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Approximation by a Class of New Type Bernstein Polynomials of one and two Variables

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Abstract

In this paper we deal with the following generalized Bernstein polynomials

$$F_{n,a,b}(f;x) = \sum_{k=0}^{n} f\left(\frac{k(n+a)}{n(n+b)}\right) q_{n,k,a,b}(x), 0 \le x \le \frac{n+a}{n+b}$$

where $q_{n,k,a,b}(x) = \left(\frac{n+b}{n+a}\right)^n \left(\begin{array}{c} n \\ k \end{array}\right) x^k \left(\frac{n+a}{n+b} - x\right)^{n-k}, 0 \le a \le b$. And also we give these polynomials for bivariate functions corresponding to the both square and triangle domain respectively. We give rate of approximation of all these polynomials and some plots and numerical tables. We show that approximation of $F_{n,a,b}$ is better than approximation of both Bernstein polynomials B_n ([3]) and one of modifications of Bernstein polynomials by Deo and his collaborates V_n (see [17]) for $b - a > 1$.

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1. Introduction

The more classical examples of linear positive operators throughout approximation process are the Bernstein polynomials, which are defined by Bernstein [3] as following:

$$B_n(f;x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) \varphi_{n,k}(x), 0 \le x \le 1 \tag{1}$$

for any $f \in C[0,1]$, where $\varphi_{n,k}(x) = \binom{n}{k} x^k(1-x)^{n-k}$. Due to the importance of Bernstein polynomials, many of their generalizations and related topics has been intensive research. [2], [5], [13], [16], [14], [8] and [17]. And also, the Bernstein polynomials and their modifications have been used in many branches and computer (for example see [15], [11]).

Stancu [17] generalized (1) as following

$$B_{n,(\alpha,\beta)}(f;x) = \sum_{k=0}^{n} f\left(\frac{k+\alpha}{n+\beta}\right)\varphi_{n,k}(x), 0 \le x \le 1 \tag{2}$$

where $\alpha, \beta \in \mathbb{R}$ and $0 \le \alpha \le \beta, n = 1, 2, \dots$.

Recently Deo and his collaborates [6] gave a modification of (1) as

$$V_n(f;x) = \sum_{k=0}^{n} f\left(\frac{k}{n+1}\right) p_{n,k}(x), 0 \le x \le \frac{n}{n+1} \tag{3}$$

where $p_{n,k}(x) = \left(1+\frac{1}{n}\right)^n \binom{n}{k} x^k \left(\frac{n}{n+1} - x\right)^{n-k}$. The operators (3) convert in classical Bernstein polynomials which are given in (1) for n sufficiently large.

In this paper we introduce a new type Bernstein polynomials as

$$F_{n,a,b}(f;x) = \sum_{k=0}^{n} f\left(\frac{k(n+a)}{n(n+b)}\right) q_{n,k,a,b}(x), 0 \le x \le \frac{n+a}{n+b} \tag{4}$$

where $q_{n,k,a,b}(x) = \left(\frac{n+b}{n+a}\right)^n \binom{n}{k} x^k \left(\frac{n+a}{n+b} - x\right)^{n-k}, 0 \le a \le b.$

It is clear that the operators (4) convert in classical Bernstein polynomials for $a = b$, and convert in (3) for $a = 0$ and $b = 1$. Therefore, the operators (4) include both (1) and (3). Although (2) and (4) may seem similar but they are different.

The first purpose of this article is to see approximation of the operators (4) in the space of continuous functions.

The second is to give definitions of operators for bivariate functions as (4) corresponding to the both square and triangle domain respectively. We also study approximation and rate of approximation of the operators which mentioned above in the space of two variables continuous functions.

The last is to give some plots and numerical examples corresponding to the obtained approximation results. We see that in plots and numerical values:

1) Approximation of $V_n(f;x)$ to f is better than approximation of $B_n(f;x)$ to f. Mean, $|V_n(f;x) - f(x)| \le |B_n(f;x) - f(x)|$
2) If $b-a < 1$, then $|V_n(f;x) - f(x)| \le \left|F_{n,a,b}(f;x) - f(x)\right| \le |B_n(f;x) - f(x)|$
3) If $b-a > 1$, then $\left|F_{n,a,b}(f;x) - f(x)\right| \le |V_n(f;x) - f(x)| \le |B_n(f;x) - f(x)|$.

For all these inequalities see figures and tables 1, 2 and 3.

2. Approximation of One Variable of New Type Bernstein Polynomials

The new type Bernstein polynomials given in (4) are sequences of linear positive operators in $C\left(\left[0, \dfrac{n+a}{n+b}\right]\right)$, where $C(A)$ denotes the class of all continuous functions defined on A. Because of this reality we can use the study of approximation the classical Korovkin's theorem [12], (see also [1])

Firstly, we give the following Lemma.

Lemma 2.1. The following equalities are hold for the polynomials (4).

$$F_{n,a,b}(1;x) = 1 \tag{5}$$

$$F_{n,a,b}(t;x) = x \tag{6}$$

$$F_{n,a,b}(t^2;x) = x^2 + \frac{x\,[(n+a)-(n+b)x]}{n(n+b)} \tag{7}$$

Proof.

$$\begin{aligned}
F_{n,a,b}(1;x) &= \left(\frac{n+b}{n+a}\right)^n \sum_{k=0}^{n} \binom{n}{k} x^k \left(\frac{n+a}{n+b}-x\right)^{n-k} \\
&= \left(\frac{n+b}{n+a}\right)^n \left(\frac{n+a}{n+b}\right)^n = 1. \\
F_{n,a,b}(t;x) &= \left(\frac{n+b}{n+a}\right)^{n-1} \sum_{k=1}^{n} \frac{k}{n}\binom{n}{k} x^k \left(\frac{n+a}{n+b}-x\right)^{n-k} \\
&= \left(\frac{n+b}{n+a}\right)^{n-1} \sum_{k=1}^{n} \binom{n-1}{k-1} x^k \left(\frac{n+a}{n+b}-x\right)^{n-k} \\
&= \left(\frac{n+b}{n+a}\right)^{n-1} \sum_{k=0}^{n+1} \binom{n-1}{k} x^{k+1} \left(\frac{n+a}{n+b}-x\right)^{n-1-k} \\
&= x\left(\frac{n+b}{n+a}\right)^{n-1} \sum_{k=0}^{n+1} \binom{n-1}{k} x^{k} \left(\frac{n+a}{n+b}-x\right)^{n-1-k} \\
&= x\left(\frac{n+b}{n+a}\right)^{n-1} \left(\frac{n+a}{n+b}\right)^{n-1} = x.
\end{aligned}$$

We can write $k^2 = k(k-1)+k$, then

$$
\begin{aligned}
F_{n,a,b}(t^2;x) &= \left(\frac{n+b}{n+a}\right)^{n-2}\sum_{k=1}^{n}\frac{k^2}{n^2}\binom{n}{k}x^k\left(\frac{n+a}{n+b}-x\right)^{n-k}\\
&= \frac{n-1}{n}\left(\frac{n+b}{n+a}\right)^{n-2}\sum_{k=2}^{n}\frac{k(k-1)}{n(n-1)}\binom{n}{k}x^k\left(\frac{n+a}{n+b}-x\right)^{n-k}\\
&\quad+\frac{1}{n}\left(\frac{n+b}{n+a}\right)^{n-2}\sum_{k=2}^{n}\frac{k}{n}\binom{n}{k}x^k\left(\frac{n+a}{n+b}-x\right)^{n-k}\\
&= \frac{n-1}{n}\left(\frac{n+b}{n+a}\right)^{n-2}\sum_{k=2}^{n}\binom{n-2}{k-2}x^k\left(\frac{n+a}{n+b}-x\right)^{n-k}\\
&\quad+\frac{1}{n}\left(\frac{n+b}{n+a}\right)^{-1}\left(\frac{n+b}{n+a}\right)^{n-1}\sum_{k=2}^{n}\frac{k}{n}\binom{n}{k}x^k\left(\frac{n+a}{n+b}-x\right)^{n-k}\\
&= \frac{n-1}{n}\left(\frac{n+b}{n+a}\right)^{n-2}\sum_{k=0}^{n-2}\binom{n-2}{k}x^{k+2}\left(\frac{n+a}{n+b}-x\right)^{n-2-k}\\
&\quad+\frac{1}{n}\left(\frac{n+a}{n+b}\right)x\\
&= \frac{n-1}{n}x^2+\frac{1}{n}\left(\frac{n+a}{n+b}\right)x = x^2+\frac{x\left[(n+a)-(n+b)x\right]}{n(n+b)}.
\end{aligned}
$$

■

Theorem 2.2. If $f \in C[0,1]$, then

$$\lim_{n\to\infty} F_{n,a,b}(f;x) = f(x)$$

for each $x \in \left[0, \frac{n+a}{n+b}\right]$.

Proof. From Lemma 2.1,

$$
\begin{aligned}
\lim_{n\to\infty}\max_{x\in\left[0,\frac{n+a}{n+b}\right]}\left|F_{n,a,b}(t^2;x)-x^2\right| &= \lim_{n\to\infty}\max_{x\in\left[0,\frac{n+a}{n+b}\right]}\left|\frac{x\left[(n+a)-(n+b)x\right]}{n(n+b)}\right|\\
&= \lim_{n\to\infty}\left(\frac{n+a}{n+b}\right)^2\frac{1}{4n} = 0.
\end{aligned}
$$

Thus we get

$$\lim_{n\to\infty}\max_{x\in\left[0,\frac{n+a}{n+b}\right]}\left|F_{n,a,b}(t^v;x)-x^v\right| = 0, v = 0,1,2.$$

Therefore the proof follows from Korovkin's theorem. ■

Lemma 2.3. Let $p-th$ degree moment for the polynomials (4) defined by

$$T_{n,p}(x) = F_{n,a,b}((t-x)^p; x), p = 0, 1, 2, \dots \tag{8}$$

Then we have $T_{n,0}(x) = 1$, $T_{n,1}(x) = 0$ and

$$nT_{n,p+1}(x) = x\left(\frac{n+a}{n+b} - x\right)\left[T^{|}_{n,p}(x) + nT_{n,p-1}(x)\right]. \tag{9}$$

Consequently, for each $x \in \left[0, \frac{n+a}{n+b}\right]$

(i) $T_{n,p}(x)$ is a polynomial in x of degree $\leq p$,

(ii) $T_{n,p}(x) = O\left(n^{-\left[\left|\frac{p+1}{2}\right|\right]}\right)$ where $[|a|]$ denotes the integral part of a.

Proof. First we get the following relation:

$$x\left(\frac{n+a}{n+b} - x\right)q^{|}_{n,k,a,b}(x) = n\left(\frac{k(n+a)}{n(n+b)} - x\right)q_{n,k,a,b}(x) \tag{10}$$

From the definition (9) we get

$$T^{|}_{n,p}(x) = \sum_{k=0}^{n} q^{|}_{n,k,a,b}(x)\left(\frac{k(n+a)}{n(n+b)} - x\right)^p - pT_{n,p-1}(x)$$

and from (9) we get

$$\begin{aligned}
&x\left(\frac{n+a}{n+b} - x\right)\left[T^{|}_{n,p}(x) + pT_{n,p-1}(x)\right] \\
&= \sum_{k=0}^{n} x\left(\frac{n+a}{n+b} - x\right)q^{|}_{n,k,a,b}(x)\left(\frac{k(n+a)}{n(n+b)} - x\right)^p \\
&= n\sum_{k=0}^{n} q_{n,k,a,b}(x)\left(\frac{k(n+a)}{n(n+b)} - x\right)^{p+1} \\
&= nT_{n,p+1}(x).
\end{aligned}$$

$$\max_{x\in\left[0,\frac{n+a}{n+b}\right]} \left|T_{n,2}(x)\right| = \frac{1}{4n}\left(\frac{n+a}{n+b}\right)^2 \tag{11}$$

Now we will measure smoothness of approximation of continuous function f by the polynomial (4). In order to do that we use modulus of continuity of function f defined by

$$\omega(f;\delta) = \sup_{\substack{x,y\in[\alpha,\beta] \\ |t-x|\leq\delta}} |f(t) - f(x)| \tag{12}$$

for any positive numbers δ. $\omega(f;\delta)$ has some useful properties which can be found, for instance [1, cf. 266-269]. ■

Theorem 2.4. If $f \in C\left(\left[0, \dfrac{n+a}{n+b}\right]\right)$, then the following inequality holds.

$$\left|F_{n,a,b}(f;x) - f(x)\right| \leq \left[1 + \frac{1}{2}\left(\frac{n+a}{n+b}\right)\right]\omega(f;\frac{1}{\sqrt{n}}).$$

Proof. From the well-known properties of modulus of continuity we have

$$|f(t) - f(x)| \leq \omega(f;\delta_n)\left[1 + \frac{|t-x|}{\delta_n}\right]$$

where (δ_n) is any sequences of positive numbers. Since the polynomials $F_{n,a,b}(f;x)$ are also linear positive operators, we have

$$\left|F_{n,a,b}(f;x) - f(x)\right| \leq \left[1 + \frac{1}{\delta_n}F_{n,a,b}(\,|t-x|\,;x)\right]\omega(f;\delta_n).$$

Use Cauchy-Schwartz inequality, (5) and (11), then we get

$$\begin{aligned}
\left|F_{n,a,b}(f;x) - f(x)\right| &\leq \left[1 + \frac{1}{\delta_n}\sqrt{\left|T_{n,2}(x)\right|}\right]\omega(f;\delta_n) \\
&\leq \left[1 + \frac{1}{\delta_n}\sqrt{\frac{1}{4n}\left(\frac{n+a}{n+b}\right)^2}\right]\omega(f;\delta_n) \\
&= \left[1 + \frac{1}{\delta_n}\frac{1}{2}\left(\frac{n+a}{n+b}\right)\sqrt{\frac{1}{n}}\right]\omega(f;\delta_n)
\end{aligned}$$

Put $\delta_n = \sqrt{\dfrac{1}{n}}$, then the we get desired result. ■

3. Approximation of Bidimensional New Type Bernstein Polynomials

As it is known, there are two type bidimensional of the polynomials (1): First we give the following polynomials corresponding to the square of (4):

$$F_{n,m,a,b}(f;x,y)=\sum_{k=0}^{n}\sum_{j=0}^{m}f\left(\frac{k(n+a)}{n(n+b)},\frac{j(m+a)}{m(m+b)}\right)Q_{n,k,m,j,a,b}(x,y),(x,y)\in D \tag{13}$$

where $D=\left[0,\frac{n+a}{n+b}\right]\times\left[0,\frac{m+a}{m+b}\right]$ and $Q_{n,k,,m,ja,b}(x,y)=q_{n,k,a,b}(x)\times q_{m,j,a,b}(y)$.

The second we give the following polynomials corresponding to the triangle of (4):

$$F^{*}_{n,a,b}(f;x,y)=\sum_{k=0}^{n}\sum_{j=0}^{n-k}f\left(\frac{k(n+a)}{n(n+b)},\frac{j(n+a)}{n(n+b)}\right)Q_{n,k,j,a,b}(x,y),(x,y)\in\Delta \tag{14}$$

where $\Delta=\left\{(x,y)\mid 0\le x+y\le\frac{n+a}{n+b}\right\}$ and

$$Q_{n,k,j,a,b}(x,y)=\left(\frac{n+b}{n+a}\right)^{n}\binom{n}{k}\binom{n-k}{j}x^{k}y^{j}\left(\frac{n+a}{n+b}-x-y\right)^{n-k-j}.$$

If $f(x,y)=g(x)+h(y)$ or $f(x,y)=g(x)\times h(y)$ then the polynomial (13) reduce $F_{n,m,a,b}(f;x,y)=F_{n,a,b}(g;x)+F_{m,a,b}(h;y)$ or $F_{n,m,a,b}(f;x,y)=F_{n,a,b}(g;x)\times F_{m,a,b}(h;y)$ respectively. Because of these facts, using well known analogous of Korovkin's theorem of Volkov [19] two dimensional and its modifications as in Theorem 2.2 we get that: If f is continuous function in the rectangle D, then

$$\lim_{\substack{n\to\infty\\ m\to\infty}}\max_{(x,y)\in D}\left|F_{n,m,a,b}(f;x,y)-f(x,y)\right|=0.$$

Similar statements come true for the polynomials which are given (14) on triangle Δ.

Now we want to measure the smoothness of the approximation by using the partial moduli of continuity of $f(x,y)$

$$\begin{aligned}\omega^{(1)}(f;\delta)&=\sup_{y}\sup_{|t-x|\le\delta}|f(t,y)-f(x,y)|,\\ \omega^{(2)}(f;\delta)&=\sup_{x}\sup_{|u-y|\le\delta}|f(x,u)-f(x,y)|,\end{aligned}$$

and the complete moduli of continuity of $f(x,y)$

$$\omega(f;\delta)=\sup_{\sqrt{(t-x)^2+(u-y)^2}\le\delta}|f(t,u)-f(x,y)|.$$

Theorem 3.1. If $f \in C(D)$, then the following inequalities holds:

$$\left|F_{n,m,a,b}(f;x,y) - f(x,y)\right| \leq \left[1 + \frac{\alpha_{nm}}{2}\right]\left[\omega^{(1)}\left(f;\frac{1}{\sqrt{n}}\right) + \omega^{(2)}\left(f;\frac{1}{\sqrt{m}}\right)\right], \quad (15)$$

$$\left|F_{n,m,a,b}(f;x,y) - f(x,y)\right| \leq \left[1 + \frac{\alpha_{nm}}{2}\right]\omega\left(f;\frac{1}{\sqrt{n}} + \frac{1}{\sqrt{m}}\right) \quad (16)$$

where $\alpha_{nm} = \max\left\{\left(\frac{n+a}{n+b}\right), \left(\frac{m+a}{m+b}\right)\right\}$.

Proof. First we prove (15). Because of linearity of $F_{n,m,a,b}$, properties of modulus of continuity, use Cauchy-Schwartz inequality and (4), (11) we have

$$\begin{aligned}
& \left|F_{n,m,a,b}(f;x,y) - f(x,y)\right| \\
= & \left|\sum_{k=0}^{n}\sum_{j=0}^{m}\left\{f\left(\frac{k(n+a)}{n(n+b)}, \frac{j(m+a)}{m(m+b)}\right) - f\left(x, \frac{j(m+a)}{m(m+b)}\right)\right.\right. \\
& \left.\left. + f\left(x, \frac{j(m+a)}{m(m+b)}\right) - f(x,y)\right\} Q_{n,k,m,j,a,b}(x,y)\right| \\
\leq & \sum_{k=0}^{n}\sum_{j=0}^{m}\left|f\left(\frac{k(n+a)}{n(n+b)}, \frac{j(m+a)}{m(m+b)}\right) - f\left(x, \frac{j(m+a)}{m(m+b)}\right)\right| Q_{n,k,m,j,a,b}(x,y) \\
& + \sum_{k=0}^{n}\sum_{j=0}^{m}\left|f\left(x, \frac{j(m+a)}{m(m+b)}\right) - f(x,y)\right| Q_{n,k,m,j,a,b}(x,y) \\
: & = I_1 + I_2
\end{aligned}$$

$$\begin{aligned}
I_1 \leq & \sum_{k=0}^{n}\sum_{j=0}^{m}\omega^{(1)}(f;\left|\frac{k(n+a)}{n(n+b)} - x\right|)Q_{n,k,m,j,a,b}(x,y) \\
\leq & \sum_{k=0}^{n}\sum_{j=0}^{m}\left[1 + \frac{1}{\delta_n}\left|\frac{k(n+a)}{n(n+b)} - x\right|\right] Q_{n,k,m,j,a,b}(x,y)\omega^{(1)}(f;\delta_n) \\
\leq & \left[1 + \frac{1}{\delta_n}\sqrt{\sum_{k=0}^{n}\left(\frac{k(n+a)}{n(n+b)} - x\right) q_{n,k,a,b}(x)}\right]\omega^{(1)}(f;\delta_n) \\
= & \left[1 + \frac{1}{\delta_n}\sqrt{\frac{1}{4n}\left(\frac{n+a}{n+b}\right)^2}\right]\omega^{(1)}(f;\delta_n) \\
= & \left[1 + \frac{1}{\delta_n}\frac{1}{2}\left(\frac{n+a}{n+b}\right)\frac{1}{\sqrt{n}}\right]\omega^{(1)}(f;\delta_n)
\end{aligned}$$

Put $\delta_n = \sqrt{\frac{1}{n}}$ then we get $I_1 \leq \left[1 + \frac{1}{2}\left(\frac{n+a}{n+b}\right)\right] \omega^{(1)}(f; \frac{1}{\sqrt{n}})$.

Similar calculations are made to obtain $I_2 \leq \left[1 + \frac{1}{2}\left(\frac{m+a}{m+b}\right)\right] \omega^{(2)}(f; \frac{1}{\sqrt{m}})$.

Now we prove (16).

$$\begin{aligned}
&\left|F_{n,m,a,b}(f; x, y) - f(x, y)\right| \\
= &\left|\sum_{k=0}^{n}\sum_{j=0}^{m} f\left(\frac{k(n+a)}{n(n+b)}, \frac{j(m+a)}{m(m+b)}\right) Q_{n,k,m,j,a,b}(x, y) - f(x, y)\right| \\
\leq &\sum_{k=0}^{n}\sum_{j=0}^{m} \left|f\left(\frac{k(n+a)}{n(n+b)}, \frac{j(m+a)}{m(m+b)}\right) - f(x, y)\right| Q_{n,k,m,j,a,b}(x, y)
\end{aligned}$$

Now use properties of modulus of continuity, then we have

$$\begin{aligned}
&\left|f\left(\frac{k(n+a)}{n(n+b)}, \frac{j(m+a)}{m(m+b)}\right) - f(x, y)\right| \\
\leq &\ \omega\left(f; \sqrt{\left(\frac{k(n+a)}{n(n+b)} - x\right)^2 + \left(\frac{j(m+a)}{m(m+b)} - y\right)^2}\right) \\
\leq &\left[1 + \frac{1}{\delta_{nm}}\sqrt{\left(\frac{k(n+a)}{n(n+b)} - x\right)^2 + \left(\frac{j(m+a)}{m(m+b)} - y\right)^2}\right] \omega(f; \delta_{nm}).
\end{aligned}$$

Use Cauchy-Schwartz inequality and (4), (11) we get

$$\begin{aligned}
&\left|F_{n,m,a,b}(f; x, y) - f(x, y)\right| \\
\leq &\left[1 + \frac{1}{\delta_{nm}}\sum_{k=0}^{n}\sum_{j=0}^{m}\sqrt{\left(\frac{k(n+a)}{n(n+b)} - x\right)^2 + \left(\frac{j(m+a)}{m(m+b)} - y\right)^2} Q_{n,k,m,j,a,b}(x, y)\right] \\
&\times \omega(f; \delta_{nm}) \\
\leq &\left[1 + \frac{1}{\delta_{nm}}\sqrt{\sum_{k=0}^{n}\sum_{j=0}^{m}\left\{\left(\frac{k(n+a)}{n(n+b)} - x\right)^2 + \left(\frac{j(m+a)}{m(m+b)} - y\right)^2\right\} Q_{n,k,m,j,a,b}(x, y)}\right] \\
&\times \omega(f; \delta_{nm}) \\
\leq &\left[1 + \frac{1}{\delta_{nm}}\sqrt{\frac{1}{4n}\left(\frac{n+a}{n+b}\right)^2 + \frac{1}{4m}\left(\frac{m+a}{m+b}\right)^2}\right] \omega(f; \delta_{nm}).
\end{aligned}$$

Now put

$$\delta_{nm} = \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{m}}$$

where

$$\alpha_{nm} = \max\left\{\left(\frac{n+a}{n+b}\right), \left(\frac{m+a}{m+b}\right)\right\},$$

then we get

$$\left|F_{n,m,a,b}(f;x,y) - f(x,y)\right| \leq \left[1 + \frac{\alpha_{nm}}{2}\right]\omega\left(f; \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{m}}\right).$$

Now we need the following property of complete modulus of continuity which proved in [9]

$$\omega(f;\lambda_1\delta_1,\lambda_2\delta_2) \leq (1+\lambda_1+\lambda_2)\,\omega(f;\delta_1,\delta_2) \tag{17}$$

where

$$\omega(f;\delta_1,\delta_2) = \sup_{|t-x|\leq\delta_1}\ \sup_{|u-y|\leq\delta_2} |f(t,u) - f(x,y)|.$$

■

Theorem 3.2. If $f \in C(\Delta)$, then the following inequalities holds:

$$\left|F^*_{n,a,b}(f;x,y) - f(x,y)\right| \leq \left[1 + \left(\frac{n+a}{n+b}\right)\right]\omega\left(f; \frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}}\right) \tag{18}$$

Proof. Because of (17) we have

$$\begin{aligned}
\left|f\left(\frac{k(n+a)}{n(n+b)}, \frac{j(n+a)}{n(n+b)}\right) - f(x,y)\right| &\leq \omega\left(f; \left|\frac{k(n+a)}{n(n+b)} - x\right|, \left|\frac{j(n+a)}{n(n+b)} - y\right|\right) \\
&\leq \left[1 + \frac{1}{\delta_1}\left|\frac{k(n+a)}{n(n+b)} - x\right| + \frac{1}{\delta_2}\left|\frac{j(n+a)}{n(n+b)} - y\right|\right] \\
&\quad \times\omega(f;\delta_1,\delta_2).
\end{aligned}$$

And because of linearity of $F_{n,a,b}(f;x,y)$

$$\begin{aligned}
\left|F_{n,a,b}(f;x,y) - f(x,y)\right| &= \left|\sum_{k=0}^{n}\sum_{j=0}^{m} f\left(\frac{k(n+a)}{n(n+b)}, \frac{j(n+a)}{n(n+b)}\right) Q_{n,k,j,a,b}(x,y) - f(x,y)\right| \\
&\leq \sum_{k=0}^{n}\sum_{j=0}^{m}\left|f\left(\frac{k(n+a)}{n(n+b)}, \frac{j(n+a)}{n(n+b)}\right) - f(x,y)\right| Q_{n,k,j,a,b}(x,y) \\
&\leq \left[1 + \frac{1}{\delta_1}\sum_{k=0}^{n}\sum_{j=0}^{m}\left|\frac{k(n+a)}{n(n+b)} - x\right| Q_{n,k,j,a,b}(x,y)\right. \\
&\quad \left. + \frac{1}{\delta_2}\sum_{k=0}^{n}\sum_{j=0}^{m}\left|\frac{j(n+a)}{n(n+b)} - y\right| Q_{n,k,j,a,b}(x,y)\right]\omega(f;\delta_1,\delta_2)
\end{aligned}$$

Now use Cauchy-Schwartz inequality, we have

$$\begin{aligned}
\left|F_{n,a,b}(f;x,y)-f(x,y)\right| &\leq \left[1+\frac{1}{\delta_1}\sqrt{\sum_{k=0}^{n}\sum_{j=0}^{m}\left(\frac{k(n+a)}{n(n+b)}-x\right)^2 Q_{n,k,j,a,b}(x,y)}\right.\\
&\left.+\frac{1}{\delta_2}\sqrt{\sum_{k=0}^{n}\sum_{j=0}^{m}\left(\frac{j(n+a)}{n(n+b)}-y\right)^2 Q_{n,k,j,a,b}(x,y)}\right]\omega\left(f;\delta_1,\delta_2\right)\\
&\leq \left[1+\frac{1}{\delta_1}\sqrt{\frac{1}{4n}\left(\frac{n+a}{n+b}\right)^2}\right.\\
&\left.+\frac{1}{\delta_2}\sqrt{\frac{1}{4n}\left(\frac{n+a}{n+b}\right)^2}\right]\omega\left(f;\delta_1,\delta_2\right)
\end{aligned}$$

Put $\delta_1=\delta_2=\frac{1}{\sqrt{n}}$ then we get desired result. ■

4. Applications

We give some animated plots and numerical tables here to see the difference of approximation of $F_{n,a,b}$ than B_n and V_n.

1) Figures.

It can be seem that if $b-a<1$, then approximation of V_n is better than approximation of both B_n and $F_{n,a,b}$ (Fig1), if $b-a>1$then approximation of $F_{n,a,b}$ is better than approximation of both B_n and V_n (Fig2, Fig3). The function we use here is $f(x)=\frac{x\sin(\pi x)}{1+x^2}$.

For the function $f(x,y)=\frac{\sin((x+y)\pi)}{\sqrt{1+x^4+y^4}}$ we have the following figures: It can be seem that if $b-a<1$, then approximation of $V_{n,m}$ is better than approximation of both $B_{n,m}$ and $F_{n,m,a,b}$ (Fig4), if $b-a>1$ then approximation of $F_{n,m,a,b}$ is better than approximation of both $B_{n,m}$ and $V_{n,m}$ (Fig5). To get a better approximation is more appropriate to use $F_{n,m,a,b}$ instead of of $B_{n,m}$ and $V_{n,m}$.

2.Numeric tables: Let $x=(x_n)$ and $y=(y_n)$ be sequences with limits x_0 and y_0 respectively. If

$$\lim_{n\to\infty}\frac{|y_n-y_0|}{|x_n-x_0|}=0,$$

then it is said that y converges faster than x (See:[4]).

(Or equivalent $\lim_{n\to\infty}\frac{|x_n-x_0|}{|y_n-y_0|}=\infty$). And if $\lim_{n\to\infty}\frac{|y_n-y_0|}{|x_n-x_0|}=M<\infty$, than it is said that x and y converges equivalent. That means that there is any n_0 such that for all

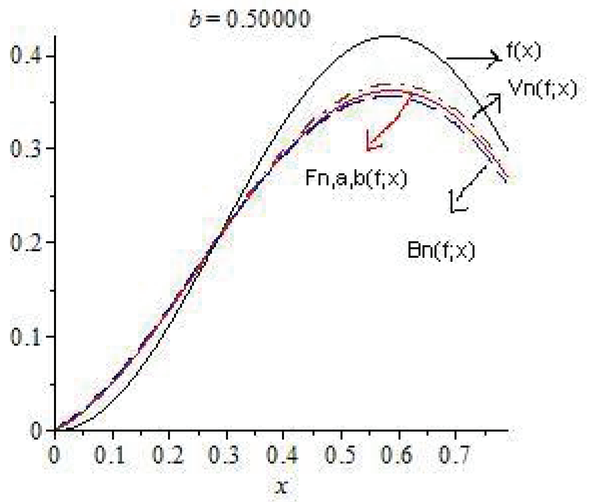

Figure 1: Comparison $F_{10,0,0.5}$ with B_{10} and V_{10}

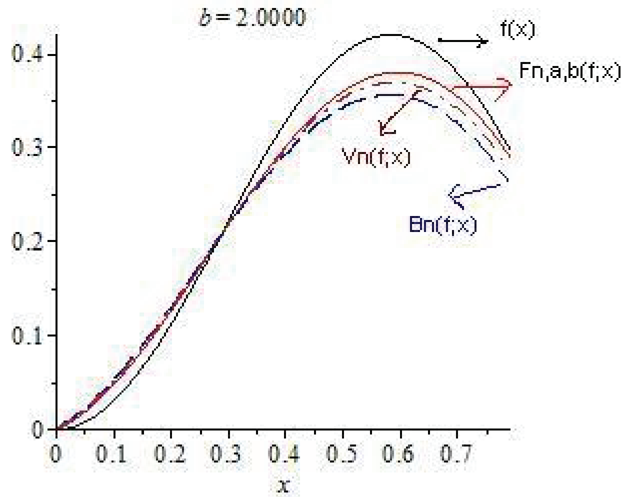

Figure 2: Comparison $F_{10,0,2}$ with B_{10} and V_{10}

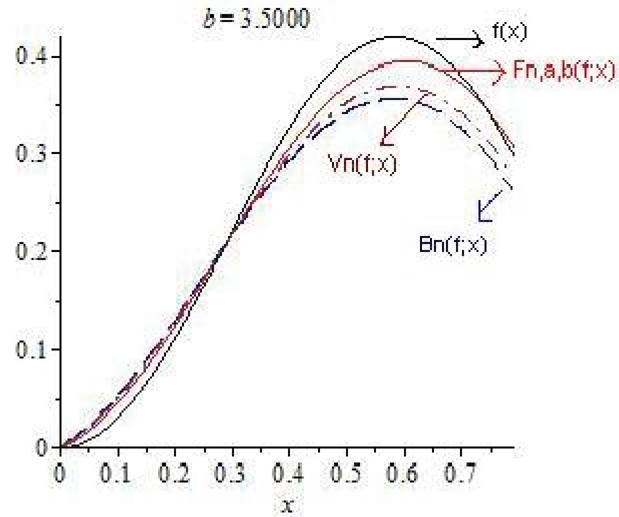

Figure 3: Comparison $F_{10,0,3.5}$ with B_{10} and V_{10}

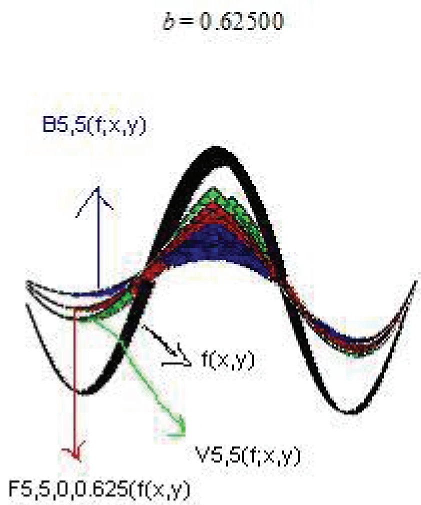

Figure 4: Comparison $F_{5,5,0,0.625}$ with $B_{5,5}$ and $V_{5,5}$

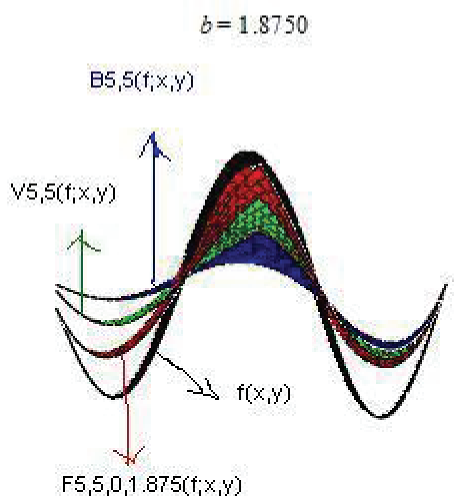

Figure 5: Comparison $F_{5,5,0,1.875}$ with $B_{5,5}$ and $V_{5,5}$

$n \geq n_0$ the inequality $|y_n - y_0| \leq M\,|x_n - x_0|$ holds. If $0 < M < 1$, then it is sa,d that convergence of y is better than convergence of x. If $1 < M < \infty$,then it is said that convergence of x is better than convergence of y (See:[18]).

Our aim was to give

$$\lim_{n\to\infty} \sup_{0\leq x\leq \frac{n+a}{n+b}} \frac{\left|F_{n,a,b}(f;x) - f(x)\right|}{|B_n(f;x) - f(x)|} \text{ and } \lim_{n\to\infty} \sup_{0\leq x\leq \frac{n+a}{n+b}} \frac{\left|F_{n,a,b}(f;x) - f(x)\right|}{|V_n(f;x) - f(x)|}$$

in order to compare approximation of $f(x)$ by $F_{n,a,b}(f;x)$ and $B_n(f;x)$, approximation of $f(x)$ by $F_{n,a,b}(f;x)$ and $V_n(f;x)$. But we could not do that by our software programma which we have. That is why we give those comparisons for some, a, b, n and x.

The following numerical values are calculated for the function

$$f(x) = (1 + x + x^2)\sin(\pi x).$$

$$\frac{|V_n(f;x) - f(x)|}{|B_n(f;x) - f(x)|}$$

$n \setminus x$	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
10	0.93200	0.10324	0.83363	0.83358	0.81321	0.77465	0.70426	0.55588
100	0.98928	0.98968	0.98520	0.98326	0.98012	0.97528	0.96713	0.95073
500	0.99778	0.99755	0.99713	0.99667	0.99601	0.99501	0.99335	0.99003
900	0.99878	0.99872	0.99837	0.99813	0.99777	0.99722	0.99629	0.99445

Table 1: Comparison of Approximation of $V_n(f;x)$ and $B_n(f;x)$ to $f(x)$

$$\frac{\left|F_{n,1,3}(f;x)-f(x)\right|}{|B_n(f;x)-f(x)|}$$

$n \setminus x$	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
10	0.88039	0.40223	0.72437	0.72108	0.68483	0.61768	0.49562	0.23818
100	0.97897	0.97972	0.97099	0.96717	0.96101	0.95152	0.93552	0.90335
500	0.99562	0.99527	0.99423	0.99334	0.99203	0.99005	0.98675	0.98013
900	0.99758	0.99744	0.99676	0.99628	0.99555	0.99445	0.99261	0.98892

Table 2: Comparison of Approximation of $F_{n,1,3}(f;x)$ and $B_n(f;x)$ to $f(x)$

$$\frac{\left|F_{n,1,3}(f;x)-f(x)\right|}{|V_n(f;x)-f(x)|}$$

$n \setminus x$	0.1	0.2	0.3	0.4	.0.5	0.6	0.7	0.8
10	0.94462	3.8959	0.86894	0.86503	0.84213	0.79737	0.70374	0.42847
100	0.98958	0.98993	0.98558	0.98364	0.98050	0.97563	0.96732	0.95016
500	0.99783	0.99772	0.99709	0.99666	0.99601	0.99502	0.99335	0.99000
900	0.99879	0.99871	0.99838	0.99814	0.99778	0.99722	0.99630	0.99444

Table 3: Comparison of Approximation of $F_{n,1,3}(f;x)$and $V_n(f;x)$ to $f(x)$

Remark 4.1. We have aims to discuss the following operators (19) and (20) which are Kantorovich [10] and Durrmeyer [7] type respectively elsewhere.

$$T_{n,a,b}(f;x)=\frac{n+b}{n+a}(n+1)\sum_{k=0}^{n}q_{n,k,a,b}(x)\int_{\frac{k(n+a)}{(n+1)(n+b)}}^{\frac{(k+1)(n+a)}{(n+1)(n+b)}}f(t)dt \tag{19}$$

$$T_{n,a,b}(f;x)=\frac{n+b}{n+a}(n+1)\sum_{k=0}^{n}q_{n,k,a,b}(x)\int_{0}^{\frac{n+a}{n+b}}q_{n,k,a,b}(t)f(t)dt \tag{20}$$

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