Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 9, Number 1 (2013), pp. 45-54

http://www.ripublication.com/gjpam.htm

Oscillation of Neutral Advanced Difference Equation

A. Murugesan

Department of Mathematics,
Government Arts College (Autonomous),
Salem - 636 007, Tamil Nadu, India.
Email: amurugesan3@gmail.com

Abstract

In this article, we establish oscillation criteria for solutions to the first order neutral advanced difference equation

$$\Delta[x(n) - p(n)x(\tau(n))] - q(n)x\big(\sigma(n)\big) = 0, \quad n \geq n_0 \qquad (*)$$

where $\{p(n)\}, \{q(n)\}$ are sequences of real numbers, $\{\sigma(n)\}$ is a sequence of positive integers such that $\sigma(n) > n + 1$ and $\{\tau(n)\}$ is a nondecreasing sequence of nonnegative integers such that $\tau(n) < n$. Sufficient conditions for existence of a positive solution for (*) when $p(n) \equiv 0$ is also established.

Mathematics Subject Classification: 39A10

Keywords: Oscillation, neutral, advanced, difference equation.

Introduction

We consider the first order neutral advanced difference equation of the form

$$\Delta[x(n) - p(n)x(\tau(n))] - q(n)x\big(\sigma(n)\big) = 0, \quad n \geq n_0 \qquad (1)$$

where $\{p(n)\}$, $\{q(n)\}$ are sequences of real numbers, $\{\sigma(n)\}$ is a sequence of positive integers such that $\sigma(n) > n + 1$, $\{\tau(n)\}$ is a nondecreasing sequence of nonnegative integers such that $\tau(n) < n$ and Δ is the forward difference operator defined by the equation $\Delta x(n) = x(n+1) - x(n)$.

We present some sufficient conditions such that every solution of (1) is either oscillatory or tends to zero as $n \to \infty$.

By a solution of equation (1), we mean a real sequence $\{x(n)\}, n \in N\big(\tau(n_0)\big) = \{\tau(n_0), \tau(n_0) + 1, \tau(n_0) + 2, \ldots\}$ satisfying (1). We consider only such solutions which are non trivial for all large n. A solution of (1) is said to oscillatory if it is neither eventually positive nor eventually negative, otherwise it is called nonoscillatory.

The qualitative properties of the solution of the advanced difference equation (1) have been the subject of our investigation. With respect to the oscillation of delay difference equation with variable coefficients, reader can refer to [3,4,8-11]. For the several background on difference equation, we can refer to [1,2,5-7].

The following conditions are assumed to be hold throughout the paper.

(C_1) $\{p(n)\}, \{q(n)\}$ are sequences of nonnegative real numbers and $\{q(n)\}$ is not identically zero.

(C_2) $\{\sigma(n)\}$ is a nondecreasing sequence of positive integers such that $\sigma(n) > n+1$ on $N(n_0)$, $\lim_{n\to\infty}\sigma(n) = \infty$ and $\lim_{n\to\infty}(\sigma(n) - n) = \infty$.

$(C_3$ $\{\tau(n)\}$ is a nondecreasing sequence of nonnegative integers such that $\tau(n) < n$ on $N(n_0) = \{n_0, n_0+1, \dots\}$ and $\lim_{n\to\infty}\tau(n) = \infty$.

(C_4) There exist a constant p such that $0 \le p(n) \le p < 1$.

Before giving the main results, we present some lemmas which will be used in the proofs of Theorems.

Lemma 1.1 Set

$$z(n) = x(n) - p(n)x(\tau(n)). \tag{2}$$

If $\{x(n)\}$ is an eventually positive solution of equation (1) such that

$$\limsup_{n\to\infty} x(n) > 0, \tag{3}$$

then $z(n) > 0$ eventually.

Proof. Let $\{x(n)\}$ be an eventually positive solution of (1) such that $\limsup_{n\to\infty} x(n) > 0$. Assume the contrary. That is, $z(n) < 0$ for large n. If $\{x(n)\}$ is unbounded, then there exists a sequence $\{n_k\}$ of integers such that

$$\lim_{k\to\infty} n_k = \infty \; and \; \lim_{k\to\infty} x(n_k) = \infty, where \; x(n_k) = \max_{n_0 \le n \le n_k} x(n).$$

Then from (2), we have

$$z(n_k) = x(n_k) - p(n_k)x(\tau(n_k)) \ge x(n_k)(1-p)$$

and hence $\lim_{k\to\infty} z(n_k) = \infty$.

This is a contradiction.

If $\{x(n)\}$ is bounded, then there is a sequence $\{n_k\}$ of integers such that

$$\lim_{k\to\infty} n_k = \infty \; and \; \lim_{k\to\infty} x(n_k) = \lim_{n\to\infty} x(n) = L > 0.$$

Since $\lim_{k\to\infty} x(\tau(n_k)) \le L$, we have

$$0 \ge \lim_{k\to\infty} z(n_k) \ \ge L(1-p) > 0.$$

This is also a contradiction and the proof is complete.

Lemma 1.2 *The sequence* $\{\sigma(n)\}$ *has the properties*

$$\sigma(\sigma(n)) \ge \sigma(n), \qquad n \ge n_0,$$
$$and \ \lim_{n\to\infty} \sigma(\sigma(n)) = \lim_{n\to\infty}(\sigma(\sigma(n)) - n) = \infty.$$

Proof. By (C_2), $\{\sigma(n)\}$ is nondecreasing, and

$$\sigma(n) > n \ for \ n \ge n_0.$$

So, we have

$$\sigma(\sigma(n)) \ge \sigma(n), \qquad for \ \ n \ge n_0.$$

Moreover, by this inequality, we can see easily that

$$\sigma(\sigma(n)) - n \ge \sigma(n) - n, \ \ n \ge n_0.$$

Taking limit as $n \to \infty$ on both sides, we obtain

$$\infty = \lim_{n\to\infty}(\sigma(n) - n) = \lim_{n\to\infty}(\sigma(\sigma(n)) - n).$$

By $\sigma(n) \to \infty \ as \ n \to \infty$, it is obvious that $\sigma(\sigma(n)) \to \infty \ as \ n \to \infty$.

Lemma 1.3 *Assume that*

$$\lim_{n\to\infty} inf \ q(n) \ne 0.$$

Then we have

$$\lim_{n\to\infty} \sum_{s=n+1}^{\sigma(n)+1} q(s) = \lim_{n\to\infty} \sum_{s=n+1}^{\sigma(\sigma(n))-1} q(s) = \infty.$$

Proof. Since $q(n) \ge 0 \ for \ n \ge n_0$, by assumption, we have

$$\liminf_{n\to\infty} q(n) > 0.$$

On the other hand, by discrete mean value theorem, we have

$$\sum_{s=n+1}^{\sigma(n)-1} q(s) \ge (\sigma(n) - n - 1) q(\bar{n}),$$

where $\bar{n} \in \{n+2, n+3, \dots, \sigma(n)-1\}$. Thus $\bar{n} \to \infty \ as \ n \to \infty$. Then it is clear that

$$\lim_{n\to\infty} \sum_{s=n+1}^{\sigma(n)-1} q(s) = \infty.$$

Since

$$\sum_{s=n+1}^{\sigma(n)-1} q(s) \le \sum_{s=n+1}^{\sigma(\sigma(n))-1} q(s),$$

we have

$$\lim_{n\to\infty} \sum_{s=n+1}^{\sigma(\sigma(n))-1} q(s) = \infty.$$

We are now in a position to state and prove our main results.

Main Results

Theorem 2.1 *Assume that*

$$\liminf_{n\to\infty} q(n) \neq 0.$$

Then every solution of (1) is either oscillatory or tends to zero.

Proof. Without loss of generality, we may assume that $\{x(n)\}$ is an eventually positive solution of (1) such that $\limsup_{n\to\infty} x(n) > 0$. Then by Lemma 1.1 , z(n) > 0 eventually, where z(n) is defined by (2). From (1) and (2), we obtain

$$\Delta z(n) - q(n) z\big(\sigma(n)\big) \ge 0, \qquad n \ge n_1 \ge n_0. \tag{4}$$

Since $\Delta z(n) \ge 0, n \ge n_1$, we have $\{z(n)\}$ is nondecreasing on $N(n_1)$. From (4), we can obtain

$$\ln \frac{z(\sigma(n))}{z(n+1)} \ge \sum_{s=n+1}^{\sigma(n)-1} q(s) \frac{z(\sigma(s))}{z(s+1)}, \qquad n \ge n_1. \tag{5}$$

Set

$$w(n) = \frac{z(\sigma(n))}{z(n+1)}, \qquad n \ge n_1. \tag{6}$$

It is clear that $w(n) \ge 1$, $n \ge n_1$.
So by the last inequality, we have

$$\ln w(n) \geq \sum_{s=n+1}^{\sigma(n)-1} q(s)w(s). \tag{7}$$

Let $l = \lim_{n\to+\infty} w(n)$, then $1 \leq l \leq +\infty$. Now we divide our discussion into the following two cases $i)$ $l \neq +\infty$ $ii)$ $l = +\infty$

l is finite

There exists a sequence $\{n_k\}$ of integers such that

$$\lim_{k\to\infty} n_k = +\infty, \quad and \quad \liminf_{n\to\infty} w(n) = \lim_{k\to+\infty} w(n_k) = 0.$$

Thus

$$l \liminf_{n\to\infty} \sum_{s=n+1}^{\sigma(n)-1} q(s) \leq \liminf_{n\to\infty} \sum_{s=n+1}^{\sigma(n)-1} w(s)q(s) \leq \liminf_{n\to\infty} \ln w(n) = \ln l$$

On the other hand, since $\{\sigma(n)\}$ is nondecreasing and $\{q(n)\}$ is nonnegative, so it follows

$$\liminf_{n\to\infty} \sum_{s=n+1}^{\sigma(n)-1} q(s) = \lim_{n\to\infty} \sum_{s=n+1}^{\sigma(n)-1} q(s).$$

Therefore we have

$$\lim_{n\to\infty} \sum_{s=n+1}^{\sigma(n)-1} q(s) \leq \frac{\ln l}{l} \leq \frac{1}{e}.$$

By Lemma 1.3, we see that it is a contradiction.

(ii) $l = +\infty$.

Thus

$$\lim_{n\to+\infty} \frac{z(\sigma(n))}{z(n+1)} = +\infty. \tag{8}$$

Summing (4) on both sides from $\sigma(n) + 1$ *to* $\sigma(\sigma(n)) - 1$, we have

$$z(\sigma(\sigma(n))) - z(\sigma(n)+1) \geq \sum_{s=\sigma(n)+1}^{\sigma(\sigma(n))-1} q(s)z(\sigma(s)), \qquad n \geq n_1$$

or

$$z\Big(\sigma\big(\sigma(n)\big)\Big) - z(\sigma(n) + 1)$$
$$\geq z\Big(\sigma\big(\sigma(n)\big)\Big) \sum_{s=\sigma(n)+1}^{\sigma(\sigma(n))-1} q(s)\ . \qquad (9)$$

Dividing both sides of this inequality by $z\Big(\sigma\big(\sigma(n)\big)\Big)$, we have

$$1 - \frac{z(\sigma(n) + 1)}{z(\sigma(\sigma(n)))} \geq \sum_{s=\sigma(n)+1}^{\sigma(\sigma(n))-1} q(s), \qquad n \geq n_1. \qquad (10)$$

And by (8), we know

$$\lim_{n\to\infty} \frac{z(\sigma(n) + 1)}{z(\sigma(\sigma(n)))} = \lim_{n\to\infty} \frac{z(n + 1)}{z(\sigma(n))} = 0.$$

Taking limit on both sides of inequality (10), in view of Lemmas 1.2 and 1.3, we have a contradiction.

The proof is complete.

Corollary 2.2 *Assume that*

$$\liminf_{n\to\infty} q(n) \neq 0.$$

Then every solution of the difference inequality

$$\Delta[x(n) - p(n)x(\tau(n))] - q(n)x\big(\sigma(n)\big) \geq 0, \ \ (or\ \leq 0), \qquad n \geq n_0, \qquad (11)$$

is either not positive (or negative) or tends to zero.

Corollary 2.3 *Assume that*

$$\lim_{n\to\infty} \sum_{s=n+1}^{\sigma(n)-1} q(s) > 1. \qquad (12)$$

Then every solution of (1) is either oscillatory or tends to zero and the every solution of the difference inequality (11) is either not positive or tends to zero.

Proof. For Corollary 2.3, we can see that in the proof of Theorem 2.1, if $\{x(n)\}$ is an eventually positive solution of (1), when l is finite, then we have

$$\lim_{n\to\infty} \sum_{s=n+1}^{\sigma(n)-1} q(s) \leq \frac{1}{e}$$

which contradicts (12). When $l = +\infty$, in view of (10), we have

$$\lim_{n\to\infty} \sum_{s=\sigma(n)+1}^{\sigma(\sigma(n))-1} q(s) \leq 1.$$

Since $\sigma(n) \to +\infty \; as \; n \to +\infty$, it follows

$$\lim_{n\to+\infty} \sum_{s=\sigma(n)+1}^{\sigma(\sigma(n))-1} q(s) = \lim_{n\to+\infty} \sum_{s=n+1}^{\sigma(n)-1} q(s) \leq 1,$$

which contradicts (12). Thus the result of Corollary 2.3 holds.

Note that the condition (12) is much weaker than the condition in Theorem 2.1. We can see this from Lemma 1.3.

Consider the following difference equation

$$\Delta[x(n) - p(n)x(\tau(n))] - f\left(n, x(\sigma(n))\right) = 0, \qquad n \geq n_0 \tag{13}$$

and the inequality

$$\Delta[x(n) - p(n)x(\tau(n))] - f\left(n, x(\sigma(n))\right) \geq 0, \qquad (or \; \leq 0), \qquad n \geq n_0 \tag{14}$$

The function $f: N \times R \to R$ satisfies the following conditions:

$a)\; f(n, v)v > 0, \;\; for \; v \neq 0, \;\; v \in R,$

$$b)\; f(n, 0) = 0,$$

$$c)\; |f(n, v)| \geq q(n)|v|, \qquad n \in N, \qquad v \in R,$$

where $\{p(n)\}$, $\{q(n)\}$, $\{\tau(n)\} \; and \; \{\sigma(n)\}$ are sequences appeared in the equation (1) and satisfies all the conditions mentioned at the beginning of this paper.

It follows a similar way to prove the following results.

Theorem 2.4 *Assume that*

$$\liminf_{n\to\infty} q(n) \neq 0.$$

Then every solution of equation (13) is either oscillatory or tends to zero and every solution of the inequality (14) is either not positive (or negative) or tends to zero.

Proof. As a matter of fact, if there exists an eventually positive solution $\{x(n)\}$ of the equation (13), then by equation (13) and the conditions on f, we have

$$\Delta[x(n) - p(n)x(\tau(n))] - q(n)x\left(\sigma(\sigma(n))\right) \geq 0, \quad n \geq n_1 \geq n_0.$$

Then the rest proof can follow the one that we have done in the proof of Theorem 2.1. It has similar steps if we have an eventually negative solution $\{x(n)\}$ of equation (13). Indeed, if $x(n) < 0, for \; n \geq n_1 \geq n_0$, we have

$$\Delta[x(n) - p(n)x(\tau(n))] - q(n)x\left(\sigma(n)\right) \leq 0, \quad n \geq n_1 .$$

Let $y(n) = -x(n)$, then $y(n) > 0$, $n \ge n_1$, it follows

$$\Delta[y(n) - p(n)y(\tau(n))] - q(n)y\big(\sigma(n)\big) \ge 0, \quad n \ge n_1 .$$

In the following, we investigate existence of positive solutions of (1) and equation (13).

Theorem 2.5 *Assume that*

$$p(n) \equiv 0 \quad on \quad N(n_0)$$

and

$$\prod_{n=n_0}^{\infty} (1 + eq(n)) \le e, \qquad n \ge n_0.$$

Then equation (1) has a positive solution.

Proof. For the convenience, we set

$$x(n) = \prod_{s=n_0}^{n-1} \lambda(s), \qquad n > n_0, \tag{15}$$

where $\{x(n)\}$ is a solution of equation (1). By this form, from equation (1), we have the following equation

$$\lambda(n) = 1 + q(n) \prod_{s=n}^{\sigma(n)-1} \lambda(s), \qquad n > n_0. \tag{16}$$

If we can prove that equation (16) has a solution $\{\lambda(n)\}$, then by the form of $x(n)$ in (15), we see that equation (1) has a positive solution. Consider a sequence $\{\lambda_k(n)\}$ as follows

$$\lambda_0(n) = 1 + eq(n),$$

$$\lambda_1(n) = 1 + q(n) \prod_{s=n}^{\sigma(n)-1} \lambda_0(s),$$

$$\dots \dots \dots \dots \dots \dots \dots$$

$$\dots \dots \dots \dots \dots \dots \dots$$

$$\dots \dots \dots \dots \dots \dots \dots$$

$$\lambda_k(n) = 1 + q(n) \prod_{s=n}^{\sigma(n)-1} \lambda_{k-1}(s),$$

Using the induction, we can prove that $\{\lambda_k(n)\}$ is a nonincreasing sequence, namely,

$$\lambda_{k-1}(n) \ge \lambda_k(n), \qquad k = 1,2,3, \dots$$

And so we have

$$1 + eq(n) \geq \lambda_k(n) \geq 0, \quad n \in N(n_0) \qquad for \qquad k = 1,2,3, \dots$$

It follows that there exists a sequence $\{\lambda(n)\}$ such that $\lambda_k(n) \to \lambda(n)$ as $n \to \infty$, and hence

$$\lambda(n) = \lim_{k\to\infty}\left\{1 + q(n)\prod_{s=n}^{\sigma(n)-1}\lambda_{k-1}(s)\right\}$$
$$= 1 + q(n)\prod_{s=n}^{\sigma(n)-1}\lambda(s).$$

It concludes that $\{\lambda(n)\}$ is a solution of equation (16).

Theorem 2.6 Assume that the function $f(n, v)$ is nondecreasing in v and

$$p(n) \equiv 0 \quad on \;\; N(n_0).$$

Suppose that

$$\prod_{n=n_0}^{\infty}[1 + f(n, e)] \leq e.$$

Then equation (13) has a positive solution.

Proof. We can prove this result by a similar way as we have done in the proof of Theorem 2.5. Set

$$x(n) = \prod_{s=n_0}^{n-1}\lambda(s), \qquad n > n_0$$

where $\{x(n)\}$ is a solution of (13). Then by (13) and the form of $x(n)$, we have the equation

$$\lambda(n) = 1 + \frac{f\left(n, \prod_{s=n_0}^{\sigma(n)-1}\lambda(s)\right)}{\prod_{s=n_0}^{n-1}\lambda(s)}, \qquad n > n_0. \tag{17}$$

If equation (17) has a solution $\{\lambda(n)\}$, then it follows that equation (13) has a positive solution. Construct a sequence $\{\lambda_k(n)\}$ as a follows

$$\lambda_0(n) = 1 + f(n, e),$$
$$\lambda_1(n) = 1 + \frac{f\left(n, \prod_{s=n_0}^{\sigma(n)-1}\lambda_0(s)\right)}{\prod_{s=n_0}^{n-1}\lambda_0(s)}$$

… … … … … …

… … … … … …

… … … … … …

$$\lambda_k(n) = 1 + \frac{f(n, \prod_{s=n_0}^{\sigma(n)-1} \lambda_{k-1}(s))}{\prod_{s=n_0}^{n-1} \lambda_{k-1}(s)} \quad for \quad n = 1,2,3,\dots$$

In view of the assumption, we see that $\lambda_k(n) \geq 0 \;\; for \;\; k = 0,1,2,3,\dots$. Furthermore by using the induction, we can prove that

$$1 + f(n, e) \geq \lambda_{k-1}(n) \geq \lambda_k(n) \geq 0, \qquad k = 1,2,3,\dots$$

It follows that there exists a sequence $\{\lambda_k(n)\}$ such that $\lambda_k(n) \to \lambda(n)$ as $k \to \infty$. So there exists a solution $\{\lambda(n)\}$ of equation (17).

The proof is complete.

References

[1] R.P. Agarwal, Difference Equations and Inequalities: Theory, Methods and Applications, Marcel Dekker, New York, 1992.
[2] S.N. Elaydi, An Introduction to Difference Equations, Springer Verlag, New York, 1996.
[3] D.A. Georgiou, E.A. Grove and G. Ladas, Oscillation of neutral difference equations with variable coefficients, Lecture notes in pure and Appl. Maths., 127, Dekker, New York(1991) 165-173.
[4] D.A. Georgiou, E.A. Grove and G. Ladas, Oscillation of neutral difference equations, Appl. Annal. 33(1980)300-309.
[5] I. Gyori and G. Ladas, Oscillation Theory of Delay Differential Equations with Applications, Clarendon Press, Oxford, 1991.
[6] W.G. Kelly and A.C. Peterson, Difference Equations: An Introduction with Applications, Academic Press, New York, 1991.
[7] V. Laksmikantham and D. Trigiante, Theory of Difference Equations: Numerical method and Applications, Academic press, New York, 1988.
[8] B.S. Lalli and B.G. Zhang, Oscillation and comparison theorems for certain neutral difference equations, J. Aust. Math. Soc. Ser. B, 34(1992) 254-256.
[9] B.S. Lalli, B.G. Zhang and J.Z. Li, On the oscillation of solutions of neutral difference equations, J. Math.Anal. Appl. 158(1991) 213-233.
[10] J.W. Li, Z.C. Wang and H.Q. Zhang, Oscillation of neutral delay difference equations, Diff. Equns. And Dyna. Syatems, 41(1996) 113-121.
[11] B.G. Zhang and S.S. Cheng, Oscillation criteria for a neutral difference equations with delay, Appl.Math.Lett., 8(1995), 13-17.