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Global Transversal sets and Global transversal Irredundant Sets

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Abstract

Let G be a simple graph. Let Ş be a collection of subsets of $V(G)$ with a common property. A subset T of $V(G)$ is called an Ş transversal if T meets every element of Ş. Many types of transversals like independent transversal, clique transversal etc have been studied. In what follows, a new type of transversal is introduced and studied. Global transversal irredundant sets are also defined and their properties are investigated.

1. Introduction

Transversal theory is highly useful in many branches of science and humanities. In sociological studies, representation of people from different sections of society is an important aspect while constituting social bodies. A graph representing a society is highly useful for the study of many aspects of the society. A transversal which meets maximum independent sets gives a representation for independent groups. A global transversal gives representation for highly independent as well as highly clustered groups. In this paper, study of global transversal sets is initiated. Minimal global transversal sets are characterized and a new concept called global irredundant set is defined and studied.

Section1.Global Transversals sets and Global transversal number.

Definition 1.1.A Clique in a graph is a maximal induced subgraph of the graph and a maximum clique is a clique of maximum cardinality.

Definition 1.2.A Transversal which meets all maximum cliques as well as maximum independent sets is called a global transversal set of G.

Note 1.3. $V(G)$ is always a global transversal set.

Definition 1.4. The minimum cardinality of global transversal set of G is the called the global transversal number of G and it is denoted by $t_g(G)$.

Example 1.5. Global transversal number of some standard graphs are given the following.

1 $t_g(G) = n, \forall n.$

2 $t_g(\overline{K_n}) = 1, \forall n.$

3 $t_g(K_{1,n}) = 2, \forall n.$

4 $t_g(K_{m,n}) = min\{m, n\} + 1.$

5 $t_g(C_n) = \left\lceil \frac{n}{2} \right\rceil + 1.$

6. $t_g(P_n) = \left\lceil \frac{n}{2} \right\rceil$

7 $t_g(W_{n+1}) = \left\lceil \frac{n}{2} \right\rceil + 1.$

8 $t_g(D_{r,s}) = 3, r, s \geq 2.$

For any Tree T

9 $2 \leq t_g(G) \leq \beta_0(G).$

Proposition1.6.$t_g(K_n) = n \; \forall \, n$ and$t_g(\overline{K_n}) = 1 \; \forall \, n$

Propositio1.7. $t_g(K_{1,n}) = 2 \; \forall \, n \geq 2.$

Proof. Let $V(K_{1,n}) = \{u, v_1, v_2, \dots, v_n\}$ where u is the centre.Then $\{u, v_1\}$ is a global transversal set and clearly a single vertex cannot consititute a global transversal set. Hence $t_g(K_{1,n}) = 2 \; \forall \, n \geq 2.$

Proposition1.8. $t_g(K_{m,n}) = min\{m, n\} + 1.$

Proof. Let $m, n \geq 2$.Since $K_{m,n}$ cannot contain any complete subgraph of order ≥ 3, $\omega(K_{m,n}) = 2$. Any clique joins a vertex of one bipartite set with a vertex of another bipartite set. Maximum independent set of $K_{m,n}$ is unique if $m \neq n$ and both partite sets are maximum independent sets,
if $m = n$.Hence$t_g(K_{m,n}) = min\{m, n\} + 1.$

llustration1.9. To illustrate this,consider the two graphs $K_{2,4}, K_{4,4}$ given in Figure 1.

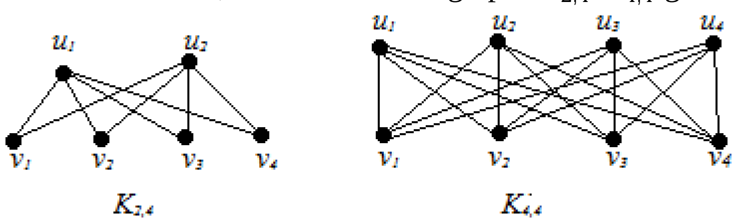

Figure 1: *Graphs $K_{2,4}$ and $K_{4,4}$*

$t_g(K_{2,4}) = min\{2,4\} + 1 = 2 + 1 = 3.\{u_1, u_2, v_1\}$ is a minimum global transversal set.Thus $t_g(K_{2,4}) = 3.t_g(K_{4,4}) = min\{4,4\} + 1 = 4 + 1 = 5. \{u_1, u_2, u_3, v_1\}$ is a minimum global transversal set.Thus $t_g(K_{4,4}) = 5.$

Theorem 1.10. For any bipartite graph G, $t_g(G) = \alpha_0(G) + 1$.

Proof:Let S be a α_0 $-$ set of G. Then $V - S$ is a β_0 set of G. Let $u \in V - S$. Consider $S \cup \{u\}$. Therefore $S \cup \{u\}$ meets all β_0 sets as well as all cliques of G.Therefore $t_g(G) \leq \alpha_0(G) + 1$.Let S_1 be a t_g $-$set of G. Suppose $|S_1| < \alpha_o(G) + 1$. If $V - S_1$ is independent, then $|V - S_1| > n - (\alpha_0(G) + 1) = \beta_0(G) + 1$,a contradiction. Therefore $V - S_1$ is not independent. Therefore $V - S_1$contains a K_2. Since G is bipartite,$\omega(G) = 2$.Therefore S_1 does not meet a maximum clique in $V - S_1$,a contradiction. Therefore $t_g(G) = \alpha_0(G) + 1$.

Corollary 1.11. For any Tree T, $t_g(T) = \alpha_0(T) + 1$.

Remark 1.12. The above property is not true for graphs which are not bipartite. To illustrate this consider the following graph G in figure 2.

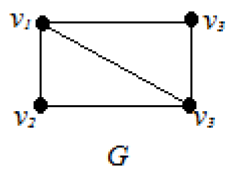

Figure 2: *Graph G*

$\{1,2\}, \{1,4\}$ are t_g $-$ sets of G.$\alpha_0(G) = 2$.Therefore $t_g(G) < \alpha_0(G) + 1$.

Remark 1.13. There are graphs which are not bipartite in which, $t_g(T) = \alpha_0(T) + 1$ holds.

For, Consider C_n,n is odd.$t_g(C_n) = \left\lceil \frac{n}{2} \right\rceil + 1$,$\alpha_0(C_n) = \left\lceil \frac{n}{2} \right\rceil$,$t_g(C_n) = \alpha_0(C_n) + 1$.

Remark 1.14. For any graph G,$t_g(G) \leq \alpha_0(G) + 1$.(Since any vertex cover C together with a vertex from $V - C$ is a global transversal of G).

Observation1.15. If $G$is any graph,$t_g(G.K_1) = |V(G)| + 1 = \alpha_0(G.K_1) + 1$

Observation1.16. Given a graph G, there are induced subgraphs S whose t_g may be equal to greater or less than that of G.

For,

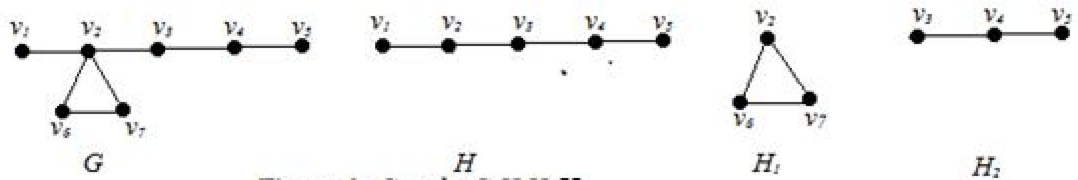

Figure 3: *Graphs G,H,H1H2*

The β_0 sets are $\{v_1, v_6, v_3, v_5\}$,$\{v_1, v_7, v_3, v_5\}$.The only clique is $\{v_2, v_6, v_7\}$$\{v_1, v_2\}$ is a t_g sets of G. $t_g(H) = 3 > t_g(G)$,$t_g(H_1) = 1 < t_g(G)$,$t_g(H_2) = 2 = t_g(G)$.

Observation.1.17. Let T be a global transversal set of G. Then any super set of T is also global transversal set of G.

Definition 1.18. T is called a minimal global transversal set of G if no proper subset of T is a global transversal set of G. Since global transversal property is super hereditary, a global transversal set is minimal iff it is 1- minimal.The minimum cardinality of a minimal global transversal number of G is the global transversal number of G and is denoted by $t_g(G)$ and the maximum cardinality of a minimal global transversal set of G is called the upper global transversal number of G and is denoted by $T_g(G)$.

Example 1.19.consider the graph given in Figure 4

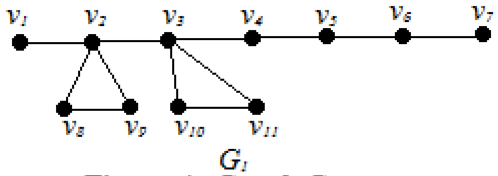

Figure 4: ***Graph G_1***

The maximum independent sets of the graph G_1 are$\{v_1, v_8, v_{10}, v_4, v_6\}$,$\{v_1, v_8, v_{10}, v_5, v_7\}$,$\{v_1, v_8, v_{10}, v_4, v_7\}$, $\{v_1, v_8, v_{11}, v_4, v_6\}$, $\{v_1, v_8, v_{11}, v_5, v_7\}$, $\{v_1, v_8, v_{11}, v_4, v_7\}$, $\{v_1, v_9, v_{10}, v_4, v_6\}$, $\{v_1, v_9, v_{10}, v_5, v_7\}$, $\{v_1, v_9, v_{10}, v_4, v_7\}$, $\{v_1, v_9, v_{11}, v_4, v_6\}$, $\{v_1, v_9, v_{11}, v_5, v_7\}$, $\{v_1, v_9, v_{11}, v_4, v_7\}$. $c$liques of G_1 are $\{v_2, v_8, v_9,\}$, and $\{v_3, v_{11}, v_{10}\}$.The minimal global transversal sets of G_1 are $\{v_1, v_2, v_3\}$, $\{v_2, v_3, v_4, v_5,\}$, $\{v_2, v_3, v_5, v_7,\}\{v_2, v_3, v_4, v_6,\}$,Therefore $t_g(G_1) = 3, T_g(G_1) = 4$.

Example1.20.

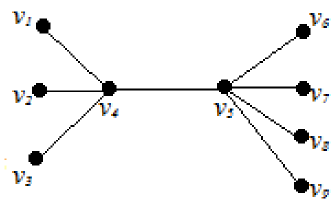

Figure 5: *The graph $G = D_{3,4}$*

The unique maximum independent set of the above given graph in the figure 5 is $\{v_1, v_2, v_3, v_6, v_7, v_8, v_9\}$.Cliquesare$\{v_4, v_1\}$, $\{v_4, v_2\}$, $\{v_4, v_3\}$, $\{v_4, v_5\}$,$\{v_5, v_6\}$, $\{v_5, v_7\}$, $\{v_5, v_8\}$, $\{v_5, v_9\}$.The minimal global transversal sets are $\{v_1, v_4, v_5\}$,$\{v_6, v_7, v_8, v_9, v_4\}$ and $\{v_1, v_2, v_3, v_5,\}$.Therefore $t_g(G) = 3, T_g(G) = 5$.

Remark1.21. Given any positive integer k, there exists a graph G such that $T_g(G) - t_g(G) = k$..

Proof. Consider $D_{r,s}$ where $s = k + 2$ and $r \leq s, t_g(D_{r,s}) = 3$ and $, T_g(D_{r,s}) = k + 3$.Therefore $T_g(G) - t_g(G) = k$.

Theorem1.22. Let T be a global transversal set of G. Then T is minimal iff for every $u \in T$,either there exists a maximum independent set S such that $S \cap T = \{u\}$ or there exists a maximum clique C such that $C \cap T = \{u\}$.

Proof. Obvious.

Definition 1.23. Let $u \in V(G)$.(i)The maximum independent set neighbourhood of u in G is defined as $\{S \subseteq V(G): S$ is a maximum independent set of G and $u \in S\}$.This is denoted by $N_{\beta_o}(u)$.(ii)The maximum clique neighbourhood of u in G is defined as $\{C \subseteq V(G): C$ is a maximum clique and $u \in V(G)\}$.This is denoted by $N_c(u)$.

Definition 1.24. Let T be a global transversal set of G. Let $u \in T$.Let S be a maximum independent set of G. S is said to be a private maximum independent neighbour of u with respect to$T$ if $S \in N_{\beta_0}(u)$ and $S \notin N_{\beta_0}(v)$ for any $v \neq u,\ v \in T$.The set of all private maximum independent set neighbours of u is denoted by $pn_{\beta_0}(u, T)$.

Definition1.25. Let T be a global transversal set of G. Let $u \in T$.Let C be a maximum clique of G. C is said to be a private maximum clique neighbour of u with respect to T if $C \in N_c(u)$ and $C \notin N_c(v)$ for all $v \neq u, v \in T$.The set of all private maximum clique neighbours of u is denoted by $pn_c(u, T)$. **Remark 1.26.** Let T be a global transversal set of G. T is minimal iff $u \in T\ pn_{\beta_0}(u, T) \neq \emptyset\ or\ pn_c(u, T) \neq \emptyset$

Section 2. $g.t$ irredundant sets

Definition 2.1. Let T be a non empty subset of $V(G)$ satisfying the following :
$\forall u \in T\ pn_{\beta_0}(u, T) \neq \emptyset\ or\ pn_c(u, T) \neq \emptyset$. Then T is called $g.t$ irredundant set of G.

Observation 2.2. If T is a $g.t$ irredundant set of G,then any subset of T is also a $g.t$ irredundant set of G.(That is $g.t$ irredundance is a hereditary property).

Proof: Let T be a $g.t$ irredundant set of G.Let $S \subseteq V$.Let $u \in S$.Therefore $u \in T$.Therefore $pn_{\beta_0}(u, T) \neq \emptyset\ or\ pn_c(u, T) \neq \emptyset$ That is there exists a maximum independent set A of G such that $A \cap T = \{u\}$ or there exists a maximum clique C of G such that $T \cap C = \{u\}$. Therefore $A \cap S = \{u\}\ or\ C \cap S = \{u\}$. Therefore $pn_{\beta_0}(u, S) \neq \emptyset\ or\ pn_c(u, S) \neq \emptyset$.Therefore S is a $g.t$ irredundant set of G.

Definition 2.3. The minimum (maximum) cardinality of a maximal $g.t$ irreduntant set of a graph G is called (upper) $g.t$ irreduntance number of G and is denoted by $ir_{g.t}(G)\left(IR_{g.t}(G)\right)$.

Example 2.4.

1.$ir_{g.t}(K_n) = IR_{g.t}(K_n) = n$.

2.$ir_{g.t}(P_{2n}) = n, IR_{g.t}(P_{2n}) = n + 1$.

3. $ir_{g.t}(P_{2n+1}) = n, IR_{g.t}(P_{2n+1}) = n + 1, n \geq 1$.

Example 2.5

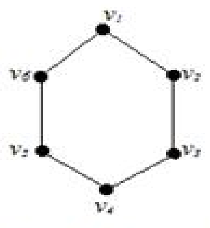

Figure 6: *Graph C_6*

Maximum independent sets of C_6 are $\{v_1, v_3, v_5\}$,$\{v_2, v_4, v_6\}$ Cliques of C_6 are $\{v_1, v_2\}, \{v_2, v_3\}, \{v_3, v_4\}, \{v_4, v_5\}, \{v_5, v_6\}, \{v_6, v_1\}, \{v_1, v_2, v_4, v_5\}, \{v_1, v_2, v_3, v_5\}$, $\{v_2, v_3, v_4, v_6\}, \{v_1, v_3, v_4, v_5\}$ are maximal $g.t$ irreduntant sets.$\{v_2, v_4, v_5, v_6\}, \{v_5, v_6, v_1, v_3\}, \{v_6, v_1, v_2, v_4\}$ are also maximal $g.t$ irrreduntant sets. For, Consider $S = \{v_1, v_2, v_4, v_5\}$. Therefore S is a $g.t$ irreduntant set of C_6.Consider $\{v_1, v_2, v_3, v_4, v_5\}$.$v_2$ belongs to $\{v_1, v_2\}, \{v_2, v_3\}, \{v_2, v_4, v_6\}$ and none of them is a private neighbour of v_2.Therefore$\{v_1, v_2, v_3, v_4, v_5\}$ is not a $g.t$ irreduntant set. Consider $\{v_1, v_2, v_6, v_4, v_5\}$.$v_5$ belongs to

$\{v_4, v_5\}, \{v_5, v_6\}, \{v_1, v_3, v_5\}$ and none of them is a private neighbour of v_5.Therefore $\{v_1, v_2, v_6, v_4, v_5\}$ is not a $g.t$ irreduntant set. Therefore$\{v_1, v_2, v_4, v_5\}$ is a maximal $g.t$ irreduntant set of C_6..Consider 3 element sets of C_6..There are 20, 3 element sets..Theyare$\{v_1, v_2, v_3\}, \{v_2, v_3, v_4\}, \{v_3, v_4, v_5\}, \{v_4, v_5, v_6\}, \{v_5, v_6, v_1\}, \{v_6, v_1, v_2\}$, $\{v_1, v_2, v_4\}, \{v_2, v_3, v_5\}, \{v_3, v_4, v_6\}, \{v_4, v_5, v_1\}, \{v_5, v_6, v_2\}, \{v_6, v_1, v_3\}, \{v_1, v_2, v_5\}$, $\{v_2, v_3, v_6\}, \{v_3, v_4, v_1\}, \{v_4, v_5, v_2\}, \{v_5, v_6, v_3\}, \{v_6, v_1, v_4\}, \{v_1, v_3, v_5\}$ and$\{v_2, v_4, v_6\}$.None of them is maximal. Any 5 element subset of $V(C_6)$ is not an$g.t$ irreduntant set.Therefore $ir_{g.t}(C_6) = IR_{g.t}(C_6) = 4$.

Theorem2.6. $ir_{g.t}(C_n) = IR_{g.t}(C_n) = \left\lceil \frac{n}{2} \right\rceil + 1$.

Proof: Case(i) 'n' $= 2n$. Let $V(C_{2n}) = \{u_1, u_2, u_3, \dots, u_{2n}\}$.
Let S=$\{u_1, u_2, u_3, u_5, u_7, u_9, \dots, u_{2n+1}\}$ .Clearly S is a maximal $g.t$ irreduntant set of C_{2n} of cardinality $(n + 1)$.Let T be any subset of $V(C_{2n})$ of cardinality $(n + 2)$. Then either T contains five consecutive terms of $V(C_{2n})$ or three consecutive terms occurring in two different places.In either case T is not a $g.t$ irreduntant set of C_{2n} .Let T be any $g.t$ irreduntant subset of $V(C_{2n})$ of cardinality n. Then either T contains all odd suffixed terms or all even suffixed terms or three consecutive terms and other $(n-3)$ terms having suffixes with the opposite pairty of the suffix of the middle term. In any case T is contained properly in a $g.t$ irredutant set of C_{2n}..Also any $g.t$ irredutant subset of $V(C_{2n})$ of cardinality less than or equal to $(n-1)$ can be proved to be contained in a maximal $g.t$ irredutant set of C_{2n}..Hence the theorem.
Case(ii) 'n' $= 2n + 1$.Let$V(C_{2n+1}) = \{u_1, u_2, \dots, u_{2n+1}\}$.
Let S=$\{u_1, u_2, u_3, u_4, u_6, u_8, \dots, u_{2n}\}$.Clearly S is a maximal $g.t$ irreduntant set of C_{2n+1} cardinality $(n + 2)$.Let T be any subset of $V(C_{2n+1})$ of cardinality $(n + 3)$. Then either T contains six consecutive terms of $V(C_{2n+1})$ or three consecutive terms occurring in three different places. In either case T is not a $g.t$ irreduntant set of C_{2n+1}..Let T be any $g.t$ irreduntant subset of $V(C_{2n+1})$ of cardinality $(n + 1)$.Then either T contains all odd suffixed terms with exactly one even suffixed term or all even suffixed terms with exactly one odd suffixed terns or three consecutive terms and other $(n-3)$ terms having suffixes with the opposite parity of the suffix of the middle term.In any case T is contained properly in a $g.t$ irreduntant set of C_{2n+1}.Also any $g.t$ ireduntant subset of $V(C_{2n+1})$ of cardinality less than or equal to n can be proved to be contained in a maximal set of C_{2n+1}. Hence the theorem.

Theorem 2.7. Let G be a simple graph. Any minimal global transversal set of G is a maximal $g.t$ irreduntant set of G.

Proof: Let S be a minimal global transversal set of G. Then S is a $g.t$ irreduntant set of G.Suppose S is not a maximal $g.t$ irreduntant set of G, then S is contained in a maximal $g.t$ irreduntant set of G say T .Let $u \in T - S$.Then $S \cup \{u\}$ is a $g.t$ irreduntant set of G. Therefore $pn_c\{u, S \cup \{u\}\} \neq \emptyset \text{ or } pn_{\beta_0}\{u, S \cup \{u\}\} \neq \emptyset$.Therfore there exists a maximum clique C such that $u \in C$ and no vertex of S belongs to C or there exists a maximum independent set I such that $u \in I$ and no vertex of S belongs to I.Therefore $S \cap C = \emptyset \text{ or } S \cap I = \emptyset$, a contradiction, since S is a global transversal set of G.Hence the theorem.

Remark 2.8 .$ir_{g.t}(G) \leq t_g(G) \leq T_g(G) \leq IR_{g.t}(G)$

Example 2.9.

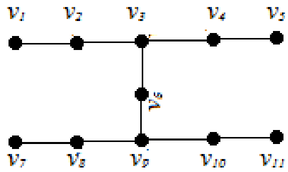

Figure 7: *Graph G2*

In the above graph G_2 in the figure 7, Each edge is a clique and $\{v_1, v_3, v_5, v_7, v_9, v_{11}\}$ is the unique maximum independent set. Any global transversal set must intersect each edge and hence contains atleast five points.If it contains exactly five points, then the transversal contains v_2, v_4, v_6, v_8 and v_{10}.The transversal also intersects the independent set and none of the pointsv_2, v_4, v_6, v_8 and v_{10} belongs to the unique independent set. Therefore any global transversal set contains atleast six vertices. $\{v_1, v_2, v_4, v_6, v_8, v_{10}\}$ is a global transversal set of cardinality 6. Therefore $t_g(G_2)$=6.$\{v_1, v_3, v_5, v_7, v_9, v_{11}\}$ is also a global transversal set. It can be seen that $\{v_2, v_3, v_4, v_8, v_9, v_{10}\}$ is a maximal $g.t$ irreduntant set of G. It can be verified that $ir_{g.t}(G_2) = 6$.

Example 2.10.

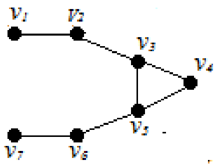

Figure 8: *Graph G3*

In the Graph $G_{3,}\{v_3, v_4, v_5\}$ is the unique maximum clique. $\{v_1, v_3, v_6\}, \{v_1, v_4, v_6\}, \{v_2, v_4, v_6\}, \{v_2, v_4, v_7\}, \{v_2, v_5, v_7\}, \{v_1, v_5, v_7\}$, $\{v_1, v_4, v_7\}, \{v_1, v_3, v_7\}$ are the maximum independent sets. Here $ir_{g.t}(G) = t_g(G) = T_g(G) = IR_{g.t}(G) = 3$.

Remark 2.11.

$(i). ir_{g.t}(K_{1,n}) = 2, IR_{g.t}(K_{1,n}) = n, \forall n.$

$(ii). ir_{g.t}(K_{m,n}) = \{ \ min\{m, n\} + 1 \quad if \ m \neq n.$

$m + n \quad if \ m = n.$

$IR_{g.t}(K_{m,n}) = \{ max\{m, n\} \qquad if \ m \neq n \ .$

$m + 1 \quad if \ m = n.$

Remark2.12.

$(i). \ ir_{g.t}(K_{1,n}) = 2, IR_{g.t}(K_{1,n}) = \alpha_0(K_{1,n}), \forall \ n \ .$

For any bipartite graph $G = K_{m,n}$,

$(ii) ir_{g.t}(K_{m,n}) = IR_{g.t}(K_{m,n}) = \{ \ \beta_0(K_{m,n}) \qquad if \ m \neq n \ .$

$\beta_0(K_{m,n}) + 1 \quad if \ m = n.$

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