2 – dominated.
b) V_2 is a dominating set for G_{sd}uv, since u is 2 – dominated.
c) Also $V_3 \cup \{ w \}$ dominates G_{sd} uv.

That is $d (G_{sd}\, uv) \geq 3$. Let $S \subseteq V = \{ uwv \}$, $< S > = P_3$ is a subgraph of G_{sd} uv. By theorem [1], we know that $d (G_{sd}\, uv) \leq 3$, which implies $d (G_{sd}\, uv) = 3$.

Theorem 3 *When $d (G) = 2$, $d (G_{sd}\, uv) = 3$ if and only if there is a domatic partition $d (G) = | \{ V_1 \}, \{ V_2 \} |$, $u \in V_1$, $v \in V_2$ such that*

i. *v is 2 – dominated with respect to V_1.*
ii. *$V_2 = V_{12} \cup V_{22}$, where*
   a) *$v \in V_{12}$, u is 2 – dominated with respect to V_{12}.*
   b) *V_{22} does not dominate atleast one of u or v, that is V_{22} is a dominating set for $G – \{ u \} – \{ v \}$.*

Proof Let $d (G) = 2$, $d (G_{sd}\, uv) = 3$. Since $d (G_{sd}\, uv) = 3$, let $d (G_{sd}\, uv) = | \{ V_1 \}, \{ V_2 \}, \{ V_3 \} |$, such that $u \in V_1$, $v \in V_2$, $w \in V_3$ [This is possible by theorem [1]].
Also

i. V_1 is a dominating set for G such that v is 2 – dominated.
ii. V_2 is a dominating set for G such that u is 2 – dominated.

V_3 – { w } can not dominate atleast one of u or v [else V_3 – { w } would be a dominating set for G, which implies d (G) = | { V_1 }, { V_2 }, { V_3 } |, a contradiction as d (G) = 2]. Hence,

iii. V_3 – { w } is a dominating set for G – { u } – { v }.

d (G) = | { V_1 }, { $V_2 \cup V_3$ } | is a domatic partition for G, such that the conditions of the theorem is satisfied. [Here $V_2 = V_{12}$, $V_3 = V_{22}$].

Conversely, assume that the condition of the theorem is satisfied.

i. Since v is 2 – dominated in V_1, u dominates w in G_{sd} uv, which implies V_1 is a dominating set for G_{sd} uv such that $u \in V_1$.
ii. Since u is 2 – dominated in V_{12}, v dominates w in G_{sd} uv, which implies V_{12} is a dominating set for G_{sd} uv
iii. Since V_{22} dominates G – { u } – { v }, $V_{22} \cup$ { w } is a dominating set for G_{sd} uv.

{ V_1, V_{12}, $V_{22} \cup$ { w } } is a domatic partition for G_{sd} uv, which implies d (G_{sd} uv) $\geq$ 3 . Also [By theorem [1]], d (G_{sd} uv) $\leq$ 3, which implies d (G_{sd} uv) = 3.

Remark d (G) = d (G_{sd} uv) = 2 if and only if there is no d (G) = | { V_1 }, { V_2 } | that satisfies the conditions of theorem [3].

Conclusion

From theorem 1, 2 and 3 we conclude that

1. If G is a graph such that d (G) = 3, then G is dss if and only if the conditions of theorem [2] is satisfied.
2. If G is a graph such that d (G) = 2, then G is dss if and only if there is no domatic partition d (G) = | { V_1 }, { V_2 } | that satisfies the conditions of theorem [3].

3.2 dss and DSS graph

In this subsection we provide conditions for a dss graph to be DSS graph. In [3], M. Yamuna and K. Karthika introduced the concept of domination subdivision stable graph. A graph G is said to be domination subdivision stable (DSS), if the γ - value of G does not change by subdividing any edge of G. They have proved the following result,

R_1. A graph G is DSS if and only if for every u, v $\in$ V (G), either there is a γ - set containing u and v or, there is a γ - set D such that either pn[u, D] = { v } or v is 2 – dominated.

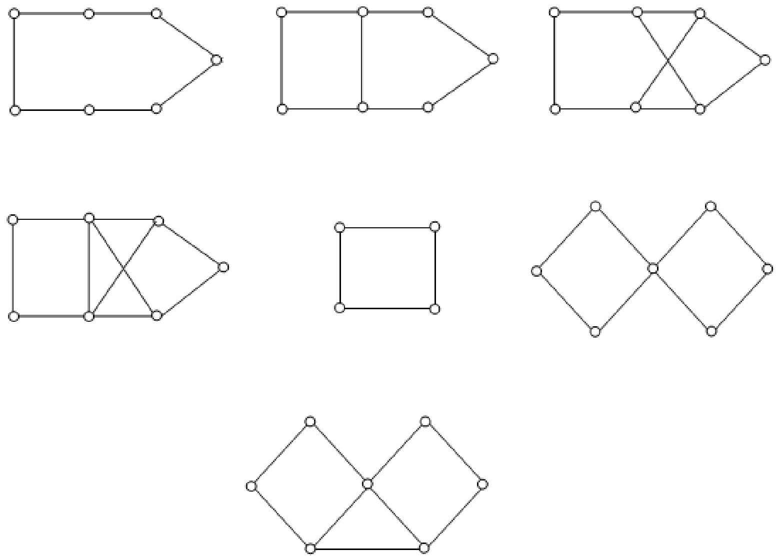
In order to characterize connected graphs with $\gamma (G) = \left\lfloor \frac{n}{2} \right\rfloor$, by defining the following six

classes of graphs. The following results were proved independently by Cockayne, Haynes and Hedetniemi and by Randerath and Volkmann [1].

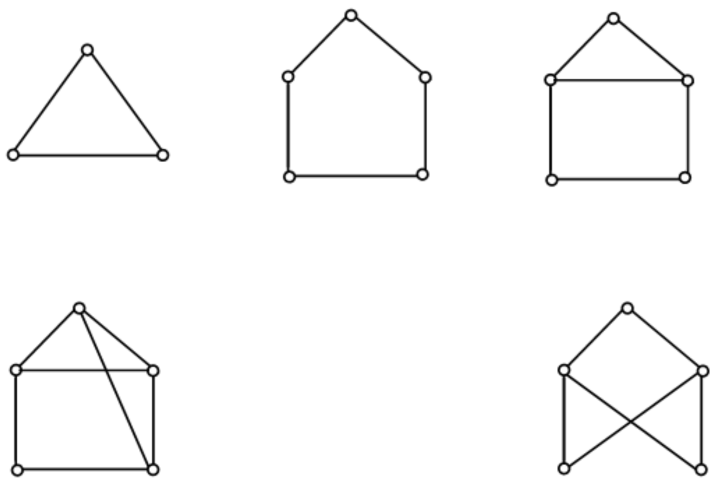
1. G_1 = { C_4 } $\cup$ { G : G = H o K_1 where H is connected }.

2. $G_2 = A \cup B - \{ C_4 \}$.

Graphs in family *A*



Graphs in family *B*



3. $G_3 = \bigcup_H S(H)$, where S(H) denote the set of connected graphs, each of which can be formed from H o K_1 by adding a new vertex x and edges joining x to one or more vertices of H.

4. $G_4 = \{ \theta(G) : G \in G_3 \}$, where $y \in V(C_4)$ and for $G \in G_3$, $\theta(G)$ is obtained by joining G to C_4 with the single edge xy, where x is the new vertex added in forming G.

5. $G_5 = \bigcup_H P(H)$, where u, v, w is a vertex sequence of a path P_3. For any graph H, let P (H) be the set of connected graphs which may be formed from H o K_1 by joining each of u and w to one or more vertices of H.

6. $G_6 = \bigcup_{H,X} R(H, X)$, where H is a graph, X ∈ B and R (H, X) is the set of connected graphs obtained formed from H o K_1 by joining each vertex of U ⊆ V (X) to one or more vertices of H such that no set with fewer than γ (X) vertices of X dominates V (X) – U.

R_2. A connected graph G satisfies $\gamma(G) = \left\lfloor \frac{n}{2} \right\rfloor$ if and only if $G \in \mathcal{G} = \bigcup_{i=1}^{6} G_i$

Theorem 4 *Let G be a dss graph such that* $|V_i| = \gamma(G)$, *for all* $V_i \in d(G)$. *Then G is DSS.*

Proof Let G be a dss graph.

Case i d (G) = 2. By [R_2], we observe that $\gamma(G) = \left\lfloor \frac{n}{2} \right\rfloor$, if and only if $G \in G_1$ [All the remaining class of graphs G_2 to G_6 contain odd number of vertices]. $G_1 = C_4 \cup \{ G : G = H \text{ o } K_1$, where H is connected }. C_4 and H o K_1 are dss graphs, which are DSS also.

Case ii d (G) = 3. We know that d (G) = d (G_{sd} uv) = 3 if and only if there is a partition Z = { V_1, V_2, V_3 } for G, where u ∈ V_1, v ∈ V_2 such that

i. When u ∈ V_1, v is 2 – dominated.
ii. When v ∈ V_2, u is 2 – dominated.
iii. V_3 dominate atleast G – { u } – { v }.

We have assumed that, $|V_i| = \gamma(G)$, i = 1, 2, …, k. For all u, v ∈ V (G), by condition (i) and (ii), there are γ - sets V_1 and V_2 such that both u and v are 2 – dominated, which implies γ (G) = γ (G_{sd} uv), for all u, v ∈ V (G), u ⊥ v [By R_1]. Hence G is DSS, when G is dss.

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Biographical Sketch

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