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Harmonic and Biharmonic Maps of Riemann Surfaces

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Abstract

We obtain expressions in terms of isothermal coordinates for harmonic non $\pm$ holomorphic maps between Riemann surfaces. We utilize the expressions to generalize the first main theorem for holomorphic maps from a bounded domain into harmonic maps with well-defined order, and the Gauss-Bonnet formula for holomorphic maps to harmonic maps with Jacobians ≥ 0 (resp. ≤ 0). We also study biharmonic maps of Riemann surfaces and their relationship with harmonic maps.

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1. Introduction

The theory of harmonic maps between Riemannian manifolds were first established by Eells and Sampson [14] in 1964. Chiang, Ratto, Sun and Wolak have studied harmonic maps and biharmonic maps in [6]–[12].

Professor Joseph H. Sampson (deceased, Chiang's Ph.D. adviser) proposed a project for me to generalize the value distribution for holomorphic maps into harmonic maps a while ago. Unfortunately, we are not successful (even for Riemann surfaces), and have encountered two main obstructions (as opposed to $\pm$ homomorphic maps): 1. The pull-back f^*h of a Hermitian metric h of a harmonic non $\pm$ holomorphic f is not necessarily Hermitian. 2. The pull-back f^* of a harmonic non $\pm$ holomorphic map f fails to commute with the ∂ and $\overline{\partial}$ operators. However, we circumnavigate part of obstructions through a careful use of the Korn-Lichtenstein theorem (cf. [4, 18]), and obtain partial results.

In Section two, we obtain expressions in terms of isothermal coordinates for harmonic non $\pm$ holomorphic maps between Riemann surfaces in Theorem 2.1, which provide a more precise form than a result was derived by Eells and Wood [15]. We then utilize the expressions to generalize the first main theorem for holomorphic maps from a bounded domain into a Riemann surface to harmonic maps with well-defined order in Theorem 3.1. Afterwards, we use the expressions to generalize the Gauss-Bonnet formula for holomorphic maps from a bounded domain into a Riemann surface to harmonic maps of Jacobians $J \geq 0$ (resp. $J \leq 0$) with isolated stationary points in Theorem 4.1 through Korn-Lichtenstein theorem. We finally investigate biharmonic maps of Riemann surfaces. We prove in Theorem 5.4 that if $f : M \to N$ is a stable biharmonic map from a compact Riemannian manifold into a Riemannian manifold N with positive constant sectional curvature $K > 0$ satisfying the conservation law, then f is a harmonic map. We then apply this theorem to discuss the relationship between biharmonic maps and harmonic maps of Riemann surfaces.

2. Expressions of harmonic maps between Riemann surfaces

Let $f : (M, g_{ij}) \to (N, h_{\alpha\beta})$ be a C^2 map between two Riemannian manifolds M and N with smooth Riemannian metrics g and h. A harmonic map $f : (M^m, g_{ij}) \to (N, h_{\alpha\beta})$ between two Riemannian manifolds is a critical point of the energy functional

$$E(f) = \frac{1}{2}\int_M |df|^2 dv = \frac{1}{2}\int_M h_{\alpha\beta} f_i^\alpha f_j^\beta g^{ij} dv, \tag{1}$$

where dv is the volume form of M determined by g. In order to compute the first variation of the energy functional, we consider a one-parameter family of maps $\{f_t\} \in C^\infty(M \times [0, 1], N)$ from a compact manifold M (without boundary) into a Riemannian manifold N such that f_t is the endpoint of a segment starting at $f(x)(= f_0(x))$ determined in length and direction by the vector field $\dot{f}(x)$. If M is a non-closed manifold, we assume that the compact support of $\dot{f}(x)$ is contained in the interior of M. In terms of the Euler-Lagrange equation, we have

$$\begin{aligned}\dot{E}(f) &= \frac{d}{dt}E(f_t)|_{t=0} = \int_M (df_t, \nabla_t df_t)|_{t=0} dv = \int_M (df, \nabla \dot{f}) dv \\ &= \int_M div(w) dv - \int_M (\tau f, \dot{f}) dv = -\int_M (\tau f, \dot{f}) dv = 0, \quad \forall \dot{f},\end{aligned} \tag{2}$$

by the divergence theorem, where $\tau(f) = trace_g(\nabla df)$, ∇ is the connection on $T^*M \otimes f^{-1}TN$ induced by the Levi-Civita connections on M and N, and $div(w) = w^j_{|j}$ with $w^j = h_{\alpha\beta} f_i^\alpha \dot{f}^\beta g^{ij}$ a vector field on M. The map $f : M \to N$ is *harmonic* iff the tension field

$$\begin{aligned}\tau^\alpha(f) &= trace_g(\nabla df) = g^{ij} f^\alpha_{i|j} = g^{ij}(f^\alpha_{i,j} + \Gamma'^\alpha_{\beta\gamma} f_i^\beta f_j^\gamma) \\ &= g^{ij}(f^\alpha_{ij} - \Gamma^k_{ij} f^\alpha_k + \Gamma'^\alpha_{\beta\gamma} f_i^\beta f_j^\gamma) = 0,\end{aligned} \tag{3}$$

where Γ^k_{ij} and $\Gamma'^{\alpha}_{\beta\gamma}$ are the Christoffel symbols of the Levi-Civita connections on M and N, respectively.

We can apply the above concept to a harmonic map $f : M \to N$ between two Riemann surfaces. In isothermal coordinates, let $g = \sigma(z)dzd\bar{z}$ ($\sigma > 0$) and $h = \lambda(w)dwd\bar{w}$ ($\lambda > 0$) be the conformal metrics on M and N, respectively, and f be represented by $z \to w(z)$. Then the energy functional is

$$E(f) = \frac{1}{2}\int_M \|df\|^2\, dv = \int_M \frac{\lambda}{\sigma}\left\{|w_z|^2 + |w_{\bar{z}}|^2\right\} dv,$$

where $z = x + iy$ and $dv = \lambda dx \wedge dy$ is the area element of M. Hence, the map $f : M \to N$ of Riemann surfaces is *harmonic* iff the tension field of f

$$\tau(f) = \frac{4}{\sigma}(w_{z\bar{z}} + \frac{\lambda_w}{\lambda} w_z w_{\bar{z}}) = \frac{4}{\sigma}[w_{z\bar{z}} + (ln\lambda)_w w_z w_{\bar{z}}] = 0. \tag{4}$$

Observe that (4) is equivalent to

$$w_{z\bar{z}} + \frac{\lambda_w}{\lambda} w_z w_{\bar{z}} = 0. \tag{5}$$

Let $f : (M^m, g) \to (N^n, h)$ be a C^2 map from an m-dimensional manifold M into an n-dimensional Riemannian manifold, $\tilde{g} = \rho^2 g$ be a conformal change of the metric g, and $\tau(f)_g$ and $\tau(f)_{\tilde{g}}$ be the tension fields of f with respect to the metrics g and $\tilde{g}$, respectively. Then

$$\tau(f)_{\tilde{g}} = \frac{1}{\rho^2}\left\{\tau(f)_g + (m-2)df(grad\, ln\rho)\right\},\ \rho \neq 0.$$

In particular, if $m = 2$, then the harmonicity of f is conformally invariant on a Riemann surface. If $m \neq 2$, then it is not necessarily true. We need the following lemma (by Sampson [23, 24], and Eells and Wood [15]) to prove Theorem 2.2.

Lemma 2.1. If $f : M \to N$ is harmonic between Riemann surfaces, then the (2,0) part of the tensor field $f^*(\lambda dwd\bar{w})$ is a holomorphic quadratic differential on M, and it is denoted by

$$Q = \phi(z)dz^2 = \lambda(w(z))w_z\bar{w}_z dz^2. \tag{6}$$

Moreover, $\phi = 0$ iff f is $\pm$ holomorphic. If f is harmonic non $\pm$ holomorphic, then $\phi(z) \neq 0$, and the zeros of w_z and $w_{\bar{z}}$ are isolated of finite order.

Theorem 2.2. If $f : M \to N$ is harmonic non $\pm$ holomorphic between Riemann surfaces, then

$$w(z) = Az^m + o(|z|^m)\ for\ some\ m \geq 1\ and\ complex\ number\ A \neq 0,\ or \tag{7}$$

$$w(z) = B\bar{z}^n + o(|z|^n)\ for\ some\ n \geq 1\ and\ B \neq 0,\ or \tag{8}$$

$$w(z) = Cz^k + D\bar{z}^k + o(|z|^k)\ for\ some\ k \geq 1\ and\ C \neq 0,\ D \neq 0. \tag{9}$$

Proof. If $f : M \to N$ is a harmonic map between Riemann surfaces such that M and N are endowed with analytic conformal metrics g and h, then f is analytic by [14]. Thus in local coordinates z, w at a point p and its image point $a = f(p)$, we can express

$$w(z) = \sum_{i,j=0}^{\infty} a_{ij} z^i \bar{z}^j, \; a_{ij} \in \mathbf{C}, \tag{10}$$

which is convergent in a neighborhood of $z = 0$. We first have $a_{00} = 0$ due to $a = f(p)$. Since f is harmonic non $\pm$ holomorphic, the zeros of w_z and $w_{\bar{z}}$ are isolated of finite order by Lemma 2.1. Let's rewrite (5) as

$$\frac{\lambda_w}{\lambda} = -\frac{w_{z\bar{z}}}{w_z w_{\bar{z}}}, \tag{11}$$

where $w_z, w_{\bar{z}}$ and $w_{z\bar{z}}$ are analytic at a (since $w(z)$ is analytic at a). Therefore, we can choose a sufficiently small neighborhood U of p with $f(p) = a$ such that it avoids all the isolated zeros of w_z and $w_{\bar{z}}$, then $\frac{\lambda_w}{\lambda}$ is analytic at a, and it can be expressed as

$$\frac{\lambda_w}{\lambda} = \sum_{k,l=0}^{\infty} b_{kl} w^k \bar{w}^l, \; b_{kl} \in \mathbf{C}, \tag{12}$$

which is convergent in a neighborhood of $w = 0$. Let $v(w) = \frac{\lambda_w}{\lambda}(w)$ and $\xi = v(w)$ be the local coordinate of $q = v(a)$(i.e. $\xi(q) = 0$). Thus we have $b_{00} = 0$. Substituting (10), (11) and (12) into (5) and by straightforward computation, we first derive $a_{11} = 0$. If $a_{10} \neq 0$, $a_{01} = 0$, then $w(z)$ is in (7); if $a_{10} = 0, a_{01} \neq 0$, then $w(z)$ is in (8) ; if $a_{10} \neq 0$, $a_{01} \neq 0$, then $w(z)$ is in (9). If $a_{10} = a_{01} = 0$, then by (5) we derive $a_{21} = a_{12} = 0$, and $w(z)$ can be put in (7), (8) or (9), etc. ■

Theorem 2.2 provides a more precise form for harmonic non $\pm$ holomorphic maps between Riemann surfaces than the following result was obtained by Eells and Wood (cf. [15], p. 265).

Corollary 2.3. [15] If $f : M \to N$ is harmonic non $\pm$ holomorphic between Riemann surfaces, then

$$\begin{aligned} w_z &= E z^{m-1} + o(|z|^{m-1}), \; for\, some\, m \geq 1\, and\, complex\, number\, E \neq 0; \\ w_{\bar{z}} &= F \bar{z}^{n-1} + o(|z|^{n-1}), \; for\, n \geq 1\, and\;\, F \neq 0. \end{aligned}$$

3. First main theorem

Let D be a compact smooth oriented domain bounded by a piece-wise smooth curve γ in a Riemann surface M, and let $f : D \to N$ be a harmonic map from D into a compact oriented Riemann surface N. Suppose $f(p) = a,\ p \in D-\gamma,\ a \in N$, and suppose that p is the only point in a neighborhood of p which is mapped into a by f. The integer $n(p, a)$ is called the *order* of f at $(p,\ a)$, which measures the number of times of a neighborhood of a covered by a neighborhood of p under f. If $f : M \to N$ is holomorphic (resp. anti-holomorphic), then $w(z) = a_l z^l + a_{l+1} z^{l+1} + \cdots = a_l z^l + o(|z|^l),\ l \geq 1$ (resp. $w(z) = b_j \bar{z}^j + b_{j+1}\bar{z}^j + 1 + \cdots = b_j \bar{z}^j + o(|z|^j),\ j \geq 1$) and $n(p,\ a) = l \in \mathbf{Z}^+$ (resp. $-j \in \mathbf{Z}^-$). If $f : M \to N$ is harmonic non $\pm$ holomorphic, then $n(p, a) = m \in \mathbf{Z}^+$ in (7); $n(p, a) = -n \in \mathbf{Z}^-$ in (8). In (9), if $|C| > |D|$, $w(z) \approx Cz^k(1 + \dfrac{D}{C}(\dfrac{\bar{z}}{z})^k)$ as $|z| \to 0$ (where the second term is strictly dominated by 1), then $n(p, a) = k \in \mathbf{Z}^+$. If $|C| < |D|$, $w(z) \approx D\bar{z}^k(\dfrac{C}{D}(\dfrac{z}{\bar{z}})^k + 1)$ as $|z| \to 0$ (where the first term is strictly dominated by 1), then $n(p, a) = -k \in \mathbf{Z}^-$. If $|C| = |D|$, then the order $n(p,\ a)$ may not be well-defined. From now on, we assume that $f : D \to N$ is harmonic with well-defined order. Suppose that $f(z_0) = a,\ z_0 \in D - \gamma,\ a \in N$, and z_0 is the only point in a neighborhood of z_0, which is mapped into a by f such that $f^{-1}(a) \cap \gamma = \emptyset$. If a is a point such that $f^{-1}(a)$ is a finite set of points in $D - \gamma$, define $n(a) = \displaystyle\sum_{z \in f^{-1}(a)} n(z, a)$.

Since $f : D \to N$ is harmonic (f is continuous), there are coordinate neighborhoods $V,\ W$ of $z_0,\ a$, respectively, such that $f(V) \subset W$. Let $z = \eta e^{i\theta},\ w = re^{i\phi}$ be the local coordinates of z_0 $(z = 0)$, a $(w = 0)$ in $V,\ W$, respectively, and let C_ρ and S_ϵ be the circles $\eta = \rho = const$ and $r = \epsilon = const$, respectively, where ρ and ϵ are sufficiently small. There is a retraction $R : W - a \to S_\epsilon$, which maps a point with the coordinate $w = re^{i\phi}$ into a point with the coordinate $\epsilon e^{i\phi}$. Since both D and N are oriented, the circles C_ρ and S_ϵ have the induced orientations. Then $R \circ f(C_\rho)$ is a cycle on S_ϵ, and is homologous to an integral multiple of S_ϵ. This multiple is exactly the order $n(z_0, a)$, which is independent of the coordinate choices.

In fact, the order $n(z_0, a)$ can be represented by an integral formula. Let Ω be a real-valued two form on the compact Riemann surface N such that $\displaystyle\int_N \Omega = c > 0$. Note that $w = re^{i\phi}$ is the local coordinate at a point in N. If g is a real-valued differentiable function in $r,\ \phi$, we have

$$dg = g_r d\phi + g_\phi d\phi = g_w dw + g_{\bar{w}} d\bar{w} = (g_w e^{i\phi} + g_{\bar{w}} e^{-i\phi})dr + ir(g_w e^{i\phi} - g_{\bar{w}} e^{-i\phi})d\phi,$$

such that $g_r = g_w e^{i\phi} + g_{\bar{w}} e^{-i\phi}$, and

$$(\partial - \bar{\partial})g = g_w dw - g_{\bar{w}} d\bar{w} = (\cdots)dr + irg_r d\phi. \tag{13}$$

Suppose that $u(z_0, a)$ is a solution of

$$\frac{1}{\pi i}\,\partial\bar{\partial}u \;=\; \frac{1}{c}\,\Omega, \tag{14}$$

where $u(z_0, a)$ is a C^2 function in $N - a$, and $u(z_0, a) - log|w_a| = g$ (i.e. $u = log|w_a| + g$)

is a C^2 function in a neighborhood of a [5]. Let

$$\Psi = \frac{i}{2\pi}(\partial - \bar{\partial})\, u \tag{15}$$

such that $d\Psi = \frac{1}{c}\Omega$. If we consider an identity map $\iota : S_\epsilon \to W - a$, then

$$\iota^* \Psi = -\frac{1}{2\pi} d\phi + o(\epsilon) \tag{16}$$

by (13), (14) and (15), where $o(\epsilon)$ denotes a differential form which tends to zero as $\epsilon \to 0$. Since we assume that f is harmonic with well-defined order (i.e., $n(z_0, a)$ and $n(a)$ are well-defined), (16) implies that

$$n(z_0, a) = -\lim_{\epsilon \to 0} \int_{Rf(C_\rho)} \iota^* \Psi = -\lim_{\epsilon \to 0} \int_{\iota Rf(C_\rho)} \Psi. \tag{17}$$

By applying the Stokes' theorem to the integral of $d\Psi$ over D, we obtain the following first main theorem for harmonic maps.

Theorem 3.1. Let $f : D \to N$ be a harmonic map from a compact smooth oriented domain D bounded by a piece-wise smooth curve γ in a Riemann surface M into a compact oriented Riemann surface N with well-defined order. If $a \in N$ is such that $f^{-1}(a)$ contains finitely many points and that $f^{-1}(a) \cap \gamma = \emptyset$, then

$$n(a) + \int_{f(\gamma)} \Psi = \frac{1}{c} \int_{f(D)} \Omega = \frac{1}{c} v(D), \tag{18}$$

where $v(D)$ is the area of f(D).

Corollary 3.2. If D has no boundary, then (18) reduces to $n(a) = \frac{1}{c} v(D)$.

Remark 3.3. Theorem 3.1 generalizes the first main theorem for holomorphic maps from a bounded domain [5, 27] to harmonic maps. If $f : D \to N$ is $\pm$ holomorphic, then $n(z_0, a)$ and $n(a)$ are automatically well-defined.

4. Gauss-Bonnet formula

Let D be a compact smooth oriented domain in a Riemannian surface M bounded by a piece-wise smooth curve γ, and let $f : D \to N$ be a harmonic map from D into a compact oriented Riemann surface N. Let z be a local coordinate at a point $p \in D$, and w be a local coordinate at the image point $a = f(p)$. In terms of the local coordinate $w = re^{i\phi}$, let $h = \lambda dw d\bar{w}$ be a Hermitian metric on N. Set $\lambda = \mu^2$ and

$$\Theta = -d\phi + i(\partial - \bar{\partial}) log \mu. \tag{19}$$

Then $d\Theta = K\Omega$, where $\Omega = \frac{i}{2}\mu^2 dw \wedge d\bar{w}$ is the area element, and

$$K = -(\frac{4}{\mu^2})\frac{\partial^2 log\mu}{\partial w \partial \bar{w}}. \tag{20}$$

is the curvature with respect to the Hermitian metric $h = \mu^2 dwd\bar{w}$ on N.

If $f : D \to N$ is harmonic non $\pm$ holomorphic, then the Jacobian of f is $J_f = |w_z|^2 - |w_{\bar{z}}|^2$. By Theorem 2.2, J_f may be positive, or negative, or zero. A point $p \in D$ is a *singular point* of f iff the Jacobian J_f vanishes at p. A point p is a *stationary point* of f iff $df|_p = 0$ iff $w_z = w_{\bar{z}} = 0$ (since $dw = w_z dz + w_{\bar{z}} d\bar{z}$). When J_f is zero, f may have isolated stationary points, or non-isolated singularities. Wood [26] studied the singularities of harmonic maps between surfaces, and Smith [25] constructed a harmonic non $\pm$ holomorphic map from a torus into a sphere, which exhibited collapsed lines. We shall not consider this degenerated case here. From now on, we assume that $f : D \to N$ is harmonic non $\pm$ holomorphic of $J \geq 0$ (resp. $J \leq 0$) with isolated stationary points S (in (7), (8) or (9) with $|C| \neq |D|$, since $\phi(z) = \lambda w_z \bar{w}_z \neq 0$ is holomorphic and the zeros of w_z and $w_{\bar{z}}$ are isolated of finite order by Lemma 2.1). Thus $S = \{p \in D : df|_p = 0\}$ is a finite set of isolated points, since D is compact. We compute

$$\begin{aligned}
f^*h &= \lambda w_z \bar{w}_z dz^2 + \lambda w_z \bar{w}_{\bar{z}} dz d\bar{z} + \lambda w_{\bar{z}} \bar{w}_z d\bar{z} dz + \lambda w_{\bar{z}} \bar{w}_{\bar{z}} d\bar{z}^2 \\
&= \phi dz^2 + \lambda(|w_z|^2 + |w_{\bar{z}}|^2) dz d\bar{z} + \bar{\phi} d\bar{z}^2 \\
&= [\lambda w_z \bar{w}_z + \lambda(|w_z|^2 + |w_{\bar{z}}|^2|) + \lambda w_{\bar{z}} \bar{w}_{\bar{z}}] dx^2 + [2i\lambda w_z \bar{w}_z - 2i\lambda w_{\bar{z}} \bar{w}_{\bar{z}}] dxdy \\
&\quad +[-\lambda w_z \bar{w}_z + \lambda(|w_z|^2 + |w_{\bar{z}}|^2) - \lambda w_{\bar{z}} \bar{w}_{\bar{z}}] dy^2 \\
&= [2Re\,\phi(z) + \lambda(|w_z|^2 + |w_{\bar{z}}|^2)] dx^2 - 4Im\,\phi(z) dxdy \\
&\quad +[-2Re\,\phi(z) + \lambda(|w_z|^2 + |w_{\bar{z}}|^2)] dy^2,
\end{aligned} \tag{21}$$

where $Re\phi$ and $Im\phi$ are harmonic functions on $D - S$. By (21) the Hermitian metric $h = \lambda dwd\bar{w}$ in N induces the metric f^*h which is Riemannian (cf. (a), (b) and (c)), but not necessarily Hermitian in D except at S where $f^*h = 0$. We circumnavigate the pull-back metric through a careful use of the Korn-Lichtenstein's theorem (cf. [4, 18]) as follows. If f^*h is Riemannian in $D - S$, then by [4] there exists an isothermal coordinate transformation $\zeta(z)$ in $D - S$ such that $f^*h = \rho^2 d\zeta d\bar{\zeta}$, and $D - S$ has a new complex structure ζ. Consequently, we can transform the harmonic non $\pm$ holomorphic map $f(z) : D \to N$ with $J \geq 0$ (resp. $J \leq 0$) into a holomorphic (resp. anti-holomorphic) map $f(\zeta) : D \to N$ in the new isothermal coordinate ζ with respect to f^*h except at S.

We want to calculate the relationship between ζ and z in $D - S$. Note that f^*h is conformal, i.e., $f^*h = \psi\bar{\psi} = \rho^2 d\zeta d\bar{\zeta}$, where

$$d\zeta = \left(\frac{1}{\rho}\right)\psi. \tag{22}$$

By [4] and (21), we write $\psi = \theta_1 + i\theta_1$, where $\theta_1 = a_1 dx + b_1 dy$, $\theta_2 = a_2 dx + b_2 dy$

are real linear differential forms, such that

$$\begin{aligned} a_1^2 + a_2^2 &= \lambda w_z \bar{w}_z + \lambda(|w_z|^2 + |w_{\bar{z}}|^2|) + \lambda w_{\bar{z}} \bar{w}_{\bar{z}} = 2Re\phi + \lambda(|w_z|^2 + |w_{\bar{z}}|^2), \\ 2(a_1b_1 + a_2b_2) &= 2i\lambda w_z \bar{w}_z - 2i\lambda w_{\bar{z}} \bar{w}_{\bar{z}} = 2i\phi - 2i\bar{\phi} = -4Im\phi, \qquad (23) \\ b_1^2 + b_2^2 &= -\lambda w_z \bar{w}_z + \lambda(|w_z|^2 + |w_{\bar{z}}|^2) - \lambda w_{\bar{z}} \bar{w}_{\bar{z}} = -2Re\phi + \lambda(|w_z|^2 + |w_{\bar{z}}|^2). \end{aligned}$$

(a) If $f(z) : D \to N$ is harmonic non $\pm$ holomorphic of Jacobian $J \geq 0$ in (7) with finite isolated stationary points S and consider $f(z) : D - S \to N$, then $((f^*h)_{ij})_{2\times 2}$ can be expressed as

$$\begin{aligned} (f^*h)_{11} &\approx \lambda|Amz^{m-1}|^2, (f^*h)_{12} = (f^*h)_{21} \approx 0, (f^*h)_{22} \\ &\approx \lambda|Amz^{m-1}|^2, \, as \, |z| \to 0. \qquad (24) \end{aligned}$$

Therefore,

$$\left(f^*h\right)_{2\times 2} \approx \begin{bmatrix} \lambda|Amz^{m-1}|^2 & 0 \\ 0 & \lambda|Amz^{m-1}|^2 \end{bmatrix}, \qquad (25)$$

as $|z| = \sigma \to 0$, we have $det(f^*h) \approx \lambda^2|A|^4m^4\sigma^{4m-4} > 0$ on $D - S$. The pull-back f^*h is Riemannian, and it is *almost Hermitian* on $D - S$. By (23) and (24), as $|z| \to 0$, for simplicity we may take $a_1 \approx \sqrt{\lambda}|Amz^{m-1}|$, $a_2 \approx -\sqrt{\lambda}|Amz^{m-1}|$, $b_1 \approx \sqrt{\lambda}|Amz^{m-1}|$, $b_2 \approx \sqrt{\lambda}|Amz^{m-1}|$. It follows from (4.4) that

$$\begin{aligned} \zeta &= \int (\frac{1}{\rho})\psi = \int \frac{1}{\rho}(\theta_1 + i\theta_2) \approx \int \frac{\sqrt{\lambda}}{\rho}|Amz^{m-1}|(1-i)dz \\ &= \int \frac{\sqrt{\lambda}}{\rho}|A|m\sigma^{m-1}(1-i)dz = c_1\sigma^{m-1}z, \qquad (26) \end{aligned}$$

as $|z| \to 0$, where ρ and λ may be assumed constants for simplicity, ζ may be viewed as the line integral of 1-form $\frac{1}{\rho}\psi$ along the line segment $l(t) = tz$, $0 \leq t \leq 1$, connecting 0 and z in $D - S$, and $c_1 = \frac{\sqrt{\lambda}}{\rho}|A|m(1-i)$. Hence, we can transform the harmonic non $\pm$ holomorphic map $f(z) : D \to N$ of $J \geq 0$ into a holomorphic map $f(\zeta) : D \to N$ except at S, and let $f(\zeta) = a_l\zeta^l + a_{l+1}\zeta^{l+1} + \cdots = a_l\zeta^l + o(|\zeta|^l)$, $l \geq 1$. If we replace $\zeta \approx c_1\sigma^{m-1}z$, then $f(\zeta(z)) = f(z) \approx a_l(c_1\sigma^{m-1}z)^l + a_{l+1}(c_1\sigma^{m-1}z)^{l+1} + \cdots = a_l(c_1\sigma^{m-1}z)^l + o((c_1\sigma^{m-1}z)^l) + \cdots = a_1z'^l + o(z'^l)$, where $z' = c_1\sigma^{m-1}z, m \geq 1$. By changing variable $z' \mapsto z$, $f(z)$ can be expressed as $a_lz^l + o(z^l)$, $l \geq 1$, which is consistent with $f(z)$ in (7).

(b) If $f(z) : D \to N$ is harmonic non $\pm$ holomorphic of $J \leq 0$ in (8) with S and consider $f : D - S \to N$, then

$$\left(f^*h\right)_{2\times 2} \approx \begin{bmatrix} \lambda|Bn\bar{z}^{n-1}|^2 & 0 \\ 0 & \lambda|Bn\bar{z}^{n-1}|^2 \end{bmatrix},$$

as $|z| = \sigma \to 0$, and f^*h is Riemannian and almost Hermitian. Similarly to (a), we can transform $f(z) : D \to N$ into an anti-holomorphic map $f(\zeta) : D \to N$ with $\zeta \approx c_2\sigma^{n-1}\bar{z}$ except at S.

(c) (i) If $f(z) : D \to N$ is harmonic non $\pm$ holomorphic of $J \geq 0$ in (9) ($|C| > |D|$) with S and consider $f : D - S \to N$, then we can write $w(z) \approx Cz^k(1 + \frac{D}{C}(\frac{\bar{z}}{z})^k) \approx Cz^k(1+\frac{D}{C}(-i)^k) = Gz^k$, where $G = C((1+\frac{D}{C}(-i)^k)$, $\frac{\bar{z}}{z} = \frac{x^2-y^2}{x^2+y^2} - i\frac{2xy}{x^2+y^2} \to -i$, as $|z| = \sqrt{x^2+y^2} \to 0$ (i.e. $x \to 0, y \to 0$, for instance, let $x = \epsilon$, $y = \epsilon$). Thus

$$(f^*h)_{2\times 2} \approx \begin{bmatrix} \lambda|Gkz^{k-1}|^2 & 0 \\ 0 & \lambda|Gkz^{k-1}|^2 \end{bmatrix},$$

as $|z| = \sigma \to 0$, and f^*h is Riemannian and almost Hermitian. Similarly to (a), we can transform $f(z) : D \to N$ into a holomorphic map $f(\zeta) : D \to N$ with $\zeta \approx c_3\sigma^{k-1}z$ except at S. (ii) If $f : D \to N$ is harmonic non $\pm$ holomorphic map $J \leq 0$ in (9) ($|C| < |D|$) with S and consider $f : D - S \to N$, then we can write $w(z) \approx D\bar{z}^k(\frac{C}{D}(\frac{z}{\bar{z}})^k + 1) \approx D\bar{z}^k(\frac{C}{D}i^k + 1) = H\bar{z}^k$, where $H = D(\frac{C}{D}i^k + 1)$, $\frac{z}{\bar{z}} \to i$, as $|z| \to 0$. Thus

$$(f^*h)_{2\times 2} \approx \begin{bmatrix} \lambda|Hk\bar{z}^{k-1}|^2 & 0 \\ 0 & \lambda|Hk\bar{z}^{k-1}|^2 \end{bmatrix},$$

as $|z| = \sigma \to 0$, and f^*h is Riemannian and almost Hermitian. Similarly to (a), we can transform $f : D \to N$ into an anti-holomorphic map $f(\zeta) : D \to N$ with $\zeta \approx c_4\sigma^{k-1}\bar{z}$ except at S.

Consequently, by (a), (b) and (c) we can transform a harmonic non $\pm$ holomorphic map $f(z) : D \to N$ of $J \geq 0$ (resp. $J \leq 0$) into a holomorphic (resp. anti-holomorphic) map $f(\zeta) : D \to N$ except at S. We now apply the Gauss-Bonnet formula to the domain D by deleting n isolated stationary points S to $f(\zeta) : D - S \to N$ leaving S in terms of z (not in ζ), and have

$$2\pi(\chi(D) - n) + \int_\gamma \Theta + \text{ integral of } \Theta \text{ over } n \text{ stationary points } S = \int_{f(D)} K\Omega.$$

If we transform the above $f(\zeta)$ back to $f(\zeta(z)) = f(z) : D - S \to N$ of $J \geq 0$ (resp. $J \leq 0$), then the Hermitian metric h in N induces the almost Hermitian metric f^*h in D except at S, and get

$$2\pi(\chi(D) - n) + \int_\gamma \Theta + \text{ integral of } \Theta \text{ over } n \text{ stationary points } S = \int_{f(D)} K\Omega.$$

For the third term in the local coordinate z, as we draw a small circle Σ_δ of radius δ around a stationary point p of $f(z)$ and attach an outward normal vector at each point on the circle, then it will be an inward normal vector of the complement of the domain. The integral of Θ in (19) over this vector field tends to $-2\pi n(p, a)$ (= $m > 0$ in (7) or $k > 0$ in (9) when $|C| > |D|$; =$n < 0$ in (8) or $k < 0$ in (9) when $|C| < |D|$), as $\delta \to 0$. Hence, we obtain

$$2\pi \chi(D) + \int_\gamma \Theta + 2\pi \sum_{i=1}^{n} (n(p_i, a) - 1) = \int_{f(D)} K\Omega.$$

Theorem 4.1. If $f : D \to N$ be a harmonic map of Jacobian $J_f \geq 0$ (resp. $J_f \leq 0$) with isolated stationary points, then

$$2\pi \chi(D) + \int_\gamma \Theta + 2\pi \sum_{i=1}^{n} (n(p_i, a) - 1) = \int_{f(D)} K\Omega, \tag{27}$$

where $\chi(D)$ is the Euler characteristic of D, $n(p_i, a) - 1$ (resp. $-n(p_i, a) + 1$), $i = 1, \cdots, n$, are the stationary indices.

Remark 4.2. If $f : D \to N$ is $\pm$ holomorphic and J_f is automatically ≥ 0 (resp. ≤ 0) with isolated stationary points, then (27) is known [5, 27].

Corollary 4.3. If D is a d-sheet covering surface of N without boundary, then by $\int_{f(D)} K\Omega = 2\pi d\chi(N)$, (27) reduces to

$$2 - 2g(D) + Ind R = d(2 - 2g(N)), \tag{28}$$

where $Ind\ R = \sum_{i=1}^{n} (n(p_i, a) - 1)$ (resp. $\sum_{i=1}^{n} (-n(p_i, a) + 1)$) is the index of ramification. This is exactly the Hurwitz's formula.

Wood [26] applied the Gauss-Bonnet formula to a harmonic map $f : M \to N$ between compact connected real analytic surfaces equipped with real analytic Riemannian metrics (N is orientable), and obtained the following inequality by applying Sampson's [22] maximum principle for a harmonic map between surfaces.

Proposition 4.4. Let $K(f(M))$ and $\chi(f(M))$ be the total curvature and the Euler characteristic of the image of $f : M \to N$. Then we have

$$2\chi(f(M)) \leq K(f(M)). \tag{29}$$

5. Biharmonic maps of Riemann surfaces

Biharmonic maps between Riemannian manifolds were first studied by Jiang [16, 17] in 1986. A map $f : M \to N$ between Riemann manifolds is *biharmonic* iff it is the critical point of the bi-energy functional

$$E_2(f) = \frac{1}{2}\int_M ||(d+d^*)^2 f||^2 = \frac{1}{2}\int_M ||\tau(f)||^2 dv.$$

In terms of the Euler-Lagrange equation, $f : M \to N$ is $biharmonic$ between Riemannian manifolds iff the $bitension\ field$

$$\begin{aligned} \tau_2(f) = J_f^\alpha(\tau f) &= g^{ij}\tau(f)^\alpha_{|i|j} + g^{ij}R'^\alpha_{\beta\gamma\mu} f_i^\beta f_j^\gamma \tau(f)^\mu \\ &= \triangle\tau(f)^\alpha + R'^\alpha(df, df)\tau(f) = 0, \end{aligned} \tag{30}$$

i.e., the tension field $\tau(f)$ is a Jacobi field, where R' is the Riemannian curvature of N.

We can apply (30) to compute the bitension field of a biharmonic map between Riemann surfaces. By [21], a map $f : (M, g = \sigma dz d\bar{z}) \to (N, h = \lambda dw d\bar{w})$ between Riemann surfaces with conformal metrics is biharmonic iff

$$\begin{aligned} \tau_2(f) &= \frac{4}{\sigma}\{\tau_{z\bar{z}} + (ln\,\lambda)_w[\tau_z f_{\bar{z}} + \tau_{\bar{z}} f_z + \frac{1}{4}\sigma\tau^2] \\ &\quad + \bar{\tau} f_z f_{\bar{z}} (ln\,\lambda)_{w\bar{w}} + \tau f_z f_{\bar{z}} (ln\,\lambda)_{ww}\} = 0. \end{aligned} \tag{31}$$

Proposition 5.1. [16] If $f : M \to N$ is a biharmonic map from a compact Riemannian manifold M into a Riemannian manifold N with non-positive curvature, then f is a harmonic map.

In particular, if $f : M \to N$ is a biharmonic map from a compact Riemann surface M into a Riemann surface N with non-positive curvature, then f is a harmonic map. Assume that N is a compact Riemann surface, by the Gauss-Bonnet formula we have $\int_N K dv = 2\pi\chi = 4\pi(1-g)$, where K is the Gaussian curvature of N, $\chi = 2(1-g)$ is the characteristic of N, and g is the genus of N. Therefore, any biharharmonic map from a compact Riemann surface M into a compact Riemann surface N of genus $g \geq 1$ is a harmonic map. Hence, there only exist biharmonic non-harmonic maps from compact Riemann surfaces into spheres.

Example 5.2. A theorem in [19] is stated as follows: Let $\psi : M \to S^n(\frac{r}{\sqrt{2}})$ be a non-constant map from a compact Riemannian manifold into an n-dimensional sphere with radius $\frac{r}{\sqrt{2}}$, and $i : S^n(\frac{r}{\sqrt{2}}) \to S^{n+1}(r)$ be an inclusion. The map $f = i \circ \psi : M \to S^{n+1}(r)$ is biharmonic non-harmonic if and only if ψ is harmonic with constant energy density $e(\psi)(= \frac{1}{2}|d\psi|^2)$. Let $\psi : T^2 = S^1 \times S^1 \to S^1(\frac{1}{\sqrt{2}})$ from a 2-torus into 2-sphere be defined by $\psi(t, s) = (\frac{1}{\sqrt{2}}cos(t+s), \frac{1}{\sqrt{2}}sin(t+s))$, and $i : S^1(\frac{1}{\sqrt{2}}) \hookrightarrow S^2$ be an inclusion. By a calculation, we have $e(\psi) = 1/2$. Hence, $f = i \circ \psi : T^2 \to S^2$ is biharmonic non-harmonic by the above theorem. The degree of the biharmonic map f is zero, since f is not onto. Eells and Wood [15] showed that there exists no harmonic map from T^2 into S^2 of degree ± 1. It is an open problem whether there exists a biharmonic non-harmonic map from T^2 into S^2 of degree ± 1 or not.

Proposition 5.3. [2, 20] Let $f : (M^2, g) \to (N^n, h)$ be a map from a Riemann surface into an n-dimensional Riemannian manifold, $\tilde{g} = \rho^2 g$ be a conformal change of the metric g, and $\tau_2(f)_g$ and $\tau_2(f)_{\tilde{g}}$ be the bitension fields of f with respect to the metrics g and $\tilde{g}$, respectively. Then

$$\tau_2(f)_{\tilde{g}} = \frac{1}{\rho^4}\{\tau_2(f)_g - 2(\triangle ln\, \rho + 2|grad\, ln\, \rho|^2 \tau(f)_g) - 4\nabla^f_{grad\, ln\, \rho}\tau(f)_g\}. \quad (32)$$

The biharmonicity of f is not necessarily conformally invariant on a Riemann surface except

$$2(\triangle ln\, \rho + 2|grad\, ln\, \rho|^2 \tau(f)_g) + 4\nabla^f_{grad\, ln\, \rho}\tau(f)_g = 0.$$

Recently, Baird, Loubeau and Oniciuc [3] studied harmonic and biharmonic maps from surfaces into Riemannian manifolds, and obtained some results.

Lemma 5.4. [16] If a biharmonic map $f : M \to N$ from an m-dimensional compact Riemannian manifold M into an n-dimensional Riemannian manifold N, then

$$\begin{aligned}\frac{1}{2}\ \frac{d^2}{dt^2}E_2(f_t)|_{t=0} &= \int_M ||\triangle V + R'(df(e_i), V)df(e_i)||^2 dv \\ &+ \int_M < V, (\nabla'_{df(e_i))}R')(df(e_k)), \tau(f))V + (\nabla'_{\tau(\bar{f})}R')(df(e_k), V)df(e_k) \\ &+ R(\tau(f), V)\tau(f) + 2R'(df(e_k), V)\nabla_{e_k}\tau(\bar{f}) + 2R'(df(e_k), \tau(f))\nabla_{e_k}V > dv \quad (33)\end{aligned}$$

where $\{e_i\}$ is a local frame at a point in M, $V = \frac{\partial f_t}{\partial t}|_{t=0}$, ∇ is the connection in $T^*(M) \otimes f^{-1}TN$, and ∇' is the Levi-Civita connection in TN.

Let $f : M \to N$ be a biharmonic map between Riemannian manifolds. If $\frac{d^2}{dt^2}E_2(f_t)|_{t=0} \geq 0$, then f is a stable biharmonic map. If we consider a harmonic map as a biharmonic map, then by (33) we have $\frac{d^2}{dt^2}E_2(f_t)|_{t=0} \geq 0$ and it is automatically stable.

Theorem 5.5. If $f : M \to N$ is a stable biharmonic map from a compact Riemannian manifold into a Riemannian manifold N with positive constant sectional curvature $K > 0$ satisfying the conservation law, then f is a harmonic map.

Proof. Since N has the constant sectional curvature, (33) reduces to

$$\begin{aligned}\frac{d^2}{dt^2}E_2(f_t)|_{t=0} &= 2\int_M ||\triangle V + R'(df(e_i),V)df(e_i)||^2 dv \\ &+2\int_M < V,\ R'(\tau(f),V)\tau(f) + 2R'(df(e_k),V)\nabla_{e_k}\tau(f) \\ &+2R'(df(e_k),\tau(f))\nabla_{e_k}\bar{V} > dv. \end{aligned} \tag{34}$$

In particular, let $V = \tau(f)$, and by the biharmonicity of f and N with positive constant sectional curvature K, (34) yields

$$\begin{aligned}\frac{d^2}{dt^2}\quad E_2(f_t)|_{t=0} &= 8\int_M < R'(df(e_i,\tau(f))\nabla_{e_k}\tau(f),\tau(f) > dv \\ = \quad 8K\int_M &\Big\{ < df(e_k),\nabla_{e_k} > |\tau(f)|^2 - < df(e_k,\tau(f) >< \tau(f),\nabla_{e_k}\tau(f) > \Big\} dv.\end{aligned} \tag{35}$$

Because f satisfies the conservation law [1], we have

$$\begin{aligned} < df(e_k),\tau(f) > &= 0, \\ < df(e_k),\nabla_{e_k}\tau(f) > &= - < \nabla_{e_k}df(e_k),\tau(f) >= -|\tau(f)|^2. \end{aligned} \tag{36}$$

Substituting (36) into (35), we obtain

$$\frac{d^2}{dt^2}E_2(f_t)|_{t=0} = -8K\int_M |\tau(f)|^4 dv \geq 0.$$

Hence, $\tau(f) = 0$ and f must be a harmonic map. ■

Corollary 5.6. If $f : M \to S^n$ is a stable biharmonic map from a compact Riemannian manifold into an n-sphere satisfying the conservation law, then f is a harmonic map.

In particular, if $f : M \to S^2$ is a stable biharmonic map from a compact Riemann surface into a 2-sphere satisfying the conservation law, then f is a harmonic map.

Let $f : M \to S^n(1)$ be a biharmonic map from an m-dimensional Riemannian manifold into an n-dimensional unit sphere. We can apply Lemma 5.3 to rewrite (33) as the Hessian of the bi-energy $E_2(f)$ as follows:

$$H(E_2(f))(X,Y) = \int_M < I_f(X),Y > dv,$$

where

$$I_f(X) = \triangle(\triangle X) + \triangle(trace(X, df\cdot)df\cdot - |df|^2X) + 2(d\tau_\Box(f), df)X + |\tau_\Box(f)|^2X - 2trace(X, d\tau_\Box(f)\cdot)df - 2trace(\tau_\Box(f), dX\cdot)df\cdot - (\tau_\Box(f), X)\tau_\Box(f) + trace(df\cdot, \triangle^f X)df\cdot + trace(df, trace(X, df\cdot)df\cdot)df\cdot - 2|df|^2 trace(df\cdot, X)df\cdot + 2(dX, df)\tau_\Box(f) - |df|^2\triangle^f X + |df|^4X, \quad (37)$$

for $X, Y \in \Gamma(f^{-1}TS^n(1))$.

Let $\iota : S^n \to S^n$ be an identity map. Then (37) reduces to

$$I(X) = \triangle(\triangle X) - 2(n-1)\triangle X + (n-1)^2X.$$

Thus $\iota : S^n \to S^n$ is stable, since $I(X) \geq 0$. The *nullity* is the dimension of the kernel of I which is exactly the dimension of the vector space $\{X \in \Gamma(TS^n) | \triangle X = (n-1)X\}$. By the Hodge decomposition theorem for S^n [13], for $X \in \Gamma(TS^n)$ we can write

$$X = Z + du, \quad d^*Z = 0, \quad u \in C^\infty(S^n),$$

where Z and du lie in orthogonal subspaces invariant by $\triangle$. Let $\triangle_i X = (\bar{\triangle} X^\flat)^\#$ preserves invariantly on these subspaces using musical isomorphism, and $\bar{\triangle}$ is the Laplacian acting on $\Lambda^1(S^n)$. It is known that $\triangle X = \triangle_i(X) - (n-1)X$ which implies $I(X) = 0 \Leftrightarrow \triangle_i(X) = 2(n-1)X$. Therefore,

$$\triangle_i(X) = 2(n-1)X \Leftrightarrow \triangle_i(Z) = 2(n-1)Z \ \& \ \triangle_i du = 2(n-1)du.$$

Hence,

$$I(X) = 0 \Leftrightarrow Z \text{ is a Killing field } \& \ \triangle u = 2(n-1)u.$$

By [13], the first eigenvalues of $\triangle$ which act on $C^\infty(S^n)$ are 0, n, $2(n+1)$, and the eigenvalue n has the multiplicity $n+1$. Hence, $2(n-1)$ is a eigenvalue iff $n = 2$, its multiplicity is 3, and $nullity(\iota) = 6$. If $n \geq 3$, we obtain

$$nullity(\iota) = dim\{Z \in \Gamma(TS^n) | Z\, is\, a\, Killing\, field\} = \frac{n(n+1)}{2}.$$

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