Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 9, Number 1 (2013), pp. 35-43

http://www.ripublication.com/gjpam.htm

The New Integral Transform ''Aboodh Transform''

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Abstract

In this paper a new integral transform namely Aboodh transform was applied to solve linear ordinary differential equations with constant coefficients.

Keywords: Aboodh transform- Differential Equations.

Introduction

Aboodh Transform is derived from the classical Fourier integral. Based on the mathematical simplicity of the Aboodh transform and its fundamental properties.

Aboodh transform was introduced by Khalid Aboodh to facilitate the process of solving ordinary and partial differential equations in the time domain.

Typically, Fourier, Laplace, ELzaki and Sumudu transforms are the convenient mathematical tools for solving differential equations,

Also Aboodh transform and some of its fundamental properties are used to solve differential equations.

A new transform called the Aboodh transform defined for function of exponential order we consider functions in the set A defined by

$$A = \{ f(t) : \exists M, k_1, k_2 > 0, |f(t)| < Me^{-vt} \} \qquad \to (1)$$

For a given function in the set A, the constant M must be finite number, k_1, k_2 may be finite or infinite.

The Aboodh transform denoted by the operator A (.) Defined by the integral equations

$$A\left[f(t)\right]=K(v)=\frac{1}{v}\int_0^\infty f(t)e^{-vt}dt \qquad ,t\geq 0,k_1\leq v\leq k_2 \qquad \rightarrow(2)$$

The variable v in this transform is used to factor the variable t in the argument of the function f . This transform has deeper Connection with the Laplace and Elzaki transform. We also present many different of properties of this new transform and Sumudu, Elzaki transforms, few properties exptent.

The purpose of this study is to show the applicability of this interesting new transform and its efficiency in solving the linear differential equations.

Aboodh Transform of the Some Functions

For any function $f(t)$, we assume that the integral equation (2) exists. The Sufficient Conditions for the existence of Aboodh transform are that $f(t)$ for $t\geq 0$ be piecewise continuous and of exponential order, Otherwise Aboodh transform may or may not exist.

In this section we find Aboodh transform of simple functions

(i) *let* $f(t)=1$ *,then*

$$A(1)=\frac{1}{v}\int_0^\infty e^{-vt}dt=\frac{1}{v}\left[\frac{1}{-v}e^{-vt}\right]_0^\infty=\frac{1}{-v^2}\left[0-1\right]=\frac{1}{v^2}$$

(ii) *let* $f(t)=t$ *then*

$$A(t)=\frac{1}{v}\int_0^\infty te^{-vt}dt$$

Integrating by parts to find that: $A(t)=\frac{1}{v^3}$

(iii) *let* $f(t)=t^n$ *then*

$$A\left(t^n\right)=\frac{1}{v}\int_0^\infty t^n e^{-vt}dt$$

Integrating by parts to find that: $A\left(t^n\right)=\frac{n!}{v^{n+2}}$

Substitution n = 0, 1, 2, 3, ……, n is an Integer numbers, $n>0$.

$$n=0,\rightarrow A\left(t^0\right)=A(1)=\frac{0!}{v^{0+2}}=\frac{1}{v^2}$$

$$n=1,\rightarrow A\left(t^1\right)=A(t)=\frac{1!}{v^{1+2}}=\frac{1}{v^3}$$

$$n=2,\rightarrow A\left(t^2\right)=A(t^2)=\frac{2!}{v^{2+2}}=\frac{2}{v^4}$$

$$iv)\qquad A(e^{at})=\frac{1}{v}\int_0^{\infty}e^{at}e^{-vt}dt=\frac{1}{v}\int_0^{\infty}e^{(a-v)t}dt=\frac{1}{v}.\frac{-1}{a-v}=\frac{1}{v}.\frac{1}{v-a}==\frac{1}{v^2-av}$$

So we can find the following transforms:-

$$a)\qquad A\left(\sin at\right)=\frac{1}{v}\int_0^{\infty}e^{-vt}\left(\frac{e^{iat}-e^{-iat}}{2i}\right)dt=\frac{1}{2iv}\int_0^{\infty}e^{(ia-v)t}-e^{(-ia-v)t}dt$$

$$=\frac{1}{2iv}\int_0^{\infty}e^{(ia-v)t}-e^{-(v+ia)t}dt=\frac{1}{2iv}.\left[\frac{1}{v-ia}-\frac{1}{v+ia}\right]=\frac{1}{2iv}\left[\frac{v+ia-v+ia}{v^2+a^2}\right]$$

$$=\frac{a}{v\left(v^2+a^2\right)}$$

$$b)\qquad A\left(\cos at\right)=\frac{1}{v}\int_0^{\infty}e^{-vt}\left(\frac{e^{iat}+e^{-iat}}{2}\right)dt=\frac{1}{2v}\int_0^{\infty}e^{(ia-v)t}+e^{(-ia-v)t}dt$$

$$=\frac{1}{2v}\int_0^{\infty}e^{(ia-v)t}+e^{-(v+ia)t}dt=\frac{1}{2v}.\left[\frac{1}{v-ia}+\frac{1}{v+ia}\right]=\frac{1}{2v}\left[\frac{v+ia+v-ia}{v^2+a^2}\right]$$

$$=\frac{1}{\left(v^2+a^2\right)}$$

$$c)\qquad A\left(\sinh at\right)=\frac{1}{v}\int_0^{\infty}e^{-vt}\left(\frac{e^{at}-e^{-at}}{2}\right)dt=\frac{1}{2v}\int_0^{\infty}e^{(a-v)t}-e^{(-a-v)t}dt$$

$$=\frac{1}{2v}\int_0^{\infty}e^{(a-v)t}-e^{-(v+a)t}dt=\frac{1}{2v}.\left[\frac{1}{v-a}-\frac{1}{v+a}\right]=\frac{1}{2v}\left[\frac{v+a-v+a}{v^2-a^2}\right]$$

$$=\frac{a}{v\left(v^2-a^2\right)}$$

$$d)\qquad A\left(\cosh at\right)=\frac{1}{v}\int_0^{\infty}e^{-vt}\left(\frac{e^{at}+e^{-at}}{2}\right)dt=\frac{1}{2v}\int_0^{\infty}e^{(a-v)t}+e^{(-a-v)t}dt$$

$$=\frac{1}{2v}\int_0^{\infty}e^{(a-v)t}+e^{-(v+a)t}dt=\frac{1}{2v}.\left[\frac{1}{v-a}+\frac{1}{v+a}\right]=\frac{1}{2v}\left[\frac{v+a+v-a}{v^2-a^2}\right]$$

$$=\frac{1}{\left(v^2-a^2\right)}$$

Theorem:

Let K(v) is the Aboodh transform of $\left[A(f(t))=K(v)\right]$*.then*

The New Integral Transform " Aboodh Transform"

$i)\quad A[f'(t)] = vk(v) - \frac{f(0)}{v}$

$ii)\quad A[f''(t)] = v^2 k(v) - \frac{f'(0)}{v} - f(0)$

$iii)\quad A\left[f^{(n)}(t)\right] = v^{(n)} k(v) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{v^{2-n+k}}$

Proof:

$i)\quad A[f'(t)] = \frac{1}{v}\int_0^\infty f'(t)e^{-vt}dt$

Integrating by parts to find that:

$A[f'(t)] = vk(v) - \frac{f(0)}{v}$

$ii)\quad A[f''(t)] = \frac{1}{v}\int_0^\infty f''(t)e^{-vt}dt$

Integrating by parts to find that:

$A[f''(t)] = v^2 k(v) - \frac{f'(0)}{v} - f(0)$

$iii)$ Can be proof by mathematical induction

Application of Aboodh Transform of Ordinary Differential Equations:-

As stated in the introduction of this paper, the Aboodh transform can be used as an effective tool. For analyzing the basic characteristics of a linear system governed by the differential equation in response to initial data. The following examples illustrate the use of the Aboodh transform in solving certain initial value problems described by ordinary differential equations.

Consider the first-order ordinary differential equation

$$\frac{dx}{dt} + px = f(t) \qquad ,t > 0 \qquad \rightarrow (3)$$

With the initial Condition

$$x(0) = a \qquad \rightarrow (4)$$

Where p and a are constants and $f(t)$ is an external input function so that its Aboodh transform exists.

Applying Aboodh transform of the equation (3) we have

$$A\bar{x} + p\bar{x}v = \bar{f}(v)$$

$$vk(v) - \frac{f(0)}{v} + p\bar{x}v = \bar{f}(v)$$

$$v\bar{x}(v) - \frac{x(0)}{v} + p\bar{x}v = \bar{f}(v)$$

$$\bar{x}(v)[v + p] = \frac{a}{v} + \bar{f}(v)$$

$$x(v) = \frac{a + v\bar{f}(v)}{v(v + p)}$$

The inverse Aboodh transform leads to the solution.

The second order linear ordinary differential equation has the general form

$$\frac{d^2y}{dx^2} + 2p\frac{dy}{dx} + qy = f(x) \qquad , x > 0 \qquad (5)$$

The initial conditions are

$$y(0) = a \qquad , \frac{dy}{dx}(0) = b \qquad (6)$$

Where p, a and b are constants. Applications of Aboodh transforms to this general initial value problem gives

$$v^2\bar{y}(v) - \frac{y'(0)}{v} - y(0) + 2pv\bar{y}(v) - \frac{2py(0)}{v} + q\bar{y}(v) = \bar{f}(v)$$

$$\bar{y}(v)\left[v^2 + 2pv + q\right] = \bar{f}(v) + \frac{b}{v} + a + \frac{2pa}{v}$$

$$\bar{y}(v) = \frac{v\bar{f}(v) + b + av + 2pa}{v(v^2 + 2pv + q)}$$

$$\bar{y}(v) = \frac{(\bar{f}(v) + a)}{(v^2 + 2pv + q)} + \frac{b + 2pa}{v(v^2 + 2pv + q)}$$

The inverse Aboodh transform gives the solution.

Example (1) consider the first order differential equation

$$\frac{dy}{dx} + y = 0 \qquad , y(0) = 1$$

Solution: take Aboodh transform to this equation gives

$$vA(y)-\frac{y(0)}{v}+A(y)=0$$

$$A(y)[v+1]=\frac{1}{v} \qquad \therefore A(y)=\frac{1}{v(v+1)}$$

Where $A(y)$ Is the Aboodh transform of the function $y(x)$

Example (2) solve the differential equation

$$y'+2y=x \qquad ,y(0)=1$$

Solution: take Aboodh transform to this equation gives

$$vA(y)-\frac{y(0)}{v}+2A(y)=A(x)$$

$$vA(y)-\frac{1}{v}+2A(y)=\frac{1}{v^3}$$

$$A(y)[v+2]=\frac{1}{v^3}+\frac{1}{v} \qquad \therefore A(y)=\frac{1+v^2}{v^3(v+2)}$$

The inverse Aboodh transform of this equation is simply obtained as

$$y(x)=\frac{1}{2}x+17e^{-2x}-16$$

Example (3) Let us consider the second-order differential equation

$$y''+y=0 \qquad ,y(0)=y'(0)=1$$

Solution: take Aboodh transforms from to this equation gives

$$v^2A(y)-\frac{y'(0)}{v}-y(0)+A(y)=0$$

$$v^2A(y)-\frac{1}{v}-1+A(y)=0$$

$$A(y)\left[v^2+1\right]=1+\frac{1}{v}$$

$$\therefore A(y)=\frac{v+1}{v(v^2+1)}$$

The inverse Aboodh transform of this equation is simply obtained as

$$y(x)=\cos x+\sin x$$

Example (4) Consider the second-order differential equation

$$y''-3y'+2y=0 \qquad ,y(0)=1 \qquad ,y'(0)=4$$

Solution: take Aboodh transforms of this equation we find that:

$$v^2A(y)-\frac{y'(0)}{v}-y(0)-3\left[vA(y)-\frac{y(0)}{v}\right]+2A(y)=0$$

$$v^2A(y)-\frac{4}{v}-1-3vA(y)+\frac{3}{v}+2A(y)=0$$

$$A(y)\left[v^2-3v+2\right]=\frac{4}{v}-\frac{3}{v}+1$$

$$A(y)\left[v^2-3v+2\right]=\frac{1+v}{v}$$

$$A(y)=\frac{1+v}{v(v-1)(v-2)}$$

$$A(y)=\frac{1}{v}\left[\frac{3}{v-2}-\frac{2}{v-1}\right]=3e^{2t}-2e^{t}$$

Example (5) Consider the second-order differential equation

$$y''+9y=\cos 2x \qquad ,y(0)=1 \qquad ,y(\frac{\pi}{2})=-1$$

Solution: since $y'(0)$ is not known, let $y'(0)=c$

Take Aboodh transforms from of this equation and using the conditions, we have

$$v^2k(v)-\frac{y'(0)}{v}-y(0)+9k(v)=\frac{1}{v^2+4}$$

$$k(v)\left[v^2+9\right]=\frac{1}{v^2+4}+\frac{c}{v}+1$$

We can write this equation in the form

$$k(v)=\frac{1}{(v^2+4)(v^2+9)}+\frac{c}{v(v^2+9)}+\frac{1}{v^2+9}$$

And invert to find the solution

$$y(x)=\frac{1}{2}\cos 2x+\frac{c}{3}\sin 3x+\frac{4}{5}\cos 3x$$

To determine c note that $y(\frac{\pi}{2})=-1$ thin we find $c=\frac{12}{5}$ then

$$y(x)=\frac{1}{5}\cos 2x+\frac{4}{5}\sin 3x+\frac{4}{5}\cos 3x$$

Example (6) Solve the differential equation

$$y'' - 3y' + 2y = 4e^{3x} \qquad , y(0) = -3 \qquad , y'(0) = 5$$

Solution: Take Aboodh transforms from of this equation and using the conditions, we have

$$v^2 k(v) - \frac{y'(0)}{v} - y(0) - 3vk(v) + 3\frac{y(0)}{v} + 2k(v) = 4.\frac{1}{v^2 - 3v}$$

$$k(v)\left[v^2 - 3v + 2\right] = \frac{4}{v^2 - 3v} - 3 + \frac{5}{v} + \frac{9}{v}$$

$$k(v) = \frac{\frac{4}{v^2 - 3v} - 3 + \frac{14}{v}}{v^2 - 3v + 2} = \frac{4}{(v^2 - 3v)(v^2 - 3v + 2)} + \frac{14 - 3v}{v(v^2 - 3v + 2)}$$

$$k(v) = \frac{1}{v}\left[\frac{2}{v-3} - \frac{4}{v-2} + \frac{2}{v-1}\right] + \frac{1}{v}\left[\frac{8}{v-2} - \frac{11}{v-1}\right]$$

Inverting to find the solution in the form

$$y(x) = 2e^{3x} + 4e^{2x} - 9e^{x}$$

Example (7) Find the solution of the following initial value problem:

$$y'' + 4y = 9x \qquad , \qquad y(0) = 0 \qquad y'(0) = 7$$

Solution: Take Aboodh transforms from of this equation and using the conditions, we have

$$v^2 k(v) - \frac{y'(0)}{v} - y(0) + 4k(v) = \frac{9}{v^3}$$

$$k(v)\left[v^2 + 4\right] = \frac{9}{v^3} + \frac{7}{v}$$

$$k(v) = \frac{7}{v(v^2+4)} + \frac{9}{v^3(v^2+4)} = \frac{7}{v(v^2+4)} + \frac{9}{v}\left[\frac{\frac{1}{4}}{v} - \frac{\frac{1}{4}v}{v^2+4}\right]$$

Inverting to find the solution in the form

$$y(x) = \frac{9}{4}x + \frac{19}{8}\sin 2x$$

Conclusion

The definition and application of the new transform " Aboodh transform " to the solution of ordinary differential equations has been demonstrated;

References

[1] Tarig M. Elzaki, The New Integral Transform “Elzaki Transform” Global Journal of Pure and Applied Mathematics, ISSN 0973-1768, Number 1(2011), pp. 57-64.

[2] Lokenath Debnath and D. Bhatta. Integral transform and their Application second Edition, Chapman & Hall /CRC (2006).

[3] G.K.watugala, simudu transform- anew integral transform to Solve differential equation and control engineering problems.Math Engrg Induct.6 (1998), no 4, 319-329.

[4] A.Kilicman and H.E.Gadain. An application of double Laplace transform and sumudu transform, Lobachevskii J. Math.30 (3) (2009), pp.214-223 .

[5] J. Zhang, Asumudu based algorithm m for solving differential equations Comp. Sci. J. Moldova 15(3) (2007), pp – 303-313 .

[6] Christian Constanda, Solution Techniques for Elementary Partial differential Equations, New York, 2002 .

[7] Dean G. Duffy, Transform Methods for solving partial differential Equations, 2nd Ed, Chapman & Hall / CRC, Boca Raton, FL, 2004 .

[8] Sunethra Weera Koon, Application of Sumudu transform to partial differential equation. INT. J. MATH. EDUC. Scl. TECHNOL, 1994, Vol.25, No2, 277-283.