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On the Convergence of Generalized Polynomials

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Abstract

We have tested the convergence of a newly defined Generalized Polynomials and so the results of Bernstein has been extended for Lebesgue integrable function in L_1-norm by our newly defined Generalized Polynomials

$$U_n^{\alpha}(f,x) = (n+1)\sum_{k=0}^{n}\left\{\int_{k/(n+1)}^{(k+1)/(n+1)} f(t)dt\right\} q_{n,k}(x;\alpha)$$

where

$$q_{n,k}(x;\alpha) = \binom{n}{k}\frac{x(x+k\alpha)^{k-1}(1-x)(1-x+(n-k)\alpha)^{n-k-1}}{(1+n\alpha)^{n-1}}$$

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Introduction and Results

If $f(x)$ is a function defined on [0, 1], the Bernstein polynomial $B_n^f(x)$of f is given as

$$B_n^f(x) = \sum_{k=0}^{n} f(k/n)\, P_{n,k}(x)\ldots \qquad \ldots \qquad (1.1)$$

Where

$$P_{n,k}(x) = \binom{n}{k} x^k(1-x)^{n-k} \qquad \ldots\ldots\ldots.(1.2)$$

Bernstein (1912-13) proved that if $f(x)$ is continuous in closed interval [0,1], then

$$B_n^f(x) \rightarrow f(x)$$

uniformly as $n \rightarrow \infty$. This Yields a simple constructive proof of Weierstrass's approximation theorem.

A slight modification of Bernstein polynomials due to Kantorovitch[5] makes it possible to approximate Lebesgue integrable function in L_1-norm by the modified polynomials

$$P_n^f(x) = (n+1)\sum_{k=0}^{n}\left\{\int_{k/(n+1)}^{(k+1)/(n+1)} f(t)dt\right\} p_{n,k}(x)\ldots \quad (1.3)$$

where$P_{n,k}(x)$ is defined by (1.2)

By Abel's formula(see Jensen [3])

$$(x+y)(x+y+n\alpha)^{n-1} = \sum_{k=0}^{n}\binom{n}{k} x(x+k\alpha)^{k-1}y(y+(n-k)\alpha)^{n-k-1}\ldots \quad .(1.4)$$

If we put$y = 1 - x$, we obtain (see Cheney and Sharma [3])

$$1 = \sum_{k=0}^{n}\binom{n}{k}\frac{x(x+k\alpha)^{k-1}(1-x)(1-x+(n-k)\alpha)^{n-k-1}}{(1+n\alpha)^{n-1}}\ldots.. \quad (1.5)$$

Thus defining

$$q_{n,k}(x;\alpha) = \binom{n}{k}\frac{x(x+k\alpha)^{k-1}(1-x)(1-x+(n-k)\alpha)^{n-k-1}}{(1+n\alpha)^{n-1}}\ldots \quad ..(1.6)$$

we have

$$\sum_{k=0}^{n} q_{n,k}(x;\alpha) = 1 \quad \ldots.(1.7)$$

We now define apolynomial (seeHabib&Saleh[2])analogous to (1.3)

$$U_n^\alpha(f,x) = (n+1)\sum_{k=0}^{n}\left\{\int_{k/(n+1)}^{(k+1)/(n+1)} f(t)dt\right\} q_{n,k}(x;\alpha) \quad \ldots(1.8)$$

where$q_{n,k}(x;\alpha)$ is defined in(1.6) and moreover when $\alpha = 0$,(1.6) and (1.8) reduces to (1.2) and (1.3) respectively.

In this paper, we shall test the convergence of our polynomial (1.8) for Lebesgue integrable function in L_1-norm. In fact we state our result as follows

Theorem: If $f(x)$is continuous Lebesgue integrable function on [0,1] then for $\alpha = \alpha_n = o(1/n)$

$$\lim_{n\rightarrow\infty} U_n^\alpha(f,x) = f(x)$$

holds uniformly on [0,1]

2. Lemma

In order to prove our result we need the following lemma (see Habib& Saleh [2])

Lemma 2.1 – For all value of x

$$\sum_{k=0}^{n} kq_{n,k}(x;\alpha) \le \frac{1+n\alpha}{1+\alpha}nx - \frac{n(n-1)x\alpha}{1+2\alpha}$$

Lemma 2.2: For all values of x

$$\sum_{k=0}^{n} k(k-1)q_{n,k}(x;\alpha) \le n(n-1)[(x+2\alpha)\{\frac{1+n\alpha}{(1+2\alpha)^2} - \frac{(n-2)\alpha}{(1+3\alpha)^2}\}$$
$$+(n-2)\alpha^2\{\frac{1+n\alpha}{(1+3\alpha)^3} - \frac{(n-3)\alpha}{(1+4\alpha)^3}\}]$$

Lemma 2.3: - For all values of $x \in |0,1|$ and for $\alpha = \alpha_n = o(1/n)$, we have

$$(n+1)\sum_{k=0}^{n}\left\{\int_{k/(n+1)}^{(k+1)/(n+1)} (t-x)^2 dt\right\} q_{n,k}(x;\alpha) \le x(1-x)/n$$

3. Proof of the Theorem

Proof :

$$\left|U_n^{\alpha}(f,x) - f(x)\right| \le (n+1)\sum_{k=0}^{n}\left\{\int_{k/(n+1)}^{(k+1)/(n+1)} |f(t)-f(x)|dt\right\} q_{n,k}(x;\alpha)$$

Now splitting above inequality into two parts corresponding to those values of t for which $|t-x| < \delta$ and those for which $|t-x| \ge \delta$, we get

$$\left|U_n^{\alpha}(f,x) - f(x)\right| \le (n+1) \sum_{|t-x|<\delta}\left(\int_{k/(n+1)}^{(k+1)/(n+1)} |f(t)-f(x)|dt\right)q_{n,k}(x;\alpha)$$
$$+ (n+1) \sum_{|t-x|\ge\delta}\left(\int_{k/(n+1)}^{(k+1)/(n+1)} |f(t)-f(x)|dt\right)q_{n,k}(x;\alpha)$$

$= I_1 + I_2$ (say) -------- (3.1)

If the function $f(x)$ is bounded say $|f(x)| \le M$ in $0 \le x \le 1$&x is a point of continuity,for a given ε> 0 , ∃ a number δ> 0, ∋: $\left|x'' - x'\right| < \delta$ implies $\left|f(x'') - f(x')\right| < \varepsilon$

and therefore

$$I_1 \leq \frac{\varepsilon}{2}(n+1)\sum_{k=0}^{n}\left\{\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} dt\right\} q_{n,k}(x;\alpha) = \frac{\varepsilon}{2}$$

&

$$I_2 \leq 2M(n+1) \sum_{|t-x|\geq\delta} \left(\int_{k/(n+1)}^{(k+1)/(n+1)} dt\right) q_{n,k}(x;\alpha)$$

$$\leq \frac{2M}{\delta^2}(n+1)\sum_{k=0}^{n}\left\{\int_{k/(n+1)}^{(k+1)/(n+1)} (t-x)^2 dt\right\} q_{n,k}(x;\alpha)$$

$\leq \frac{2M}{4n\delta^2}$, by lemma 3 and the fact $x(1-x) \leq \frac{1}{4}$on $[0,1]$

On substituting the value of I_1 & I_2 in (3.1) we get

$$\left|U_n^{\alpha}(f,x) - f(x)\right| \leq \frac{\varepsilon}{2} + \frac{2M}{4n\delta^2}$$

for$\delta = (\frac{M}{n\varepsilon})^{1/2}$, we get

$$\left|U_n^{\alpha}(f,x) - f(x)\right| < \varepsilon$$

which complete the proof of the theorem.

Conclusion

The result of Bernstein has been extended for Lebesgue integrable function in L_1-norm by our newly defined Generalized Polynomials.

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